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Nonlinear integral operators on $C(S, E)$

by

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Abstract. We investigate the class of operators T on the space $C(S, E)$ of vector-valued continuous functions on a compact Hausdorff space S which are uniformly continuous on bounded sets and satisfy the algebraic relation

$$T(f+f_1+f_2) = T(f+f_1)+T(f+f_2)-Tf$$

for all $f, f_1, f_2 \in C(S, E)$ with f_1 and f_2 having disjoint support. We derive integral representations and prove universal properties of these operators. Special attention is given to the (weakly) compact operators.

Introduction. Since A. Grothendieck's celebrated treatment [22] of this subject in 1953, linear bounded transformations on spaces of continuous functions and their universal properties have been of interest for many writers. Our present study is related to the work of A. Pełczyński [24], C. C. Brown [10], E. Thomas [27] and H. H. Schaefer [26] and especially to the results obtained in connection with generalizations of the Riesz Theorem (see [7] for a detailed account) via integral representations for the operators on the basis of the well-known paper of R. Bartle, N. Dunford and J. Schwartz [1] in 1955, namely the results of C. Foiaş and I. Singer [17], P. W. Lewis [23], I. Dobrakov [14], J. K. Brooks and D. R. Lewis [9] and of the author [2] [3] [4]. In 1965 N. A. Friedman, R. V. Chacon and M. Katz began to derive representation theorems for real, so-called "additive" (not necessarily linear) functionals on spaces of real-valued continuous functions in a series of three papers [11] [18] [19] with successive improvements. It was a natural question to ask which results an integration theory could yield in the investigation of nonlinear operators on spaces of continuous functions. Let E and F be Banach spaces and $C(S, E)$ the space of continuous functions on the compact Hausdorff space S with values in E (with the uniform norm): In this note we present what we think is the adequate extension of the theory of the linear bounded transformations to the class of nonlinear transformations $T: C(S, E) \to F$ which are at all representable as integrals with respect to additive "nonlinear" set functions (hereby we understand a set function which takes its values in a space of operators from one Banach space into another which are uniformly continuous on bounded sets). This class consists of those transformations T which are uniformly continuous on bounded sets in $C(S, E)$ and have the algebraic property

$$T(f+f_1+f_2) = T(f+f_1)+T(f+f_2)-Tf$$

for all $f, f_1, f_2 \in C(S, E)$ with f_1 and f_2 having disjoint support; this property was termed "additivity" by N. A. Friedman, "strong additivity" in [6]— we prefer here to call it the "Hammerstein property" in view of the fact that the class of Hammerstein operators satisfy this condition(1).

Every new type of transformation needs the development of a new integral for its representation ([13], p. 1095). The definition of the integral is the subject of Section 1. We remark that, after the completion of this part of the manuscript, essentially the same idea of an integral, but without explicit development of the concept of a nonlinear measure has been published by N. A. Friedman and A. E. Tong [20] for the special case $E = \mathbf{R}$. In Sections 2 and 3 we represent the functionals and the transformations on $C(S, E)$ by integrals with respect to nonlinear measures. In Section 4 we derive a Radon–Nikodým Theorem for our nonlinear measures hereby making it possible to give kernel representations for the functionals with Lebesgue- or Bochner-integrable functions (kernel representations for transformations without further assumptions seem to be very artificial). The progress in comparison with [11] [18] [19] [20] is this: 1. We show that the detour via a compact metric space S [18] [19] is not in fact necessary; we work on a compact Hausdorff space from the beginning. 2. Our approach is not limited to the case of real functions f; making use of the theory of a lifting we can handle the case of E-valued functions f. 3. We can answer uniqueness questions and 4. we can give direct evaluation of the norm and modulus of continuity of the transformations represented.

In Section 5 we characterize and study the compact and weakly compact transformations on $C(S, E)$ and $C(S)$. In particular we completely extend the results of R. Bartle, N. Dunford and J. Schwartz ([15], pp. 492–497) to our class of nonlinear operators. It is our special interest to apply the integration theory to the derivation of universal properties of these transformations (that is, properties which are independent from any integral representation). Among these we show that $C(S)$ has the strict Dunford–Pettis property ([16], p. 633) also with respect to the considered class of nonlinear transformations and that any T of this type on $C(S)$ with values in a Banach space no subspace of which is isomorphic to c_0 is weakly compact. In the last section we apply our results to transformations $T: C(S) \to C(Q)$ where Q is another compact Hausdorff space. An essential tool of our study is the theory of nonlinear compact mappings and their adjoints as developped in [5].

(1) The name was suggested to the author by Professor C. Foiaş.

A great part of our results was announced in [6] and has been presented on the conference on "Function spaces and modular spaces" in October 1971 which was arranged by Prof. W. Orlicz and Prof. J. Musielak of the A. Mickiewicz University in Poznań. The author gratefully acknowledges the honor of the invitation and wants to thank Prof. W. Orlicz for the interest in his work.

1. The integral. In this section we introduce the integral needed for the representations in Sections 2 and 3. Let E and F be Banach spaces and $M(E, F)$ the linear space of all transformations $U: E \to F$ with the following properties:

(i) $U0 = 0$,

(ii) If U_a is the restriction of U to the ball $E_a := \{x \in E: \|x\| \leqslant a\}$ then $\|U_a\| := \sup_{x \in E_a} \|Ux\| < \infty$, $a > 0$,

(iii) $D_\delta U_a := \sup_{\substack{x, x' \in E_a \\ \|x-x'\| \leqslant \delta}} \|Ux - Ux'\| \to 0 \ (\delta \to 0)$, $a > 0$

((i) is assumed for convenience and (ii) is of course a consequence of (iii)). The spaces $M_a(E, F) := \{U_a: U \in M(E, F)\}$ are linear and considered to be normed by $\|\cdot\|: U_a \to \|U_a\|$. Let us also agree always to denote the a-ball of a normed space by the index a affixed to the symbol for the space and the restriction of an operator on this space to the a-ball by the index a affixed to the symbol for the operator.

Now let S be an abstract set, $\mathscr{A}$ be an algebra of subsets of S and $U: \mathscr{A} \to M(E, F)$ be an additive set function. If g is an element of $\mathscr{E}_E(\mathscr{A})$, the space of the $\mathscr{A}$-simple functions on S with values in E ([12], p. 82) and if g has the representation

$$g = \sum_{j=1}^{r} x_j \chi_{A_j}$$

where $x_1, \ldots, x_r \in E$ and $A_1, \ldots, A_r$ is an $\mathscr{A}$-partition of S (that is, a finite system of disjoint sets $A_1, \ldots, A_r \in \mathscr{A}$ whose union is S) then we may and shall define the integral of g with respect to U by

$$\int_S g\, dU := \sum_{j=1}^{r} U(A_j) x_j.$$

In fact, it is easily seen that the integral for two representations of g is the same (as long as the sets A_j are disjoint). Consider $\mathscr{E}_E(\mathscr{A})$ to be normed by the uniform norm. Then the integral is continuous on $\mathscr{E}_E(\mathscr{A})$ if and only if U (as a function from $\mathscr{A}$ into $M(E, F)$) is of bounded semivariation on S, that is,

$$sv(U_a, S) := \sup\Big\{\Big\|\sum_{j=1}^{r} U(A_j)x_j\Big\|: \{A_1, \ldots, A_r\}\ \mathscr{A}\text{-part.}, x_1, \ldots, x_r \in E_a\Big\} < \infty$$

$$sv_\delta(U_a, S) := \sup\Big\{\Big\|\sum_{j=1}^{r}\big(U(A_j)x_j - U(A_j)x_j'\big)\Big\| : \{A_1, \ldots, A_r\}\ \mathscr{A}\text{-part.},$$
$$x_j, x_j' \in E_a,\ \|x_j - x_j'\| \leqslant \delta,\ j = 1, \ldots, r\Big\} \to 0,$$
$$(\delta \to 0), \qquad a > 0.$$

