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Some Mazur–Orlicz–Brudno like theorems

by

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Abstract. In this paper the Mazur–Orlicz–Brudno consistency Theorem is extended to certain BK-spaces. These spaces, named W-hump spaces, generalize the Solid spaces and Hump spaces. Examples of spaces possessing the W-hump property, but none of the Hump property and the Solid property are given. Results which relate cores and diameters of cores of matrix transforms, are also obtained.

1. Introduction. This paper is concerned with properties of limit-points assigned to a sequence by "regular" matrices. We obtain these properties by using a method due to G. M. Petersen [9]. In Section 3 we define the "W-hump" property of BK spaces and we show that this property generalizes the "Hump" property [6] and the "Solid" property [4].

Examples of BK spaces possessing the W-hump property, but none of the Hump property and the Solid property, are given. Also we study (μ^*, E) spaces defined by A. Jakimovski and A. Livne in [5]. In § 4 we obtain a necessary and sufficient condition for the existence of the inequality

$$\lim_{n\to\infty}\Big|\sum_{k=0}^{\infty} a_{nk}x_k\Big| \leqslant a \tag{1.1}$$

when $A = (a_{nk})$ is a given generalized "regular" matrix [4], $a \geqslant 0$ a given real constant and x a sequence belonging to a (μ^*, E) space. By using this inequality, we generalise the well known Mazur–Orlicz–Brudno's theorem to W-hump spaces. This result is known only for Solid spaces (see [4]). Theorems are obtained, which relate cores [3] and diameters of cores of matrix transforms of sequences belonging to a given (μ^*, E) space by different generalized "regular" matrices.

§ 2. Terminology, notations and assumptions. We denote by E a closed subspace of co-dimension one, of a given BK space $(D, \|\cdot\|_D)$. If we denote

$$e = (1, 1, 1, \ldots), \quad e^0 = (1, 0, 0, \ldots), \quad e^1 = (0, 1, 0, \ldots), \ \ldots$$

and

$$\varphi = \{e^0, e^1, \ldots\}, \quad \varphi^+ = \{e, e^0, e^1, \ldots\},$$

then we assume that φ and φ^+ are Schauder bases for E and D respectively.

For a given BK space E, the space (λ^*, E) is defined as the space of all infinite sequences y satisfying:

$$(2.1)\qquad \|y\|_{\lambda^*} \equiv \|y\|_{(\lambda^*,E)} \equiv \sup_{n\geqslant 0}\sup_{\|x\|_E\leqslant 1}\Big|\sum_{k=0}^{n} x_k y_k\Big| < \infty$$

(see A. Jakimovski and A. Livne [5]); similarly the space (μ^*, E) and $\|\cdot\|_{\mu^*}$ are defined by $\mu^* \equiv (\mu^*, E) = (\lambda^*(\lambda^*, E))$.

It is obvious that

$$(2.2)\qquad \sup_{n\geqslant 0}\Big|\sum_{k=0}^{n} x_k y_k\Big| \leqslant \|x\|_E\|y\|_{\lambda^*},\qquad \sup_{n\geqslant 0}\Big|\sum_{k=0}^{n} y_k z_k\Big| \leqslant \|y\|_{\lambda^*}\|z\|_{\mu^*}$$

for $x\in E$, $y\in\lambda^*$ and $z\in\mu^*$.

If $E = c_0$, then $\lambda^* = l_1$ and $\mu^* = m$.

For an infinite matrix $A = (a_{n,k})_{n,k\geqslant 0}$ and an infinite sequence $x = \{x_k\}_{k\geqslant 0}$ of complex numbers denote $Ax = \{(Ax)_n\}_{n\geqslant 0}$, $(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k$, where by the existence of Ax we mean the convergence of this last series for each $n\geqslant 0$.

Definition. The matrix A is called *conservative* with respect to D if $D\subset c_A$ and is called *regular* with respect to D if it is conservative with respect to D, $\lim_A e = 1$ and $\lim_A x = 0$ for each $x\in E$.

By Theorem 5.1 of [5] we have:

Theorem 2.1. *A matrix $A \equiv (a_{mn})_{mn\geqslant 0}$ is regular with respect to D if and only if the following conditions are satisfied:*

(i) $\lim_A e^j = 0$ *for* $j\geqslant 0$,

(ii) $\exists (Ae)_m$ $(m\geqslant 0)$ *and* $\lim_A e = 1$,

(iii) $\sup_{m\geqslant 0}\|\{a_{mn}\}_{n\geqslant 0}\|_{\lambda^*} \equiv M < \infty$.

