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A sub-regularity inequality for conjugate systems on local fields

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Abstract. In this paper a sub-regularity inequality is established for conjugate systems over local fields, from which a classical F. and M. Riesz Theorem can be established. The F. and M. Riesz Theorem referred to is one which identifies a singular integral transformation $f \to \tilde{f}$ such that whenever f and $\tilde{f}$ are both finite Borel measure then each is an L^1 function. These results depend on generalizations of the properties of harmonic and sub-harmonic functions, least harmonic majorants, area integral operators and non-tangential boundary behavior of harmonic functions for local fields as developed in [1] and [8].

§ 1. Let K be a local field. That is, K is a locally compact, non-discrete, complete, totally disconnected field. (Such a field is a p-adic field, a finite algebraic extension of a p-adic field, or a field of formal Laurent series over a finite field. See [6] for details.) In a series of papers ([6], [7], [8], [9], [4], [5], and [3]) the details of harmonic analysis on local fields, K, and on the n-dimensional vector spaces over K, K^n, have been extensively treated.

In particular, the foundations for a treatment of H^p-spaces over K and K^n have been developed. From [8] we have the notion of regular functions on $K^n \times \boldsymbol{Z}$ and regularization of distributions on K^n. These generalize the notion of harmonic functions on euclidean half-spaces, $E^{n+1,+}$ and Poisson integrals of distributions on E^n. From [9], [4] and [5] we have the notion of singular integral operators and multipliers on local fields.

Recent work of J-A. Chao [1] has shown that the properties of regular functions essential to a treatment of H^p-space theory are valid. We have that there is a valid notion of sub-regular functions, least regular majorants of non-negative sub-regular functions, that regular functions on "bounded domains" are determined by their "boundary" values, and (most crucially) that the local field variant of the Lusin area integral theorem holds and that convergence of a regular function to boundary values on a set $E \subset K^n$ is equivalent, a.e., to local boundedness.

The central result for the theory of H^p-spaces that is then required is that if $(f, \tilde{f})$ is a "conjugate pair" in some appropriate sense, then there exists a p, $0 < p < 1$, for which $|(f, \tilde{f})|^p$ is sub-regular. One immediate consequence (see Chao [1]) of such a result is that if f and $\tilde{f}$ are both finite Borel measures then each is an L^1 function.

We will show in § 2, by a specific example, how such results can be established in local fields. In § 3 we will demonstrate how the idea of the proof of Lemma 1 in § 2 can be used to establish the sub-harmonicity of $|(f, \tilde{f})|^p$ for all $p > 0$, where $(f, \tilde{f})$ is a conjugate pair of harmonic functions in the complex plane, without reference to the harmonicity of $\log|f + i\tilde{f}|$, the concept of a Laplacian or, for that matter, the notion of differentiability.

§ 2. Let K be the 3-adic or 3-series field, $\mathfrak{o}$ the ring of integers in K, $\mathfrak{p}$ the (unique) maximal (principal) ideal in $\mathfrak{o}$. Then $\mathfrak{o}/\mathfrak{p} \cong GF(3)$. Let π be a non-trivial multiplicative character on K^* which is homogeneous of degree zero and ramified of degree one. (The reader is referred to [6] for notation and details.)

Then π is determined by its values on the non-zero elements of $GF(3)$. In this case, $\pi(1) = 1$ and $\pi(-1) = -1$ (an obvious "relation" of the sgn function on $\boldsymbol{R}$.)

One then considers the multiplier transform on the space of distributions, $f \to \tilde{f}$, defined by $(\tilde{f})^\wedge = \pi\hat{f}$, where $f \to \hat{f}$ is the Fourier transform operation on the space of distributions on K.

Let $F(x, k) = (f(x, k), \tilde{f}(x, k))$ where $x \in K$, $k \in \boldsymbol{Z}$, $f(x, k), \tilde{f}(x, k)$ are the regularization of f and $\tilde{f}$ respectively. The vector valued function $F(x, k)$ is called a conjugate pair. We note that $(f(\cdot, k))^\sim = \tilde{f}(\cdot, k)$.