If this is the case we extend the integral to $\mathscr{M}_E(\mathscr{A})$, the space of the totally $\mathscr{A}$-measurable functions on S, with values in E (the uniform limits of functions in $\mathscr{E}_E(\mathscr{A})$), normed by the uniform norm ([12], p. 83). If $g \in \mathscr{M}_E(\mathscr{A})$ and $\{g_n\}_{n=1}^{\infty}$ is a sequence in $\mathscr{E}_E(\mathscr{A})$ with $\|g - g_n\| \to 0$, we may and shall define

$$\int_S g\,dU := \lim_{n\to\infty} \int_S g_n\,dU.$$

In fact, if $\sup_n \|g_n\| =: a$, it then follows from

$$\Big\|\int_S g_n\,dU - \int_S g_m\,dU\Big\| \leqslant sv_{\|g_n - g_m\|}(U_a, S)$$

that $\{\int_S g_n\,dU\}_{n=1}^{\infty}$ is a Cauchy sequence in F. The integral thus defined is linear in U; with respect to g it has the following property: For all $g, g_1, g_2 \in \mathscr{M}_E(\mathscr{A})$ such that $\{g_1(t) \neq 0\} \cap \{g_2(t) \neq 0\} = \emptyset$ we have

$$\int_S (g+g_1+g_2)\,dU = \int_S (g+g_1)\,dU + \int_S (g+g_2)\,dU - \int_S g\,dU. \tag{1}$$

Indeed, it is sufficient to prove this relation for functions in $\mathscr{E}_E(\mathscr{A})$. If we let

$$\int_A g\,dU = \int_S g\cdot\chi_A\,dU, \qquad g \in \mathscr{M}_E(\mathscr{A}),\ A \in \mathscr{A}$$

and if $A_1, A_2 \in \mathscr{A}$ are such that $\{g_1(t) \neq 0\} \subset A_1$, $\{g_2(t) \neq 0\} \subset A_2$ and $A_1 \cup A_2 = S$, then

$$\begin{aligned}\int_S (g+g_1+g_2)\,dU &= \int_{A_1} (g+g_1)\,dU + \int_{A_2} (g+g_2)\,dU \\ &= \int_{A_1} (g+g_1)\,dU + \int_{A_2} g\,dU + \\ &\quad + \int_{A_2} (g+g_2)\,dU + \int_{A_1} g\,dU - \int_S g\,dU\end{aligned}$$

and this equals the right side of (1). Finally we remark that for $a > 0$

$$\Big\|\int_S g\,dU\Big\| \leqslant sv(U_a, S), \qquad g \in \mathscr{M}_E(\mathscr{A})_a, \tag{2}$$

$$\Big\|\int_S g\,dU - \int_S g'\,dU\Big\| \leqslant sv_\delta(U_a, S), \qquad g, g' \in \mathscr{M}_E(\mathscr{A})_a,\ \|g - g'\| \leqslant \delta. \tag{3}$$

The second inequality follows from the continuity of $sv_\delta(U_a, S)$ as a function of δ.

Besides the semivariation we consider for an additive set function μ: $\mathscr{A} \to M(E, C)$ the quantities

$$v(\mu_a, S) := \sup\Big\{\sum_{j=1}^{r} \|\mu_a(A_j)\|,\ \{A_1, \ldots, A_r\}\ \mathscr{A}\text{-part.}\Big\}$$

(which is the variation on S of the set function μ_a: $\mathscr{A} \to M_a(E, C)$ defined by $\mu_a(A) = \mu(A)_a$, $A \in \mathscr{A}$) and

$$v_\delta(\mu_a, S) := \sup\Big\{\sum_{j=1}^{r} D_\delta\mu_a(A_j),\ \{A_1, \ldots, A_r\}\ \mathscr{A}\text{-part.}\Big\}.$$

For such an additive μ: $\mathscr{A} \to M(E, C)$ we have

$$sv(\mu_a, S) \leqslant v(\mu_a, S) \leqslant 4\,sv(\mu_a, S), \tag{4}$$

$$sv_\delta(\mu_a, S) \leqslant v_\delta(\mu_a, S) \leqslant 4\,sv_\delta(\mu_a, S). \tag{5}$$

This follows from the fact that for any finite set $z_1, \ldots, z_r$ of complex numbers we have

$$\sum_{j=1}^{r} |z_j| \leqslant \sum_{i=1}^{4} \Big|\sum_{j\in I_i} z_j\Big|, \tag{6}$$

with $I_1 = \{j\colon \operatorname{Re} z_j \geqslant 0\}$, $I_2 = \{j\colon \operatorname{Re} z_j < 0\}$, $I_3 = \{j\colon \operatorname{Im} z_j \geqslant 0\}$, $I_4 = \{j\colon \operatorname{Im} z_j < 0\}$.

2. Functionals on $C(S, E)$ with the Hammerstein property. Throughout the paper, S is a compact Hausdorff space and $C(S, E)$ the space of all continuous functions f on S with values in a Banach space E with the uniform norm (if $E = C$ we shall write $C(S)$). For $f \in C(S, E)$, $S[f] = \{f(t) \neq 0\}$ is the support of f. We say a functional A: $C(S, E) \to C$ has the *Hammerstein property* (abbreviated as HP) whenever

$$A(f+f_1+f_2) = A(f+f_1) + A(f+f_2) - Af \tag{7}$$

for all $f, f_1, f_2 \in C(S, E)$ with $S[f_1] \cap S[f_2] = \emptyset$ (for a justification of this term see Section 6). This section consists of the proof for the representation theorem for the functionals $A \in M\big(C(S, E), C\big)$ with the HP.

We shall denote by $\mathscr{B}$ the σ-algebra of the Borel sets in S. For subsets M in S and P in E, $C(M, P)$ is the subset of all $f \in C(S, E)$ with $\{f(t) \neq 0\} \subset M$ and $f(M) \subset P$. If $B \in \mathscr{B}$ the set $\pi(B)$ of all open sets containing B is a directed set under the partial ordering $\leqslant$ if we define $G_1 \leqslant G_2$ for two sets $G_1, G_2 \in \pi(B)$ to mean $G_1 \supset G_2$. Similarly, the set $\pi_0(B)$ of all compact sets K contained in B is a directed set under the partial ordering $\leqslant$ if we define $K_1 \leqslant K_2$ for two sets $K_1, K_2 \in \pi_0(B)$ to mean $K_1 \subset K_2$.