In the sequel we assume that $D\subset\mu^*$.

§ 3. The W-hump property.

Definition. The matrix $A \equiv (a_{nk})_{n,k\geqslant 0}$ is a *block matrix* if the following conditions are satisfied:

$$(3.1)\quad \begin{cases} \text{(i) There exist two sequences of positive integers } \lambda(n) \text{ and } \mu(n) \\ \quad\text{such that for every } n\geqslant 0: \\ \quad\lambda(n) < \mu(n),\ \lambda(n+1)\geqslant\mu(n) \text{ and } \lambda(n)\uparrow\infty. \\ \text{(ii) } a_{nk} = 0 \text{ for } k>\mu(n) \text{ and } k\leqslant\lambda(n)\ (n\geqslant 0). \\ \text{(iii) } \sup_{n\geqslant 0}\sum_{k=0}^{\infty}|a_{n,k}-a_{n,k+1}| \equiv M < \infty. \end{cases}$$

It is clear that $|a_{nk}|\leqslant M/2$ for $n, k\geqslant 0$.

We denote by $x^{(n)}$ the sequence $\{a_{nk}x_k\}_{k\geqslant 0}$.

Definition. A BK space F with $\varphi\subset F$, has the *W-hump property* if for each sequence $x\in F$ and for each block matrix W, there exists a sequence of indices $n_k\uparrow\infty$ such that:

$$(3.2)\qquad \sup_{p\geqslant 0}\Big\|\sum_{k=0}^{p} x^{(n_k)}\Big\|_F \leqslant C_W\|x\|_F.$$

Definition. If $x = \{x_k\}_{k\geqslant 0}$, set ${}_t(x) \equiv \{x_0, x_1, \ldots, x_t, 0, \ldots\}$.

We need some lemmas:

Lemma 3.1. *Let E be a BK space and φ be a Schauder basis for E. Then:*

$$(3.3)\qquad \mu^* \equiv \{x \,|\sup_{t\geqslant 0}\|{}_t(x)\|_E \equiv M(x) < \infty\}.$$

Proof. If $x\in\mu^*$, $\|{}_t(x)\|_{\mu^*}\leqslant\|x\|_{\mu^*}<\infty$, and by Lemma 5.4 of [5] we have

$$\|{}_t(x)\|_E \leqslant \frac{\|{}_t(x)\|_{\mu^*}}{c} \leqslant \frac{\|x\|_{\mu^*}}{c} < \infty.$$

Conversely:

If $\|{}_t(x)\|_E\leqslant M(x)<\infty$, by the same Lemma 5.4 [5],

$$\|{}_t(x)\|_{\mu^*}\leqslant\|{}_t(x)\|_E\leqslant M(x)<\infty$$

and by the definition of the norm in μ^*, $x\in\mu$. ∎

Definition. A BK space E is *Solid* (with constant $C\geqslant 1$) if for each $x\in E$ and each real sequence $\theta = (\theta_0, \theta_1, \ldots)$, $\{x_k e^{i\theta_k}\}_{k\geqslant 0}\in E$ and $\|\{x_k e^{i\theta_k}\}_{k\geqslant 0}\|_E \leqslant C\|x\|_E$ (see [4]).

If E is Solid, then for each $\{y_k\}_{k\geqslant 0}\in m$ and $\{x_k\}_{k\geqslant 0}\in(\mu^*, E)$, we obtain $\{x_k y_k\}_{k\geqslant 0}\in(\mu^*, E)$ and:

$$(3.4)\qquad \|\{x_k y_k\}_{k\geqslant 0}\|_{(\mu^*,E)}\leqslant C_Y\|x\|_{(\mu^*,E)}.$$

If E is Solid, then (μ^*, E) is Solid (see [4]).