For each k, $f(x, k)$ and $\tilde{f}(x, k)$ are constant on cosets of $\mathfrak{p}^{-k}$ in K^+ ($\{\mathfrak{p}^{-k}\}$, $k \in \boldsymbol{Z}$, are the fractional ideals in K^+). Let $x_0 + \mathfrak{p}^{-(k+1)}$ be a fixed coset and $x_l + \mathfrak{p}^{-k}$ $(l = 1, 2, 3)$ three mutually disjoint cosets whose union is $x_0 + \mathfrak{p}^{-(k+1)}$.

Let

$$\left.\begin{aligned} f(x_0, k+1) &= a;\ f(x_l, k) = a + \varepsilon_l \\ \tilde{f}(x_0, k+1) &= b;\ \tilde{f}(x_l, k) = b + \delta_l \end{aligned}\right\}\ l = 1, 2, 3.$$

From the regularity of $f(x, k), \tilde{f}(x, k)$ we obtain

$$\sum_{l=1}^{3} \varepsilon_l = \sum_{l=1}^{3} \delta_l = 0. \tag{2.1}$$

Furthermore, since $(f(x, k), \tilde{f}(x, k))$ is a conjugate pair, one obtains (see Chao [1])

$$\sum_l |\varepsilon_l|^2 = \sum_l |\delta_l|^2, \quad \text{and} \tag{2.2}$$

$$\sum_l \varepsilon_l \delta_l = 0. \tag{2.3}$$

Let us repeat here the definition of regularity, using the notation above. A function, $f(x, k)$, defined on $K \times \boldsymbol{Z}$, is regular if $f(x, k)$ is constant on cosets of $\mathfrak{p}^{-k}$ and whenever x_0, x_1, x_2, x_3 are related as above, then $f(x_0, k+1) = (1/3)\sum_l f(x_l, k)$. It is sub-regular if $f(x_0, k+1) \leqslant (1/3)\sum_l f(x_l, k)$.

It is easy to see that whenever $p \geqslant 1$, then $|F(x, k)|^p$ is sub-regular, where $|F(x, k)| = \{|f(x, k)|^2 + |\tilde{f}(x, k)^2\}^{\frac{1}{2}}$. This follows from (2.1) and standard convexity arguments. Our object is to extend this fact to a p, $0 < p < 1$, when $F(x, k)$ is a conjugate pair.

THEOREM 1. *Let $F(x, k)$ be a conjugate pair on $K \times \boldsymbol{Z}$. Then there is a p_0, $0 < p_0 < 1$, p_0 independent of F, such that $|F(x, k)|^p$ is sub-regular for all $p \geqslant p_0$.*

Proof. Using the notation above, we need to show that

$$|(a, b)|^p \leqslant \tfrac{1}{3} \sum_{l=1}^{3} |(a + \varepsilon_l, b + \delta_l)|^p,$$

where $(a, b) \in C^2$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3), (\delta_1, \delta_2, \delta_3)$ are 3-tuples satisfying (2.1), (2.2) and (2.3).

Theorem 1 is then an immediate consequence of the next theorem, which properly belongs in the domain of elementary linear algebra.

THEOREM 2. *Let $n \geqslant 3$ be a fixed integer. For $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in C^n$ let $\|\varepsilon\| = [\sum_k |\varepsilon_k|^2]^{\frac{1}{2}}$. Let $a, b \in C$; $\varepsilon, \delta \in C^n$ be such that*

$$\sum_k \varepsilon_k = \sum_k \delta_k = 0, \tag{2.4}$$

$$\|\varepsilon\| = \|\delta\|, \tag{2.5}$$

$$\sum_k \varepsilon_k \delta_k = 0. \tag{2.6}$$

Then there exists a p_0, $0 < p_0 < 1$, such that

$$|(a, b)|^p \leqslant (1/n) \sum_k |(a + \varepsilon_k, b + \delta_k)|^p \tag{2.7}$$

for all $p \geqslant p_0$, where p_0 is independent of a, b, ε and δ.