If K is a compact set in S and G is an open set containing K $u(K, G')$ will always denote a function in $C(S)$ with the property that $u(S) \subset [0, 1]$, $u(K) = 1$ and $u(G') = 0$, where G' is the complement of G.

THEOREM 1. *There exists an algebraic isomorphism between the space $M_{\mathrm{HP}}(C(S, E), C)$ of all $A \in M(C(S, E), C)$ with the Hammerstein property* (7) *and the space of all additive nonlinear set functions $\mu\colon \mathscr{B} \to M(E, C)$ with the following properties:*

(i) $sv(\mu_a, S) < \infty$ *and* $sv_\delta(\mu_a, S) \to 0$ $(\delta \to 0)$ *for* $a > 0$,

(ii) $\mu_a\colon \mathscr{B} \to M_a(E, C)$ *(and hence $v(\mu_a)$) is regular (and therefore σ-additive) for* $a > 0$.

The correspondence is given by

$$Af = \int_S f\,d\mu, \qquad f \in C(S, E), \tag{8}$$

and for corresponding A and μ we have for each compact K

$$\mu(K)x = \lim_{G \in \pi(K)} A\big(xv(K, G')\big) \tag{9}$$

uniformly for $x \in E_a$ with arbitrary $v = v(K, G') \in C(S)$ such that $v(S) \subset [0, 1]$, $v(K) = 1$, $v(G') = 0$ and

$$\|A_a\| = sv(\mu_a, S), \tag{10}$$

$$D_\delta A_a = sv_\delta(\mu_a, S), \qquad a, \delta > 0. \tag{11}$$

Let us remark that the HP implies

$$A(f_1 + f_2) = Af_1 + Af_2$$

for all $f_1, f_2 \in C(S, E)$ with $S[f_1] \cap S[f_2] = \emptyset$ but is in general a stronger condition (as the simple example $Af := \inf_{t \in [0,1]} |f(t)|$ on $C([0, 1])$ shows — such a functional can hence not be represented as an integral with respect to an additive nonlinear set function).

The proof of Theorem 1 will require a series of 8 lemmas. Let us agree that K will always denote compact, G and U open sets in S (with or without subscripts). Two functions $f, g \in C(S, E)$ have disjoint support whenever there exist sets K, G such that $\{f(t) \neq 0\} \subset K \subset G \subset \{g(t) = 0\}$. Up to Lemma 8, A will denote a fixed element in $M_{\mathrm{HP}}(C(S, E), C)$.

LEMMA 1. *If $K \subset U$, $g \in C(K, E)$ and $p \in C(S)$ with $U \subset \{p(t) = 1\}$, then for all $f \in C(S, E)$ we have*

$$A(f - g) - Af = A(pf - g) - A(pf).$$

Proof. We note that $\{g(t) \neq 0\} \subset K \subset U \subset \{p(t) - 1 = 0\}$. Hence $S[g] \cap S[p-1] = \emptyset$ and therefore by (7)

$$\begin{aligned} A(pf - g) &= A\big(f + (p-1)f - g\big) \\ &= A\big(f + (p-1)f\big) + A(f - g) - Af \\ &= A(pf) + A(f - g) - Af. \end{aligned}$$

This is the only time we use (7) for $f \neq 0$.

LEMMA 2. *For each K and $U_1 \supset K$, each $g \in C(K', E)$ and $\varepsilon > 0$ there exists U: $K \subset U \subset U_1$ and $q \in C(U', [0, 1])$ such that for all $f \in C(S, E)_{\|g\|}$ we have*

$$|A(f + g) - A(f + qg)| < \varepsilon.$$

Proof. Let $\|g\| =: a$. By assumption, there exists $\delta > 0$ such that $D_\delta^{2a} A < \varepsilon$. Let $K_0 = \{\|g(t)\| \geqslant \delta\}$. K_0 is compact and disjoint from K. Hence there exist disjoint U_0, U such that $K_0 \subset U_0$ and $K \subset U \subset U_1$. Let $q = u(K_0, U_0')$. Then $U \subset U_0' \subset \{q(t) = 0\}$ so that $q \in C(U', [0, 1])$. Furthermore $K_0 \subset \{g(t) - q(t)g(t) = 0\}$ and $K_0' \subset \{\|g(t) - q(t)g(t)\| = \|g(t)(1 - q(t))\| < \delta\}$ so that $\|(f+g) - (f+qg)\| = \|g - qg\| < \delta$. Hence the last assertion follows from the choice of δ.

LEMMA 3. *For each K, for each $a > 0$ and $\varepsilon > 0$ there exists $U \supset K$ such that for all $f \in C(S, E)_a$ and $g \in C(U - K, E)_a$ we have*

$$|A(f + g) - Af| < \varepsilon$$

(or such that for all $f_1, f_2 \in C(S, E)_a$ with $\{f_1(t) \neq f_2(t)\} \subset U - K$ we have

$$|Af_1 - Af_2| < \varepsilon).$$

Proof. If the assertion were false, there exist K and $a, \varepsilon > 0$ such that for all $U \supset K$ we have $|A(f + g) - Af| \geqslant \varepsilon$ for some corresponding f, g. Given $U_1 \supset K$, there exists $f_1 \in C(S, E)_a$ and $g_1 \in C(U_1 - K, E)_a$ such that

$$|A(f_1 + g_1) - Af_1| \geqslant \varepsilon.$$

For K and $U_1 \supset K$, for $g_1 \in C(K', E)$ and $\varepsilon/2$ we determine $U_{1,1}$: $K \subset U_{1,1} \subset U_1$ and $q_1 \in C(U_{1,1}', [0, 1])$ according to Lemma 2. Then still

$$|A(f_1 + q_1 g_1) - Af_1| \geqslant \varepsilon/2$$

and $q_1 g_1 \in C(U_1 \cap U_{1,1}', E)_a$. There exists U_2: $K \subset U_2 \subset \overline{U}_2 \subset U_{1,1}$. Given U_2, we can find f_2, g_2, q_2 as before. By induction we construct sequences of sets $U_n, U_{n,1}$ such that

$$K \subset U_{n+1} \subset \overline{U_{n+1}} \subset U_{n,1} \subset U_n, \qquad n = 1, 2, \ldots$$

and sequences of functions $f_n \in C(S, E)_\alpha$, $h_n \in C(\overline{U_n} \cap U'_{n,1}, E)_\alpha$ such that

$$|A(f_n+h_n)-Af_n| \geqslant \varepsilon/2, \qquad n = 1, 2, \ldots$$

Determine now $U_{n,2}$, $U_{n,3}$ such that

$$U_{n+1} \subset \overline{U_{n+1}} \subset U_{n,3} \subset \overline{U_{n,3}} \subset U_{n,2} \subset \overline{U_{n,2}} \subset U_{n,1} \subset U_n, \qquad n = 1, 2, \ldots$$

Then $\overline{U_n} \cap U'_{n,1} \subset U_{n-1,3} - U_{n,2}$, $n = 1, 2, \ldots$ (if we let $U_{0,3} = S$). If now $p_n := u(\overline{U_n} \cap U'_{n,1}, (U_{n-1,3} - U_{n,2})')$, then $p_n(\overline{U_n} \cap U'_{n,1}) = 1$ and it follows from Lemma 1 that

$$|A(p_n f_n + h_n) - A(p_n f_n)| \geqslant \varepsilon/2, \qquad n = 1, 2, \ldots$$