Theorem 3.1. *Suppose E is Solid and that φ is a Schauder basis for E. Then (μ^*, E) is W-hump.*

Proof. Let $x\in(\mu^*, E)$ and $W = (a_{ij})$ a block matrix. Define a sequence $\{\beta_j\}_{j\geqslant 0}$ in the following way: $\beta_j = a_{ij}$ if there is an i such that $a_{ij}\neq 0$ (there is at most one such i), and $\beta_i = 0$ otherwise. The sequence $\{\beta_j\}_{j\geqslant 0}$ is bounded, hence $\{\beta_k x_k\}_{k\geqslant 0}$ belongs to (μ^*, E), and then:

$$\sup_{p\geqslant 0}\Big\|\sum_{k=0}^{p} x^{(k)}\Big\|_{\mu^*}\leqslant C_W\|x\|_{\mu^*}<\infty$$

when $x^{(k)} = \beta_k x_k e^k$. ∎

DEFINITION. A BK space E has the *Hump property* with a hump constant C, $0 < C < \infty$, if, from each sequence $\{x^n\}_{n\geqslant 0}$ of elements $x^n \in E$, $\|x^n\|_E \leqslant 1$, of the form $x^n = (0;\ldots;0;x^n_{\lambda(n)+1};\ldots;x^n_{\mu(n)};0;\ldots)$ where $\lambda(n) < \mu(n)$ and $\lim\limits_{n\to\infty}\lambda(n) = +\infty$, it is possible to extract an infinite subsequence $\{x^{n_k}\}_{k\geqslant 0}$ such that $\lambda(n_{k+1}) \geqslant \mu(n_k)$ $(k \geqslant 0)$ and:

$$\sup_{m\geqslant 0}\Big\|\sum_{k=0}^{m} x^{n_k}\Big\|_E \leqslant C. \tag{3.5}$$

It is trivial to see that if E is hump, then (μ^*, E) is hump (see [6]).

LEMMA 3.2. *Let E be a BK space with φ a Schauder basis in it. If $x \in (\mu^*, E)$ and $\theta = \{\theta_k\}_{k\geqslant 0} \in bv_0$ (the space of all bounded variation sequences which tend to zero), then $\bar{x} = \{\theta_k x_k\}_{n\geqslant 0} \in E$ and $\|\bar{x}\|_E \leqslant K_0\|x\|_{\mu^*}$.*

Proof. Set:

$$ {}^m_n x = (0;0;\ldots;0;\ \theta_m x_m;\ldots;\ \theta_n x_n;\ 0;\ldots). \tag{3.6}$$

$$\begin{aligned} \|{}^m_n x\|_{(\mu^*,E)} &= \max_{m\leqslant l\leqslant n}\ \sup_{\|y\|_{\lambda^*}\leqslant 1}\Big|\sum_{k=m}^{l}\theta_k(y_k x_k)\Big| \\ &= \max_{m\leqslant l\leqslant n}\ \sup_{\|y\|_{\lambda^*}\leqslant 1}\Big\{\Big|\sum_{k=m}^{l-1}(\theta_k-\theta_{k+1})\Big(\sum_{j=0}^{k} y_j x_j\Big)+\theta_l\Big(\sum_{j=0}^{l} y_j x_j\Big) - \\ &\qquad -\theta_m\Big(\sum_{j=0}^{m-1} y_j x_j\Big)\Big|\Big\} \\ &\leqslant \sum_{k=m}^{l-1}|\theta_k-\theta_{k+1}|\,\|x\|_{\mu^*}+\max_{m\leqslant l\leqslant n}|\theta_l|\cdot\|x\|_{\mu^*}+|\theta_m|\cdot\|x\|_{\mu^*} \\ &= \|x\|_{\mu^*}\Big(\sum_{k=m}^{l-1}|\theta_k-\theta_{k+1}|+\max_{m\leqslant l\leqslant n}|\theta_l|+|\theta_m|\Big). \end{aligned} \tag{3.6'}$$

Since $\theta \in bv_0$, the sequence $\{{}_n(\bar{x})\}_{n\geqslant 0}$ is a Cauchy sequence in (μ^*, E). By Lemma 5.4 from [5], this sequence is also a Cauchy sequence in E and since by the same Lemma, E is a closed subspace in (μ^*, ε), $\bar{x} \in E$. Taking $m = 0$ in (3.6′), we obtain $\|\bar{x}\|_E \leqslant \|x\|_{\mu^*}(|\theta_0| + \sum_{k=0}^{\infty}|\theta_k-\theta_{k+1}|)$. ■

THEOREM 3.2. *Suppose E has the Hump property, and that φ is a Schauder-basis for E, then (μ^*, E) has the W-hump property.*