Before proceeding to a proof we prove the following:

LEMMA 1. *Let $a, b, \varepsilon, \delta$ be as in Theorem 2. Given p_1, $0 < p_1 < 1$, there exists a constant $A_{p_1} > 0$ such that (2.7) holds for all $p \geqslant p_1$, provided* $\|\varepsilon\| \leqslant A_{p_1}|(a, b)|$.

Proof. The lemma holds trivially if $|(a,b)| = 0$ or $p \geqslant 1$. Thus, we may assume that $|(a,b)| \neq 0$ and $0 < p_1 \leqslant p \leqslant 1$.

(2.8)
$$\sum_k |(a+\varepsilon_k, b+\delta_k)|^p = \sum_k [|a+\varepsilon_k|^2 + |b+\delta_k|^2]^{p/2}$$
$$= \sum_k \{|a|^2+|b|^2+2\operatorname{Re}(\bar{a}\varepsilon_k+\bar{b}\delta_k)+|\varepsilon_k|^2+|\delta_k|^2\}^{p/2}$$
$$= |(a,b)|^p \sum_k \left\{1+\frac{2\operatorname{Re}(\bar{a}\varepsilon_k+\bar{b}\delta_k)}{|(a,b)|^2}+\frac{|\varepsilon_k|^2+|\delta_k|^2}{|(a,b)|^2}\right\}^{p/2},$$

(2.9)
$$\sum_k [\operatorname{Re}(\bar{a}\varepsilon_k+\bar{b}\delta_k)]^2 \leqslant \sum_k |\bar{a}\varepsilon_k+b\bar{\delta}_k|^2$$
$$= |a|^2\sum_k|\varepsilon_k|^2+|b|^2\sum_k|\delta_k|^2+2\operatorname{Re}(\bar{a}\bar{b}\sum_k\varepsilon_k\delta_k)$$
$$= |a|^2\|\varepsilon\|^2+|b|^2\|\delta\|^2 = |(a,b)|^2\|\varepsilon\|^2,$$

as follows from (2.5) and (2.6).

Assume now that $\|\varepsilon\| \leqslant (1/3)|(a,b)|$. It is easy to see that

$$\left|\frac{2\operatorname{Re}(\bar{a}\varepsilon_k+\bar{b}\delta_k)}{|(a,b)|^2}+\frac{|\varepsilon_k|^2+|\delta_k|^2}{|(a,b)|^2}\right| \leqslant 2\frac{\|\varepsilon\|}{|(a,b)|}+2\frac{\|\varepsilon\|^2}{|(a,b)|^2}$$
$$\leqslant 2\left(\frac{1}{3}+\frac{1}{9}\right) = \frac{8}{9} < 1.$$

Therefore it is valid to expand each summand of the last term of (2.8) by the binomial theorem, obtaining that the sum in the last term of (2.8) is equal to

$$(2.10)\quad \sum_k \Big\{1+p\frac{\operatorname{Re}(\bar{a}\varepsilon_k+\bar{b}\delta_k)}{|(a,b)|^2}+\frac{p}{2}\frac{|\varepsilon_k|^2+|\delta_k|^2}{|(a,b)|^2}$$
$$-\frac{p(2-p)}{8|(a,b)|^4}\big[4\big(\operatorname{Re}(\bar{a}\varepsilon_k+\bar{b}\delta_k)\big)^2+(|\varepsilon_k|^2+|\delta_k|^2)^2$$
$$+4\big(\operatorname{Re}(\bar{a}\varepsilon_k+\bar{b}\delta_k)\big)(|\varepsilon_k|^2+|\delta_k|^2)\big]+R_{3k}\Big\},$$

where R_{3k} are the third Taylor remainders.