There exists a subsequence such that

$$\lim_{k'\to\infty} \Big| \sum_{k=1}^{k'} \big(A(p_{n_k} f_{n_k} + h_{n_k}) - A(p_{n_k} f_{n_k})\big) \Big| = \infty .$$

On the other hand, for all $n, m = 1, 2, \ldots$ we have

$$\begin{aligned} \{p_{n+m}(t) \neq 0\} &\subset U_{n+m-1,3} - U_{n+m,2} \\ &\subset \overline{U_{n+m-1,3}} \subset U_{n,3} \subset \overline{U_{n,2}} \\ &\subset \overline{U_{n,2}} \cup U'_{n-1,3} = (U_{n-1,3} \cap \overline{U_{n,2}}')' \\ &= (U_{n-1,3} - \overline{U_{n,2}})' \subset \{p_n(t) = 0\}. \end{aligned}$$

Hence the p_n have disjoint support. Because also $S[h_n] \subset \overline{U_n} \cap U'_{n,1} \subset U_{n-1,3} - U_{n,2} \subset S[p_n]$, we have

$$\Big\| \sum_{k=1}^{k'} (p_{n_k} f_{n_k} + h_{n_k}) \Big\| \leqslant 2\alpha, \qquad \Big\| \sum_{k=1}^{k'} p_{n_k} f_{n_k} \Big\| \leqslant \alpha$$

and therefore

$$\begin{aligned} &\Big| \sum_{k=1}^{k'} \big(A(p_{n_k} f_{n_k} + h_{n_k}) - A(p_{n_k} f_{n_k})\big) \Big| \\ &\qquad = \Big| A\Big(\sum_{k=1}^{k'} (p_{n_k} f_{n_k} + h_{n_k})\Big) - A\Big(\sum_{k=1}^{k'} p_{n_k} f_{n_k}\Big) \Big| \leqslant \|A_{2\alpha}\| + \|A_\alpha\| \end{aligned}$$

for all k', which is a contradiction.

LEMMA 4. *For each K and G_1, G_2 with $K \subset G_1 \cup G_2$ and for each $\varepsilon > 0$ there exist $v_1, v_2 \in C(S)$ with disjoint support, with $v_k(S) \subset [0,1]$ and $\{v_k(t) \neq 0\} \subset G_k$, $k = 1, 2$, such that for all $f \in C(S, E)_\alpha$ with $\{\|f(t)\| \geqslant \delta\} \subset K$ we have*

$$\big|Af - \big(A(v_1 f) + A(v_2 f)\big)\big| < \varepsilon,$$

if α and δ satisfy the condition $D_\delta^\alpha A < \varepsilon/2$.

Proof. Let $K_1 = K \cap G'_2$. Then $K_1 \subset G_1$. For K_1, α and $\varepsilon/2$ we determine $U_1 \supset K_1$ according to Lemma 3. We may assume $U_1 \subset G_1$. There exist U_2, U_3 such that

$$K \subset U_3 \subset \overline{U_3} \subset U_2 \subset \overline{U_2} \subset U_1 .$$

Let $K_2 = K \cap U'_1$. Then $K_2 \cap G'_2 = K \cap U'_1 \cap G'_2 = K_1 \cap U'_1 = \emptyset$, so that $K_2 \subset G_2$ and $K = (K \cap U_1) \cup (K \cap U'_1) \subset U_1 \cup K_2 \subset U_1 \cup G_2$. Let $u' = u(K_1, (G_2 \cup U_3)')$. Then we have $\|f - u'f\| < \delta$, because $K \subset \{u'(t) = 1\}$ and $K' \subset \{\|f(t)\| < \delta\}$. Hence

$$|Af - A(u'f)| \leqslant \varepsilon/2 .$$

Now let $u_1 = u(K_1, U'_3)$ and $u_2 = u(U'_1, \overline{U_2})$. Then $\{u_1(t) \neq 0\} \cap \{u_2(t) \neq 0\} \subset U_3 \cap \overline{U}'_2 = \emptyset$. Hence $\{u_1(t) + u_2(t) = 1\} = \{u_1(t) = 1\} \cup \{u_2(t) = 1\}$ and therefore $\{u_1(t) + u_2(t) \neq 1\} = \{u_1(t) \neq 1\} \cap \{u_2(t) \neq 1\} \subset K'_1 \cap U_1$. Therefore $\{u'(t) \neq u'(t)(u_1(t) + u_2(t))\} = \{u'(t) \neq 0\} \cap \{u_1(t) + u_2(t) \neq 1\} \subset K'_1 \cap U_1$ and thus

$$|A(u'f) - A(u'[u_1 + u_2]f)| < \varepsilon/2 .$$

Because $\{u_1(t) \neq 0\} \subset \overline{U_3} \subset U_2 \subset \{u_2(t) = 0\}$ the u_1, u_2 have disjoint support and thus

$$A(u'[u_1 + u_2]f) = A(u'u_1 f) + A(u'u_2 f).$$

Let us now put $v_1 = u'u_1$, $v_2 = u'u_2$. Then $\{v_1(t) \neq 0\} \subset \{u_1(t) \neq 0\} \subset U_3 \subset U_1 \subset G_1$ and $\{v_2(t) \neq 0\} \subset \{u'(t) \neq 0\} \cap \{u_2(t) \neq 0\} \subset (G_2 \cup U_3) \cap \overline{U}'_2 = (G_2 \cap \overline{U}'_2) \cup (U_3 \cap U'_2) \subset G_2$, so that v_1, v_2 satisfy the assertions of the lemma.

If $\alpha > 0$, let us define for each G in the lattice $\mathscr{L}$ of all open sets

$$\lambda_\alpha(G) := \sup \sum_{i=1}^{n} |Af_i| , \tag{12}$$

where the supremum is extended over all finite systems of functions $f_1, \ldots, f_n \in C(G, E)_\alpha$ with $\|f_i(t)\| \cdot \|f_k(t)\| = 0$ for $t \in S$ and $i \neq k$; $i, k = 1, \ldots, n$.

LEMMA 5. *We have, for $\alpha > 0$,*

(i) $\lambda_\alpha(G_1 \cup G_2) \leqslant \lambda_\alpha(G_1) + \lambda_\alpha(G_2)$ for $G_1, G_2 \in \mathscr{L}$,

(ii) $\lambda_\alpha(G_1 \cup G_2) = \lambda_\alpha(G_1) + \lambda_\alpha(G_2)$ for $G_1, G_2 \in \mathscr{L}$ and $G_1 \cap G_2 = \emptyset$,

(iii) λ_α is regular on $\mathscr{L}$.