Proof. Let $x \in (\mu^*, E)$ and W a block matrix. By Lemma 3.2, and by Lemma 5.4 of [5]:

$$\|x^{(n)}\|_E \leqslant \frac{\|x^{(n)}\|}{C}\,\mu^* \leqslant C'\,\|x\|\,\mu^* < \infty.$$

By the Hump property, we can find a sequence $\{n_k\}_{k\geqslant 0}$ such that:

$$\sup_{p\geqslant 0}\Big\|\sum_{k=0}^{p} x^{(n_k)}\Big\|_E < \infty.$$

Since $\|\sum_{k=0}^{p} x^{(n_k)}\|_{(\mu^*,E)} \leqslant \|\sum_{k=0}^{p} x^{(n_k)}\|_E$, E has the W-hump property. ■

THEOREM 3.3. *Assume that for a BK space E with φ a Schauder basis for E, (μ^*, E) has the W-hump property. If W is a block matrix, $x \in (\mu^*, E)$ and $x^{(n_k)}$ is the sequence given by the W-hump property, then the sequence $\bar{x} = \sum_{k=0}^{\infty} x^{(n_k)}$ belongs to (μ^*, E).*

Proof. Let $x \in (\mu^*, E)$, then by the assumptions we have

$$\sup_{p\geqslant 0}\Big\|\sum_{k=0}^{p} x^{(n_k)}\Big\|_{\mu^*} < \infty.$$

$\|_t(\bar{x})\|_{\mu^*} \leqslant \|\sum_{k=0}^{t} x^{(n_k)}\|_{\mu^*} \leqslant \sup_{p\leqslant 0}\|\sum_{k=0}^{p} x^{(n_k)}\|_{\mu^*} < \infty$ by Lemma 3.1 and by Lemma 5.4 of [5] $x \in (\mu^*, E)$. ■

Now we give one example of a W-hump space which is not Solid and not Hump. We need two more lemmas.

LEMMA 3.3. *Let E_1, E_2 be two BK spaces with φ a Schauder basis in each space. Then:*

$$(\mu^*, E_1 \cap E_2) = (\mu^*, E_1) \cap (\mu^*, E_2). \tag{3.7}$$

Proof. Let $\|x\|_{E_1\cap E_2} \equiv \|x\|_{E_1} + \|x\|_{E_2}$ be the norm of $E_1 \cap E_2$. Let $x \in (\mu^*, E_1 \cap E_2)$. By Lemma 3.1

$$\|_t(x)\|_{E_1\cap E_2} = \|_t(x)\|_{E_1} + \|_t(x)\|_{E_2} \leqslant M(x) < \infty.$$

Then $\|_t(x)\|_{E_1} \leqslant M(x)$, $\|_t(x)\|_{E_2} \leqslant M(x)$, and by the same Lemma 3.1 $x \in (\mu^*, E_1)$ and $x \in (\mu^*, E_2)$.

If $x \in (\mu^*, E_1) \cap (\mu^*, E_2)$, then $\|_t(x)\|_{E_1} \leqslant M_1(x) < \infty$, $\|_t(x)\|_{E_2} \leqslant M_2(x) < \infty$, and then:

$$\|_t(x)\|_{E_1\cap E_2} = \|_t(x)\|_{E_1} + \|_t(x)\|_{E_2} \leqslant M_1(x) + M_2(x) < \infty.$$

By Lemma 3.1, $x \in (\mu^*, E_1 \cap E_2)$. ■

LEMMA 3.4. *Suppose the assumptions of Lemma 3.3 are satisfied, and that E_1 is a Solid space and E_2 a Hump space. Then $(\mu^*, E_1 \cap E_2)$ has the W-hump property.*

Proof. Let $x \in (\mu^*, E_1 \cap E_2)$ and W a block matrix. By Lemma 3.2, $\|x^{(n)}\|_{\mu^*} \leqslant K\|x\|_{\mu^*}$. By the Hump property of E_2, there is a sequence $\{n_k\}$ such that:

$$\sup_{p\geqslant 0} \Big\| \sum_{k=0}^{p} x^{(n_k)} \Big\|_{E_2} < \infty.$$

Hence, by Theorem, 3.3 $\bar{x} = \sum_{k=0}^{\infty} x^{(n_k)}$ belongs to (μ^*, E_2), where $\bar{x} = \{a_k x_k\}_{k\geqslant 0}$ with $|a_k| \leqslant 1$ $(k \geqslant 0)$. By virtue of by the Solid property of E_1, $\bar{x} \in (\mu^*, E_1)$, and so by Lemma 3.3 $\bar{x} \in (\mu^*, E_1 \cap E_2)$.