Let us observe that

$$(2.11)\qquad p(2-p) \leqslant 1 \quad \text{for all } p.$$

$$(2.12)\qquad \left|\binom{p/2}{3}\right| = \frac{p(2-p)(4-p)}{48} \leqslant \frac{1}{9\sqrt{3}}, \qquad 0 \leqslant p \leqslant 1$$

$$(2.13)\qquad \sum_k p\frac{\operatorname{Re}(\bar{a}\varepsilon_k+\bar{b}\delta_k)}{|(a,b)|^2} = 0, \quad \text{as follows from (2.4).}$$

$$(2.14)\qquad \sum_k \frac{p}{2}\frac{|\varepsilon_k|^2+|\delta_k|^2}{|(a,b)|^2} = p\frac{\|\varepsilon\|^2}{|(a,b)|^2}, \quad \text{as follows from (2.5).}$$

$$(2.15)\qquad \frac{p(2-p)}{8|(a,b)|^4}\sum_k 4[\operatorname{Re}(\bar{a}\varepsilon_k+\bar{b}\delta_k)]^2 \leqslant \frac{p(2-p)}{2}\frac{\|\varepsilon\|^2}{|(a,b)|^2},$$

as follows from (2.9).

It follows from (2.11), (2.12), (2.14) and (2.15) (together with easy applications of Hölder's inequality) that the remaining terms are bounded by

$$(2.16)\qquad B\frac{\|\varepsilon\|^3}{|(a,b)|^3},\ B > 0,\ B \text{ independent of } p,\ 0 < p < 1,$$

provided $\|\varepsilon\| \leqslant (1/3)|(a,b)|$.

We have now that the sum in the last term of (2.8) dominates

(2.17)
$$n+\left[p-\frac{p(2-p)}{2}\right]\frac{\|\varepsilon\|^2}{|(a,b)|^2}-B\frac{\|\varepsilon\|^3}{|(a,b)|^3} \geqslant n+\frac{p_1^2}{2}\frac{\|\varepsilon\|^2}{|(a,b)|^2}-B\frac{\|\varepsilon\|^3}{|(a,b)|^3} \geqslant n$$

provided $[\|\varepsilon\|/|(a,b)|] \leqslant A_{p_1} = \min[1/3, p_1^2/2B]$.

This completes the proof of Lemma 1.

Proof of Theorem 2. Suppose $\sum_k |(a+\varepsilon_k, b+\delta_k)| = 0$. Then $\varepsilon_k = -a$, $\delta_k = -b$ for all k and (2.4) implies that $a, b, \varepsilon, \delta$ are all zero and then (2.7) is immediate. Hence we may assume that $\sum_k |(a+\varepsilon_k, b+\delta_k)| \neq 0$ and from the homogeneity of (2.7) that

$$(2.18)\qquad (1/n)\sum_k |(a+\varepsilon_k, b+\delta_k)| = 1.$$

We now fix any p_1, $0 < p_1 < 1$ and we may assume that

$$(2.19)\qquad \|\varepsilon\| \geqslant A_{p_1}|(a,b)|$$

as follows from Lemma 1.

Let $\beta = \{(a+\varepsilon_k, b+\delta_k)\}_{k=1}^n$. The collection, D, of vectors β satisfying (2.4), (2.5), (2.6), (2.18) and (2.19) is easily seen to be compact (in the usual C^{2n} topology).

Note that if $\beta \in D$ then $\|\varepsilon\| \neq 0$. For if $\|\varepsilon\| = 0$ then $|(a, b)| = 1$ (from (2.18)). This contradicts (2.19).

We will show that

(2.20) There is an α, $0 < \alpha < 1$, α independent of $\beta \in D$, such that

$$|(a, b)| \leqslant \alpha(1/n) \sum_k |(a+\varepsilon_k, b+\delta_k)|, \beta \in D.$$

If (2.20) holds, a simple computation shows that (2.7) holds for $\beta \in D$, $p \geqslant p_2 = \left(1+[\log(1/\alpha)/\log n]\right)^{-1}$, and so (2.7) is valid for $p \geqslant p_0 = \max[p_1, p_2]$ and β satisfying (2.4), (2.5) and (2.6).