Proof. (i) Let $\varepsilon > 0$ and for $G_1 \cup G_2$ determine a corresponding system $f_1, \ldots, f_n$ such that

$$\lambda_\alpha(G_1 \cup G_2) < \sum_{i=1}^{n} |Af_i| + \varepsilon .$$

Let $\delta > 0$ be such that $D_\delta^\alpha A < \varepsilon/2n$. If $K_i := \{\|f_i(t)\| \geqslant \delta\}$, then $K^i \subset G_1 \cup G_2$, $i = 1, \ldots, n$. For K_i and G_1, G_2 and ε/n we determine v_1^i, v_2^i according to Lemma 4, $i = 1, \ldots, n$. Then

$$\left|Af_i - \left(A(v_1^i f_i) + A(v_2^i f_i)\right)\right| < \varepsilon/n, \quad i = 1, \ldots, n.$$

It follows

$$\sum_{i=1}^n |Af_i| + \varepsilon < \sum_{i=1}^n \left(A(v_1^i f_i) + A(v_2^i f_i) + \varepsilon/n\right) + \varepsilon \leqslant \lambda_\alpha(G_1) + \lambda_\alpha(G_2) + 2\varepsilon,$$

and because $\varepsilon > 0$ is arbitrary we have (i).

(ii) follows from (i) and the definition of λ_α.

(iii). We have to show: For every $\varepsilon > 0$ and $G_0 \in \mathscr{L}$ there exists $K \subset G_0$ such that for all G: $K \subset G \subset G_0$ we have $\lambda_\alpha(G) < \lambda_\alpha(G_0) + \varepsilon$. For G_0 there exists a corresponding system of functions $f_1, \ldots, f_n$ such that

$$\lambda_\alpha(G_0) < \sum_{i=1}^n |Af_i| + \varepsilon/2.$$

Let $\delta > 0$ be such that $D_\delta^\alpha A < \varepsilon/2n$. If $K_i := \{\|f_i(t)\| \geqslant \delta\}$ and $K = \bigcup_{i=1}^n K_i$, then $K \subset G_0$. Let $G \in \mathscr{L}$ such that $K \subset G \subset G_0$ and put $u := u(K, G')$. Then $uf_1, \ldots, uf_n$ is a system corresponding to G and we have

$$\sum_{i=1}^n |A(uf_i)| \leqslant \lambda_\alpha(G).$$

Furthermore $\|f_i - uf_i\| < \delta$ because $K_i \subset \{f_i(t) - u(t)f_i(t) = 0\}$ and $K_i' \subset \{\|f_i(t) - u(t)f_i(t)\| < \delta\}$. Hence

$$|Af_i - A(uf_i)| \leqslant \varepsilon/2n, \quad i = 1, \ldots, n.$$

It follows

$$\sum_{i=1}^n |A(uf_i)| + \varepsilon/2 < \sum_{i=1}^n \left(|A(uf_i)| + \varepsilon/2n\right) + \varepsilon/2 \leqslant \lambda_\alpha(G_0) + \varepsilon$$

and hence $\lambda_\alpha(G_0) < \lambda_\alpha(G) + \varepsilon$.

Similarly we define for $\alpha > 0$, $\delta > 0$ and $G \in \mathscr{L}$

$$\lambda_\delta^\alpha(G) = \sup \sum_{i=1}^n |Af_i - Af_i'|$$

where the supremum is extended over all finite systems of functions $f_1, \ldots, f_n, f_1', \ldots, f_n' \in C(G, E)_\alpha$ with $\|f_i(t)\| \cdot \|f_k(t)\| = 0$, $\|f_i'(t)\| \cdot \|f_k'(t)\| = 0$ for $t \in S$ and $i \neq k$; $i, k = 1, \ldots, n$ and

$$\left\|\sum_{i=1}^n f_i - \sum_{i=1}^n f_i'\right\| \leqslant \delta.$$

One can prove in a similar way

LEMMA 6. *For $\alpha, \delta > 0$ λ_δ^α satisfies the assertions of Lemma 5.*

LEMMA 7. *λ_α and λ_δ^α can be extended from the lattice $\mathscr{L}$ of the open sets in S to uniquely determined regular nonnegative Borel measures on S. For $\delta \leqslant \delta'$ we have*

$$\lambda_\delta^\alpha(B) \leqslant \lambda_{\delta'}^\alpha(B) \leqslant 2\lambda_\alpha(B), \quad B \in \mathscr{B}. \tag{13}$$

Proof. $\mathscr{L}$ is dense in the system of all subsets of S ([12], p. 302), and because λ is nonnegative and increasing by definition and subadditive, additive and regular by Lemma 5, the possibility of the extension and its uniqueness follow from a well-known theorem ([12], p. 347). The same holds for λ_δ^α. The inequalities (13) are true for all $G \in \mathscr{L}$ and follow for all $B \in \mathscr{B}$ by regularity.

LEMMA 8. *If K_1 and K_2 are disjoint then for all $f, g \in C(S, E)_\alpha$ with $K_1 \cup K_2 \subset \{f(t) = g(t)\}$ and $(K_1 \cup K_2)' \subset \{\|f(t) - g(t)\| < \delta\}$ we have*

$$|Af - Ag| \leqslant \lambda_\delta^\alpha(K_1' \cap K_2').$$

Proof. Let $\varepsilon > 0$. For K_1, $\alpha > 0$ and $\varepsilon > 0$ we determine $U \supset K_1$ according to Lemma 3. We may assume $K_1 \subset U \subset K_2'$. There are U_1, U_2 such that $K_1 \subset U_1 \subset \overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset U$. Let $u_1 := u(K_1, U_1')$ and $u_2 := u(U', \overline{U}_2)$. Then $\{u_2(t) \neq 0\} \subset U_2' \subset \overline{U}_1' \subset \{u_1(t) = 0\}$, hence $S[u_1] \cap S[u_2] = \emptyset$. Because also $(U - K_1)' = K_1 \cup U' \subset \{1 - [u_1(t) + u_2(t)] = 0\}$, that is, $(1 - [u_1 + u_2]) \in C(U - K_1)$, we have

$$\left|Af - \left(A(u_1 f) + A(u_2 f)\right)\right| = \left|Af - A\left(f - f(1 - [u_1 + u_2])\right)\right| < \varepsilon,$$

similar relations holding for g. Because $\{u_1(t)f(t) \neq u_1(t)g(t)\} \subset \{u_1(t) \neq 0\} \cap \{f(t) \neq g(t)\} \subset U_1 \cap (K_1' \cap K_2') \subset U - K_1$ we have

$$|A(u_1 f) - A(u_1 g)| < \varepsilon.$$

Now let us assume that f and g both vanish on K_2. Because u_2 vanishes on K_1 we have

$$|A(u_2 f) - A(u_2 g)| \leqslant \lambda_\delta^\alpha(K_1' \cap K_2').$$

It follows

$$|Af - Ag| < 3\varepsilon + \lambda_\delta^\alpha(K_1' \cap K_2'),$$

and $\varepsilon > 0$ being arbitrary, we have proven so far that

$$|Af - Ag| \leqslant \lambda_\delta^\alpha(K_1' \cap K_2') \tag{14}$$

for any pair f, g of functions satisfying the assumptions and vanishing on K_2 or (by symmetry) on K_1. But, if f and g do not vanish on K_2, $u_2 f$ and $u_2 g$ are such a pair (vanishing on K_1); therefore by (14)

$$|A(u_2 f) - A(u_2 g)| \leqslant \lambda_\delta^\alpha(K_1' \cap K_2').$$

The other estimates of the first part of the proof remaining valid we have (14) also in this case.