THEOREM 3.4. *There exists a BK space E such that (μ^*, E) has the W-hump property and not the Solid and Hump property.*

Proof. Let:

$$E_1 = \Big\{x \Big| \sum_{n=1}^{\infty} \Big|\frac{x_n}{n^2}\Big| < \infty\Big\}, \quad E_2 = c^0_{(C,1)}. \tag{3.8}$$

Denote $E = E_1 \cap E_2$. It is easy to see that φ is a Schauder basis for E_1 and E_2. Then by Lemma 3.3

$$(\mu^*, E_1 \cap E_2) = (\mu^*, E_1) \cap (\mu^*, E_2).$$

E_1 is Solid, and by Lemma 3.1, $(\mu^*, E_1) = E_1$. By the same lemma, $(\mu^*, E_2) = m_{(C,1)}$. While it was proved in [6] that E_2 and (μ^*, E_2) have the Hump property. Now the norms of the spaces (μ^*, E_1) and (μ^*, E_2) are:

$$\|x\|_{(\mu^*, E_1)} = \sum_{n=1}^{\infty} \Big|\frac{x_n}{n^2}\Big|, \quad \|x\|_{(\mu^*, E_2)} = \sup_{n\geqslant 1} \left|\frac{\sum_{k=1}^{n} x_k}{n}\right|. \tag{3.9}$$

Hence Lemma 3.4 implies that $(\mu^*, E_1 \cap E_2)$ has the W-hump property.

Let $x = \{x_k\}$ where $x_k = (-1)^k\sqrt{k}$ $(k \geqslant 1)$, then $x \in E_1$.

Since $\left|\frac{\sum_{k=1}^{n} (-1)^k\sqrt{k}}{n}\right| < 1$, $x \in (\mu^*, E_2)$ and so $x \in (\mu^*, E_1 \cap E_2)$. But the sequence $y_k = |x_k| = \sqrt{k}$ $(k \geqslant 1)$ has the property that $\lim_{(C,1)} y = +\infty$, hence $y \notin (\mu^*, E_1 \cap E_2)$. This proves that $(\mu^*, E_1 \cap E_2)$ is not Solid.

For the following sequence of blocks:

$$x_j^{(n)} = \begin{cases} (-1)^j j & 2^n \leqslant j < 2^{n+1}, \\ 0 & \text{elsewhere}, \end{cases}$$

$$\|x^{(n)}\|_{(\mu^*, E_1)} = \sum_{j=2^n}^{2^{n+1}-1} 1/j, \text{ and we obtain } \tfrac{1}{2} < \|x^{(n)}\|_{(\mu^*, E_1)} < 1,$$

$$\|x^{(n)}\|_{(\mu^*, E_2)} = \sup_{l<2^{n+1}} \left|\frac{\sum_{k=2^n}^{l} (-1)^k k}{l}\right| \leqslant 1.$$

Therefore $\|x^{(n)}\|_{(\mu^*, E_1\cap E_2)} \leqslant 2 < \infty$.

For each subsequence of disjoint block, we have

$$\Big\| \sum_{k=1}^{l} x^{(n_k)} \Big\|_{(\mu^*, E_1\cap E_2)} \geqslant \Big\| \sum_{k=1}^{l} x^{(n_k)} \Big\|_{(\mu^*, E_1)} > l/2$$

and this proves that $(\mu^*, E_1 \cap E_2)$ is not a Hump space. ∎

§ 4. A theorem concerning dispersion of limit points. We need the following lemma:

LEMMA 4.1. *Let $A = (a_{mn})_{m,n\geqslant 0}$ be a regular matrix with respect to D, and suppose that φ is a Schauder basis for E and (λ^*, E). Then there exists a matrix $B = (b_{mn})$, regular with respect to D, with the following properties:*

(i) *There exist two sequence of positive integers $\{\lambda(m)\}$, $\{\mu(m)\}$ such that $\lambda(m) < \mu(m)$ $(m \geqslant 0)$ and $\lambda(m) \uparrow \infty$,*

(ii) *$b_{mn} = 0$ for $n \leqslant \lambda(m)$ and $n > \mu(m)$,*

(iii) $\lim_{m\to a} \big(\sum_{n=0}^{\infty} a_{mn}x_n - \sum_{n=0}^{\infty} b_{mn}x_n \big) = 0$ *for all $x \in (\mu^*, E)$.*

Proof. The proof follows by Lemma 2.1 of [4] and Lemma 5.2 of [5].