We now establish (2.20).

Clearly, $|(a, b)| \leqslant (1/n) \sum_k |(a+\varepsilon_k, b+\delta_k)|$. If (2.20) does not hold then there is a $\beta \in D$ such that

$$|(a, b)| = (1/n) \sum_k |(a+\varepsilon_k, b+\delta_k)| = 1. \tag{2.21}$$

Hence, there exist $\{\lambda_k\}_{k=1}^n$ such that

$$\left.\begin{aligned} (a+\varepsilon_k, b+\delta_k) &= \lambda_k(a, b) \\ \lambda_k &\geqslant 0 \end{aligned}\right\} k = 1, 2, \ldots, n. \tag{2.22}$$

We have now, $\varepsilon_k = (\lambda_k - 1)a$, $\delta_k = (\lambda_k - 1)b$, $k = 1, 2, \ldots, n$, with $\lambda_k - 1$ real and so $(\lambda_k - 1)^2 \geqslant 0$ for all k.

From (2.6) we obtain

$$0 = \sum_k \varepsilon_k \delta_k = ab \sum_k (\lambda_k - 1)^2. \tag{2.23}$$

Hence, $\lambda_k = 1$, $k = 1, \ldots, n$; $a = 0$; or $b = 0$. Each of these three conditions implies $\|\varepsilon\| = 0$, a contradiction.

This completes the proof of Theorem 2.

Note. The main idea of the proof; namely, to reduce it to the estimate (2.20) on a compact subset of admissible vectors, is due to A. P. Calderón, and is exploited here in a manner similar to its use by Coifman and Weiss [2].

§ 3. We will now give yet another proof of the following result: *Let $f(z)$ be analytic in a domain of the complex plane. If $p > 0$ then $|f(z)|^p$ is sub-harmonic on the domain.*

Our proof will follow closely the idea of Lemma 1.

Let z_0 be a point where f is analytic, $f = u+iv$ be the decomposition of f into its real and imaginary parts. Let $u(z_0) = a$, $v(z_0) = b$, and $u(z_0+re^{i\theta}) = a+\varepsilon(r, \theta)$, $v(z_0+re^{i\theta}) = b+\delta(r, \theta)$ in a neighborhood of z_0. The sub-harmonicity result now follows from:

Lemma 2. *Let z_0 be a point where f is analytic. Fix $p > 0$. Then there is an $R(p, z_0) > 0$ such that if $0 \leqslant r \leqslant R(p, z_0)$ then*

$$|(a, b)|^p \leqslant \frac{1}{2\pi} \int_0^{2\pi} |(a+\varepsilon(r, \theta), b+\delta(r, \theta))|^p d\theta. \tag{3.1}$$

Proof. The result is trivial if $p \geqslant 1$ or $|(a, b)| = 0$ so we may assume that $0 < p \leqslant 1$, $|(a, b)| \neq 0$. Let

$$\|\varepsilon(r)\| = \left[\int_0^{2\pi} |\varepsilon(r, \theta)|^2 d\theta\right]^{1/2}, \quad \|\delta(r)\| = \left[\int_0^{2\pi} |\delta(r, \theta)|^2 d\theta\right]^{1/2}.$$

The following properties are immediate from the fact that $\varepsilon(r, \theta)$, $\delta(r, \theta)$ are harmonic conjugates and the fact that $\varepsilon(0, \theta) = \delta(0, \theta) = 0$.

$$\int_0^{2\pi} \varepsilon(r, \theta) d\theta = \int_0^{2\pi} \delta(r, \theta) d\theta = 0, \tag{3.2}$$

$$\|\varepsilon(r)\| = \|\delta(r)\|, \tag{3.3}$$

$$\int_0^{2\pi} \varepsilon(r, \theta)\delta(r, \theta) d\theta = 0. \tag{3.4}$$

Since ε and δ are continuous we may choose $R_1(z_0) > 0$ so

$$|\varepsilon(r, \theta)| < (1/5)|(a, b)|, \quad |\delta(r, \theta)| < (1/5)|(a, b)|; \quad \text{all } \theta, \ 0 \leqslant r \leqslant R_1(z_0). \tag{3.5}$$