Together with Lemma 7 we get the following

COROLLARY. *If $K \subset G$ we have for all $f, g \in C(S, E)_a$ with $\{f(t) \neq g(t)\} \subset G-K$ the inequality*

$$|Af - Ag| \leqslant 2\lambda_a(G-K).$$

Proof of Theorem 1. Let $A \in M_{\mathrm{HP}}(C(S, E), C)$ be given. For each K the limit

$$\mu(K)x := \lim_{G\in\pi(K)} A(xv(K, G'))$$

exists uniformly for $x \in E_a$. Indeed, for all $G_0, G_1, G_2 \in \pi(K)$ with $K \subset G_1$, $G_2 \subset G_0$ we have according to the last corollary

$$\left|A(xv(K, G_1')) - A(xv(K, G_2'))\right| \leqslant 2\lambda_a(G_0 - K), \qquad x \in E_a,$$

and the last quantity can be made smaller than ε for a suitable G_0 by the regularity of λ_a. Because $x \to A(xv(K, G'))$ for $G \in \pi(K)$ is an element of $M(E, C)$ it follows that $\mu(K) \in M(E, C)$. We have

$$\left|\mu(K)x - A(xu(K, G'))\right| \leqslant 2\lambda_a(G-K), \qquad x \in E_a, G \in \pi(K), \tag{15}$$

and using the HP of A one sees that μ is additive on the lattice of compact sets. For each G we define $\mu(G) = \mu(S) - \mu(G')$. If $B \in \mathscr{B}$ we can define and have

$$\mu(B)x := \lim_{K\in\pi_0(B)} \mu(K)x = \lim_{G\in\pi(B)} \mu(G)x, \qquad x \in E,$$

the limits existing uniformly on $x \in E_a$. In fact, if $K_0 \subset K_1, K_2 \subset B \subset G_0$ it follows

$$\begin{aligned}
&|\mu(K_1)x - \mu(K_2)x| \\
&\quad\leqslant \left|\mu(K_1)x - A(xv(K_1, G_0'))\right| + \left|A(xv(K_1, G_0')) - A(xv(K_2, G_0'))\right| + \\
&\qquad + \left|A(xv(K_2, G_0')) - \mu(K_2)x\right| \\
&\quad\leqslant 2\lambda_a(G_0 - K_1) + 2\lambda_a(G_0 - K_0) + 2\lambda_a(G_0 - K_2) \\
&\quad\leqslant 6\lambda_a(G_0 - K_0);
\end{aligned}$$

if $K_0 \subset B \subset G_1, G_2 \subset G_0$ (so that $G_0' \subset G_1', G_2' \subset B' \subset K_0'$) we have

$$\begin{aligned}
|\mu(G_1)x - \mu(G_2)x| &= |\mu(G_1')x - \mu(G_2')x| \\
&\leqslant 6\lambda_a(K_0' - G_0') = 6\lambda_a(G_0 - K_0),
\end{aligned}$$

and if $K_0 \subset K_1 \subset B \subset G_1 \subset G_0$ then

$$\begin{aligned}
|\mu(G_1)x - \mu(K_1)x| &= |A(x\chi_S) - \mu(K_1 \cup G_1')x| \\
&\leqslant 2\lambda_a(S - (K_1 \cup G_1')) = 2\lambda_a(G_1 - K_1) \\
&\leqslant 2\lambda_a(G_0 - K_0),
\end{aligned}$$

uniformly for $x \in E_a$. The quantity $\lambda_a(G_0 - K_0)$ can be made smaller than ε for suitable K_0, G_0 by the regularity of λ_a. Hence $\mu\colon \mathscr{B} \to M(E, C)$ and μ is additive. Passing to the limit with $K_1 \in \pi(B)$ and using $\mu(B) - \mu(K) = \mu(K') - \mu(B')$ one has

$$\begin{aligned}
&\|\mu_a(G) - \mu_a(B)\| \leqslant 2\lambda_a(G - B), && G \in \pi(B) \\
&\|\mu_a(B) - \mu_a(K)\| \leqslant 2\lambda_a(B - K), && K \in \pi_0(B), a > 0.
\end{aligned}\tag{16}$$

This proves the regularity of μ_a. Hence

$$\begin{aligned}
sv(\mu_a, S) &= \sup\left\{\left|\sum_{j=1}^{r} \mu_a(K_j)x_j\right| : K_1, \ldots, K_r \text{ disjoint}, x_1, \ldots, x_r \in E_a\right\} \\
&\leqslant \|A_a\|
\end{aligned}\tag{17}$$

and

$$\begin{aligned}
sv_\delta(\mu_a, S) = \sup\Big\{&\Big|\sum_{j=1}^{r} \big(\mu_a(K_j)x_j - \mu_a(K_j)x_j'\big)\Big|, K_1, \ldots, K_r \text{ disjoint}, \\
&x_j, x_j' \in E_a, \|x_j - x_j'\| \leqslant \delta, j = 1, \ldots, r\Big\} \leqslant D_\delta A_a.
\end{aligned}\tag{18}$$

We now show that μ represents A. Given $f \in C(S, E)_a$ and $\varepsilon > 0$, there exist $\delta > 0$ such that $D_\delta A_a < \varepsilon/5$ and $g \in \mathscr{E}_E(\mathscr{B})_a$ such that $\|f - g\| < \delta$. Then

$$\left|\int_S f\,d\mu - \int_S g\,d\mu\right| \leqslant sv_\delta(\mu_a, S) < \varepsilon/5.$$

Let $g = \sum_{j=1}^{r} x_j\chi_{B_j}$ for a $\mathscr{B}$-partition $\{B_1, \ldots, B_r\}$ and $x_1, \ldots, x_r \in E_a$. Choose sets $K_1, \ldots, K_r$ such that $K_j \subset B_j$ and $2\lambda_a(B_j - K_j) < \varepsilon/5r$, $j = 1, \ldots, r$. Then with (16)

$$\left|\int_S g\,d\mu - \sum_{j=1}^{r} \mu(K_j)x_j\right| < \varepsilon/5.$$

There exist $G_1, \ldots, G_r$ such that $G_j \supset K_j$, $j = 1, \ldots, r$ and the $\bar{G}_1, \ldots, \bar{G}_r$ are still disjoint with $\|f(t) - x_j\| < \delta$, $t \in G_j$, $j = 1, \ldots, r$. If we then let $u_j = u_j(K_j, G_j')$ the u_j have disjoint support. We have with (15)

$$\left|\sum_{j=1}^{r} \mu(K_j)x_j - \sum_{j=1}^{r} A(x_ju_j)\right| \leqslant \sum_{j=1}^{r} 2\lambda_a(G_j - K_j) \leqslant 2\sum_{j=1}^{r} \big(\lambda_a(B_j) - \lambda_a(K_j)\big) < \varepsilon/5$$

and with (7)

$$\Big|\sum_{j=1}^{r}\big(A(x_j u_j)\big)-\sum_{j=1}^{r}A(fu_j)\Big| = \Big|A\Big(\sum_{j=1}^{r}x_j u_j\Big)-A\Big(\sum_{j=1}^{r}fu_j\Big)\Big| \leqslant D_\delta A_\alpha < \varepsilon/5.$$