Remark. In the rest of this paper, we assume that the matrices are regular and the sequences from (μ^*, E). If φ is a Schauder basis in (λ^*, E), then by Lemma 4.1, we may study only row-finite matrices. Suppose therefore that the matrices are row-finite.

Moreover, it may be assumed that the matrices are *triangulares*, for otherwise, equivalent matrices satisfying this condition can be constructed by repetition of the rows. In this case $\mu(n) = n$. In the sequel we also assume that (μ^*, E) has the W-hump property.

THEOREM 4.1. *Let A be a regular matrix with respect to D. Let $\{\lambda(n)\}$ be the sequence given by Lemma 4.1, and let $a \geqslant 0$. Then $x \in (\mu^*, E)$ satisfies:*

$$\overline{\lim_{n\to\infty}} \Big| \sum_{k=0}^{\infty} a_{nk}x_k \Big| \leqslant a, \tag{4.2}$$

if, and only if, for each sequence $\{\zeta_k x_k\}_{k\geqslant 0}$ belonging to (μ^*, E) with $|\zeta_k| \leqslant 1$ $(k \geqslant 0)$, $\lim_{k\to\infty} |\zeta_k - \zeta_{k+1}| = 0$ and having at most one jump (change) of ζ_k when

$\lambda(n) < k \leqslant n \ (n \geqslant 0)$

$$\overline{\lim_{m,n\to\infty}} \Big|\sum_{k=0}^{\infty} a_{mk}\zeta_k x_k - \sum_{k=0}^{\infty} a_{nk}\zeta_k x_k\Big| \leqslant 2a. \tag{4.3}$$

Proof. Necessity. Since we may suppose that A is a triangle, we have:

$$t_m = \sum_{k=\lambda(m)+1}^{m} a_{mk}\zeta_k x_k = \sum_{k=\lambda(m)+1}^{m} (-\zeta_m+\zeta_k) a_{mk}x_k + \zeta_m \sum_{k=\lambda(m)+1}^{m} a_{mk}x_k.$$

By the properties of the sequence $\{\zeta_k\}$:

$$\Big|\sum_{k=\lambda(m)+1}^{m} (-\zeta_m+\zeta_k) a_{mk}x_k\Big| \leqslant |-\zeta_i+\zeta_{i-1}|\cdot \|\{a_{mk}\}_{k\geqslant 0}\|_{\lambda^*}\|x\|_{\mu^*}$$

(for some i, which may depend on m)

$$\leqslant |-\zeta_i+\zeta_{i-1}|\sup_{n\geqslant 0}\|\{a_{nk}\}_{k\geqslant 0}\|_{\lambda^*}\|x\|_{\mu^*} \underset{m\to\infty}{\to} 0.$$

Consequently

$$\overline{\lim_{m,n\to\infty}} |t_m-t_n| = \overline{\lim_{m,n\to\infty}} \Big|\zeta_m \sum_{k=\lambda(m)+1}^{m} a_{mk}x_k - \zeta_n \sum_{k=\lambda(n)+1}^{n} a_{nk}x_k\Big| \leqslant 2a. \tag{4.4}$$

Remark. $\lim_{m\to\infty}\Big|t_m-\zeta_m \sum_{k=\lambda(m)+1}^{m} a_{mk}x_k\Big| = 0,$

$$\overline{\lim_{m\to\infty}} |t_m| = \overline{\lim_{m\to\infty}} \Big|\sum_{k=\lambda(m)+1}^{m} a_{mk}x_k\Big| \leqslant a. \tag{4.5}$$

Remark. There are sequences ζ such that the set of the limit points of t_m is a disc.