As before we compute,

$$\begin{aligned} &\int_0^{2\pi} |(a+\varepsilon(r, \theta), b+\delta(r, \theta))|^p d\theta \\ &= |(a, b)|^p \int_0^{2\pi} \left\{1 + \frac{2(a\varepsilon(r, \theta)+b\delta(r, \theta))}{|(a, b)|^2} + \frac{|\varepsilon(r, \theta)|^2+|\delta(r, \theta)|^2}{|(a, b)|^2}\right\}^{p/2} d\theta \\ &= |(a, b)|^p I(r). \end{aligned} \tag{3.6}$$

In a computation similar to that following (2.9) we obtain

$$\left|\frac{2(a\varepsilon(r, \theta)+b\delta(r, \theta))+|\varepsilon(r, \theta)|^2+|\delta(r, \theta)|^2}{|(a, b)|^2}\right| \leqslant \frac{22}{25} < 1 \quad \text{if } 0 \leqslant r \leqslant R_1(z_0).$$

We may then expand the expression in "curly" brackets in (3.6) by the binomial theorem and obtain

$$(3.7)\quad I(r)=\int_0^{2\pi}\Big\{1+p\,\frac{2\big(a\varepsilon(r,\theta)+b\delta(r,\theta)\big)}{|(a,b)|^2}+\frac{p}{2}\,\frac{|\varepsilon(r,\theta)|^2+|\delta(r,\theta)|^2}{|(a,b)|^2}$$
$$-\frac{p(2-p)}{8|(a,p)|^4}\Big[4\big(a\varepsilon(r,\theta)+b\delta(r,\theta)\big)^2+\big(|\varepsilon(r,\theta)|^2+|\delta(r,\theta)|^2\big)^2$$
$$+4\big(a\varepsilon(r,\theta)+b\delta(r,\theta)\big)\big(|\varepsilon(r,\theta)|^2+|\delta(r,\theta)|^2\big)\Big]+R_3(r,\theta)\Big\}\,d\theta.$$

Following the argument of Lemma 1 we obtain

$$(3.8)\quad I(r)\geqslant 2\pi+\frac{p^2}{2}\,\frac{\|\varepsilon(r)\|^2}{|(a,b)|^2}-\Big|\int_0^{2\pi}\Big\{\frac{p(2-p)}{8|(a,b)|^4}\Big[\big(|\varepsilon(r,\theta)|^2+|\delta(r,\theta)|^2\big)^2$$
$$+4\big(a\varepsilon(r,\theta)+b\delta(r,\theta)\big)\big(|\varepsilon(r,\theta)|^2+|\delta(r,\theta)|^2\big)\Big]+R_3(r,\theta)\Big\}\,d\theta\Big|.$$

Let $E(r)=\sup_\theta[|\varepsilon(r,\theta)|,\ |\delta(r,\theta)|]$.

Then it is easy to check that the integral remainder in (3.8) is dominated by

$$(3.9)\qquad BE(r)\,\frac{\|\varepsilon(r)\|^2}{|(a,b)|^2},$$

where B is a positive constant that depends only on p and z_0 and $E(r)=o(1)$ as $r\to 0$.

It follows that

$$(3.10)\qquad I(r)\geqslant 2\pi+\left[\frac{p^2}{2}-BE(r)\right]\frac{\|\varepsilon(r)\|^2}{|(a,b)|^2}.$$

Choose $R_2(p,z_0)>0$ so $0\leqslant r\leqslant R_2(p,z_0)$ implies

$$(p^2/2)-BE(r)\geqslant 0.$$

Let $R(p,z_0)=\min[R_1(z_0),R_2(p,z_0)]$ and the proof is completed.

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