Finally, because we have $\bigcup_{j=1}^{r} K_j \subset \{f(t) = \sum_{j=1}^{r} f(t)u_j(t)\}$ it follows from the corollary of Lemma 8

$$\Big|\sum_{j=1}^{r}A(fu_j)-Af\Big| = \Big|A\Big(\sum_{j=1}^{r}fu_j\Big)-Af\Big| \leqslant 2\lambda_\alpha\Big(S-\bigcup_{j=1}^{r}K_j\Big)$$
$$= 2\sum_{j=1}^{r}\lambda_\alpha(B_j-K_j) < \varepsilon/5.$$

Hence we have

$$\Big|\int_S f\,d\mu - Af\Big| < \varepsilon$$

and $\varepsilon > 0$ being arbitrary, (8) follows. For an additive $\mu\colon \mathscr{B}\to M(E, C)$ satisfying (i) and (ii) the A generated by (8) has the HP according to (1) and we have by (2) and (3)

$$\|A_\alpha\| \leqslant sv(\mu_\alpha, S), \qquad D_\delta A_\alpha \leqslant sv_\delta(\mu_\alpha, S),$$

and with the same arguments, (17) and (18).

It remains to prove uniqueness. Assume for an additive

$$\mu\colon \mathscr{B}\to M(E, C)$$

we have (i) and (ii). Then, if $K\subset G$ and $x\in E_\alpha$ we have

$$\mu(K)x - A\big(xu(K, G')\big) = \int_S x\chi_K\,d\mu - \int_S xu(K, G')\,d\mu = -\int_{G-K} xu(K, G')\,d\mu,$$

so that if v is as in (9)

(19) $$\big|\mu(K)x - A\big(xv(K, G')\big)\big| \leqslant v(\mu_\alpha, G-K).$$

From the regularity of $v(\mu_\alpha)$ it follows that (9) is valid, that μ is determined by A on the compact sets and hence on all of $\mathscr{B}$ again by regularity. Theorem 1 is now completely established.

3. Transformations on $C(S, E)$ with the Hammerstein property. In this section we represent the transformations $T\in M\big(C(S, E), F\big)$ with the Hammerstein property:

THEOREM 2. *There exists an algebraic isomorphism between the space $M_{\mathrm{HP}}\big(C(S, E), F\big)$ of all $T\in M\big(C(S, E), F\big)$ with the Hammerstein property*

$$T(f+f_1+f_2) = T(f+f_1)+T(f+f_2)-Tf$$

for all $f, f_1, f_2\in C(S, E)$ with $S[f_1]\cap S[f_2] = \emptyset$ and the space of all additive nonlinear set functions $U\colon \mathscr{B}\to M(E, F'')$ with the following properties:

(i) *$sv(U_\alpha, S) < \infty$ and $sv_\delta(U_\alpha, S)\to 0$ $(\delta\to 0)$, for $\alpha > 0$,*

(ii) *if $\mu_{y'}\colon \mathscr{B}\to M(E, C)$, $y'\in F'$, is given by*

$$\mu_{y'}(B)x = \langle U(B)x, y'\rangle, \qquad B\in\mathscr{B}, x\in E,$$

then $\mu_{y',\alpha}\colon \mathscr{B}\to M_\alpha(E, C)$ is regular (and hence σ-additive) for $\alpha > 0$, $y'\in F'$,

(iii) *$\langle y'_\varrho, y\rangle\to\langle y', y\rangle$ (for a generalized sequence $\{y'_\varrho\}$ and y' in F' and all $y\in F$) implies $\langle\int_S f\,dU, y'_\varrho\rangle\to\langle\int_S f\,dU, y'\rangle$ for all $f\in C(S, E)$.*

The correspondence is given by

(20) $$Tf = \int_S f\,dU, \qquad f\in C(S, E)$$

and for corresponding T and U we have for each compact K

(21) $$\langle U(K), y'\rangle = \lim_{G\in\pi(K)}\big\langle y', T\big(xv(K, G')\big)\big\rangle, \qquad y'\in F'$$

uniformly for $x\in E_\alpha$ and functions v as in (9). Furthermore

(22) $$\|T_\alpha\| = sv(U_\alpha, S),$$

(23) $$D_\delta T_\alpha = sv_\delta(U_\alpha, S), \qquad \alpha, \delta > 0,$$

(24) $$T'y' = \mu_{y'}, \qquad y'\in F'$$

in the sense of the isomorphism established in Theorem 1.

Proof. Let $T\in M_{\mathrm{HP}}\big(C(S, E), F\big)$. By assumption, T takes bounded sets into bounded sets, hence its adjoint T' is well defined ([5], p. 9) and maps F' into $M_{\mathrm{HP}}\big(C(S, E), C\big)$. For $y'\in F'$, let $\mu_{y'}$ be the unique measure corresponding to $T'y'$ according to Theorem 1. For each $B\in\mathscr{B}$ and $x\in E$, the function $U(B)x\colon F'\to C$ given by $\langle U(B)x, y'\rangle = \mu_{y'}(B)x$, $y'\in F'$ is an element of F''. Furthermore $U\colon \mathscr{B}\to M(E, F'')$, U is additive and we have, if σ' denotes the unit ball in F',

(25) $$sv(U_\alpha, S) = \sup_{y'\in\sigma'} sv(\mu_{y',\alpha}, S) = \sup_{y'\in\sigma'}\|(T'y')_\alpha\|$$
$$= \sup_{\substack{y'\in\sigma'\\ f\in C(S,E)_\alpha}}|\langle y', Tf\rangle| = \|T_\alpha\|,$$

that is, (22), and (23) is proven similarly. We have

$$\langle y', Tf\rangle = (T'y')f = \int_S f\,d\mu_{y'}, \qquad y'\in F', f\in C(S, E)$$

and (iii) follows. Conversely, any additive function $U: \mathscr{B} \to M(E, F'')$ with (i), (ii), (iii) defines an element $T \in M_{\mathrm{HP}}(C(S, E), F'')$. But T takes values in F, because

$$\int_S f\, d\mu_{y'} = \Big\langle \int_S f\, dU, y' \Big\rangle, \qquad y' \in F', f \in C(S, E)$$

and by (iii) the integral with respect to U is continuous in $\sigma(F', F)$. (21) follows from (ii) and (24) together with (9).

4. Kernel representations. In this section we derive an analogue of the Radon–Nikodým Theorem for nonlinear measures (Lemma 9) to deduce an integral representation with a Lebesgue- or Bochner-integrable kernel (Theorem 3 and Lemma 10).