Sufficiency. Assume that the matrix A satisfies the conditions of the theorem, and that there is a sequence $x \in \mu^*$, a constant b such that:

$$\overline{\lim_{n\to\infty}} \Big|\sum_{k=0}^{\infty} a_{nk}x_k\Big| = |b| > a$$

and a sequence of positive integers $\{n_l\}$ such that:

$$n_l < \lambda(n_l+1) \ (l \geqslant 0) \quad \text{and} \quad \lim_{l\to\infty} \sum_{k=0}^{\infty} a_{n_l,k}x_k = b.$$

For each k there is at most one n_l such that $a_{n_l,k} \neq 0$. Denote by $\{c_n\}_{n\geqslant 0}$ the following sequence:

$$\Big(1;\frac{1}{2};0;\frac{1}{3};\frac{2}{3};1;\frac{3}{4};\frac{2}{4};\frac{1}{4};0;\frac{1}{5};\ldots\Big). \tag{4.6}$$

Let the sequence $\{\zeta_m\}$:

$$\zeta_m = c_i; \quad \lambda(n_{i-1}) < m \leqslant \lambda(n_i) \ (i \geqslant 0).$$

It is clear that $\lim_{m\to\infty}|\zeta_m-\zeta_{m-1}| = 0$, $|\zeta_m| \leqslant 1$ $(m \geqslant 0)$ and there is at most one jump in ζ_m if $\lambda(m) < m \leqslant n$ $(n \geqslant 0)$. In the sequence $\{c_i\}$, there are infinitely many zeroes. Let $W = (a_{ij})$ where

$$a_{ij} = \begin{cases} \zeta_j & \lambda(n_{(i-1)(2i-1)}) < j \leqslant \lambda(n_{i(2i+1)}) \ (i > 0), \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

W is a block matrix with $\sup_{n\geqslant 0}\sum_{k=0}^{\infty}|a_{nk}-a_{nk+1}| = 2$. By the W-hump property, once with regard to the even rows of W, and second, to the odd rows of W, and by Theorem 3.3, there exist two disjoint sequences x', $x'' \in (\mu^*, E)$ such that $\{(Ax')_m\}_{m\geqslant 0}$ and $\{(Ax'')_m\}_{m\geqslant 0}$ are two disjoint subsequences of $\{(Ax)_{n_l}\}_{l\geqslant 0}$. The sequence $x = x'-x''$ belongs to (μ^*, E) and has the form $\{\zeta_k x_k\}_{k\geqslant 0}$ where $\{\zeta_k\}_{k\geqslant 0}$ has all the properties needed in the theorem. Hence

$$\overline{\lim_{m,n\to\infty}} |(A\overline{x})_m-(A\overline{x})_n| \leqslant 2a.$$

But there exists a subsequence of $(A\overline{x})_m$ with limit b and another with limit $-b$. Therefore $\overline{\lim_{m,n\to\infty}} |(A\overline{x})_m-(A\overline{x})_n| \geqslant 2|b| > 2a$, which is a contradiction.

§ 5. Consistency theorems. In this part we obtain some results about limit-points of transforms by regular matrices with respect to D, when (μ^*, E) is W-hump.

The first result is an extension of the well-known consistency theorem due to Mazur–Orlicz–Brudno (see [7], [8], [9], [10]).

THEOREM 5.1. *Let A, B two regular matrices with respect to D, such that $c_A \supset c_B \cap (\mu^*, E)$. Then, A and B are consistent in (μ^*, E).*

Proof. We may assume that A and B are triangles with the same sequence $\lambda(n)$ (see Lemma 4.1).

If A and B are not consistent in (μ^*, E), there is a sequence $x \in (\mu^*, E)$ such that $\lim_A x$, $\lim_B x$ exist and $\lim_A x \neq \lim_B x$. By regularity, we may assume that $\lim_A x \neq 1$, and $\lim_B x = 0$. By Theorem 4.1, each sequence $\{\zeta_k x_k\}$ belonging to (μ^*, E) with the specials $\{\zeta_k\}$ is summable by B. Then $\{\zeta_k x_k\}$ is also limitable by A and by Theorem 4.1, $\{x_k\}$ is also limitable by A to 0, which is a contradiction. ■

Notation. We denote by $K(x)$ the core of a sequence x and by $\Omega(x) = \overline{\lim_{m,n\to\infty}} |x_m-x_n|$ (see [3]).

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THEOREM 5.2. *Let A and B be two regular matrices with respect to D. If $\Omega(Ax) \leqslant \Omega(Bx)$ for all $x \in (\mu^*, E)$, then*

$$K(Ax) \subset K(Bx) \quad \textit{for all } x \in (\mu^*, E). \tag{5.1}$$

Proof. By the same procedure as in Theorem 5.1, we take $\lambda(n)$ identical for both matrices. Let $x \in (\mu^*, E)$ and denote $\overline{\lim_{m\to\infty}} \left|\sum_{k=0}^{\infty} b_{mk}x_k\right| = R$. By Theorem 4.1, the family of sequences $\bar{x} = \{\zeta_k x_k\}$ has the property that $\Omega(B\bar{x}) \leqslant 2R$, then $\Omega(A\bar{x}) \leqslant 2R$ and by the same theorem, $\overline{\lim_{m\to\infty}} \left|\sum_{k=0}^{\infty} a_{mk}x_k\right| \leqslant R$.

We proved that for each $x \in (\mu^*, E)$ each circle around the origin which includes $K(Bx)$, also includes $K(Ax)$.

If $x \in (\mu^*, E)$ and z_0 is a complex number, by regularity of B we have:

$$\lim_{m\to\infty} \left|\sum_{k=0}^{\infty} b_{mk}(x_k - z^0) - \sum_{k=0}^{\infty} b_{mk}x_k + z^0\right| = 0$$

and a similar equality for A.

By the above equality, a circular region around z^0, which includes $K(Bx)$, under the transformation $x - z^0 e$ will become to a circular region around the origin which will include $K(B(x - z^0 e))$. Then, this region will include $K(A(x - z^0 e))$ and by the inverse transformation, we conclude that the circular region around z^0 includes also $K(Ax)$.

Then by Lemma 1 of A. Robinson in [3], p. 149: $K(Ax) \subset K(Bx)$ for all $x \in (\mu^*, E)$. ∎

Remark. We can obtain a little more than Theorem 5.2 if we remember the remark of Theorem 4.1.

THEOREM 5.3. *Let A and B be two regular matrices with respect to D. If for all $x \in (\mu^*, E)$, such that the set of limit points of (Bx) is connected we have $\Omega(Ax) \leqslant \Omega(Bx)$, then for all $x \in (\mu^*, E)$*

$$\Omega(Ax) \leqslant \Omega(Bx). \tag{5.2}$$

Proof. Suppose that there is a sequence $x \in (\mu^*, E)$ such that the set of limit points of (Bx) is not connected, and $\Omega(Ax) > \Omega(Bx)$. Then, we may assume that there is a circular region around the origin (by a suitable transformation $\alpha x + \beta e$) which includes $K(Bx)$, but not all $K(Ax)$.

By Theorem 4.1 and its remark, there is a sequence $x = \{\zeta_k x_k\} \in (\mu^*, E)$ with $\Omega(A(\zeta x)) > \Omega(B(\zeta x))$ and $K(B(\zeta x))$ is a disk, which is a contradiction. ∎

Theorems 5.2 and 5.3 give:

THEOREM 5.4. *Suppose the assumptions of Theorem 5.3 are satisfied. Then $K(Ax) \subset K(Bx)$ for all $x \in (\mu^*, E)$.*

THEOREM 5.5. *Let A and B be two regular matrices with respect to D.*

If:

$$x \in (\mu^*, E) \quad \textit{and} \quad \Omega(Ax) < \Omega(Bx) \Rightarrow \exists \lim_A x.$$

Then

$$\Omega(Ax) \geqslant \Omega(Bx) \quad \textit{for all } x \in (\mu^*, E).$$

Proof. First it will be shown that from $x \in (\mu^*, E)$ and $\Omega(Ax) < \Omega(Bx)$ we may conclude that $\lim_A x \in K(Bx)$. Suppose that there is a sequence $x \in (\mu^*, E)$, $\Omega(Bx) > 0$, $\lim_A x \notin K(Bx)$. By a linear transformation we may obtain a sequence from (μ^*, E) such that $\lim_A x = -d$, $(d > 0)$ and $\lim_{n\to\infty} |(Bx)_n| > 2d$.

By Theorem 4.1 we obtain a sequence $\bar{x} = \{\zeta_k x_k\} \in (\mu^*, E)$ with $\Omega(B\bar{x}) > 4d$ and $\Omega(A\bar{x}) = 2d$, which contradict the hypothesis.

Now suppose there is a sequence $x \in (\mu^*, E)$ with $\Omega(Ax) < \Omega(Bx)$. Then $\lim_A x$ exists, belongs to $K(Bx)$ and $\Omega(Bx) > 0$. By a similar proof as before we obtain the contradiction.

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