LEMMA 9. *Let $(Q, \mathscr{A}, \omega)$ be a measure space with a bounded, nonnegative measure $\omega \neq 0$. Then there exists an algebraic isomorphism between the additive nonlinear set functions $\mu: \mathscr{A} \to M(E, \boldsymbol{C})$ with the following properties:*

1) *μ_a is σ-additive, $v(\mu_a, Q) < \infty$ and $v_\delta(\mu_a, Q) \to 0$ $(\delta \to 0)$ for $a > 0$,*

2) *$v(\mu_a, A) \leqslant L_a \omega(A)$, $A \in \mathscr{A}$ for constants $L_a = L_a(\mu) \geqslant 0$ and $a > 0$,*

and the functions $u: Q \to M(E, \boldsymbol{C})$ (except for at most an ω-null set) such that

(i) *$u(\cdot, x): t \to u(t)x$ is an element of $\mathscr{L}^\infty(\omega)$ for $x \in E$,*

(ii) *$u(\cdot, x) = \varrho(u(\cdot, x))$ for a lifting ϱ of $\mathscr{L}^\infty(\omega)$ and all $x \in E$,*

(iii) *$\|u(\cdot)_a\|: t \to \|u(t)_a\|$ is bounded for each $a > 0$,*

(iv) *$\|u(\cdot)_a\|$ and $D_\delta u(\cdot)_a: t \to D_\delta u(t)_a$ are ω-measurable and there exists an ω-null set N such that*

$$\lim_{\delta \to 0} D_\delta u(t)_a = 0, \qquad t \in Q - N, a > 0.$$

The correspondence is given by

$$\mu(A)x = \int_A u(t, x)\, d\omega(t), \qquad A \in \mathscr{A}, x \in E \tag{26}$$

and we have for corresponding μ and u

$$v(\mu_a, A) = \int_A \|u(t)_a\|\, d\omega(t), \qquad A \in \mathscr{A}, a > 0, \tag{27}$$

$$v_\delta(\mu_a, A) = \int_A D_\delta u(t)_a\, d\omega(t), \qquad A \in \mathscr{A}, a, \delta > 0, \tag{28}$$

$$\int_Q g\, d\mu = \int_Q u(t, g(t))\, d\omega(t), \qquad g \in \mathscr{M}_E(\mathscr{A}). \tag{29}$$

Proof. Let μ be given. We have for $x \in E_a$

$$|\mu(A)x| \leqslant v(\mu(\cdot)x, A) \leqslant v(\mu_a, A) \leqslant L_a \cdot \omega(A), \qquad A \in \mathscr{A}.$$

By the Radon–Nikodým Theorem there exists $u(\cdot, x) \in \mathscr{L}^1(\omega)$ such that

$$\mu(A)x = \int_A u(t, x)\, d\omega(t), \qquad A \in \mathscr{A}.$$

It follows

$$\int_A |u(t, x)|\, d\omega(t) = v(\mu(\cdot)x, A) \leqslant v_a(\mu, A) \leqslant L_a \cdot \omega(A), \qquad A \in \mathscr{A},$$

hence $|u(t, x)| \leqslant L_a$ for ω-a.e. $t \in Q$ and $u(\cdot, x) \in \mathscr{L}^\infty(\omega)$, $x \in E$. Because ω is bounded it possesses the direct sum property ([12], p. 179) and because also $\omega \neq 0$ there exists a lifting ϱ of $\mathscr{L}^\infty(\omega)$ ([12], p. 206). After changing $u(\cdot, x)$ on a set of measure zero if necessary we can assume $\varrho(u(\cdot, x)) = u(\cdot, x)$, $x \in E$. Because $\mu(A)0 = 0$ for all $A \in \mathscr{A}$ it follows

$$u(t, 0) = 0, \qquad \text{for a.e. } t \in Q \tag{30}$$

and with ([12], p. 200) we have

$$|u(t, x)| \leqslant \sup_{t \in Q} |u(t, x)| = \omega\text{-ess.}\sup |\varrho(u(\cdot, x))|$$
$$= \omega\text{-ess.}\sup |u(\cdot, x)| \leqslant L_a, \qquad t \in Q, x \in E_a.$$

Hence $\|u(t)_a\| := \sup_{x \in E_a} |u(t, x)| \leqslant L_a$, $t \in Q$. We have proven (i)–(iii). To show that (i)–(iii) imply the first part of (iv), we let $a > 0$ and consider the set $\mathscr{F}$ of all functions g of the form

$$g = \sum_j \chi_{B_j}(\cdot) \cdot |u(\cdot, x_j)|,$$

where $\{B_j\}$ is a finite system of disjoint sets in $\mathscr{A}$ with $\varrho(B_j) = B_j$ and $x_j \in E_a$. Then one can show

(a) $\|u(t)_a\| = \sup_{g \in \mathscr{F}} g(t)$, $t \in Q$,

(b) $\varrho(g) = g$, $g \in \mathscr{F}$,

(c) $\mathscr{F}$ is a directed set of functions g in $\mathscr{L}^\infty(\omega)$ with $0 \leqslant g(t) \leqslant L_a$, $t \in Q$.

(The details of the proof are similar to the arguments in ([12], pp. 212–214) and will be omitted here.) It follows from a well-known lemma of A. I. Tulcea and C. I. Tulcea ([12], p. 209 and p. 139) that then $\|u(\cdot)_a\|$ is ω-measurable, hence ω-integrable and

$$\sup_{g \in \mathscr{F}} \int_Q g(t)\, d\omega(t) = \int_Q \|u(\cdot)_a\|\, d\omega(t),$$

that is,

$$\sup \int_Q \sum_j \chi_{B_j}(t)\, |u(t, x_j)|\, d\omega(t) = \int_Q \|u(t)_a\|\, d\omega(t), \tag{31}$$

where the supremum is taken over all systems $\{B_j\}$ occuring in the definition for $\mathscr{F}$ and $x_j \in E_a$. If $\varepsilon > 0$, there exists a finite system $\{A_j\}$ of disjoint sets in $\mathscr{A}$ and $x_j \in E_a$ such that

$$\begin{aligned} v(\mu_a, Q) &< \sum_j |\mu(A_j)x_j| + \varepsilon \\ &= \sum_j \Big| \int_{A_j} u(t, x_j)\, d\omega(t) \Big| + \varepsilon \\ &\leqslant \int_Q \sum_j \chi_{A_j}(t)\, |u(t, x_j)|\, d\omega(t) + \varepsilon \\ &= \int_Q \sum_j \chi_{B_j}(t)\, |u(t, x_j)|\, d\omega(t) + \varepsilon \end{aligned}$$

with $B_j := \varrho(A_j)$. Because $B_j \cap B_i = \varrho(B_j) \cap \varrho(B_i) = A_j \cap A_i = \emptyset$, $i \neq j$, the last integrand is an element of $\mathscr{F}$ and $\varepsilon > 0$ being arbitrary we have

$$v(\mu_a, Q) \leqslant \int_Q \|u(t)_a\|\, d\omega(t).$$

The converse inequality follows from (31) and hence we have the equality. The same proof holds for each $A_0 \in \mathscr{A}$ with $\varrho(A_0) = A_0$ (instead of Q) and we have (27) for all $A \in \mathscr{A}$ because A and $\varrho(A)$ differ at most by an ω-null set. The relation (28) is proven in a similar way by replacing $|u(t, x_j)|$ by $|u(t, x_j) - u(t, x_j')|$ for $x_j, x_j' \in E_a$ with $\|x_j - x_j'\| \leqslant \delta$ in the definition of $\mathscr{F}$.

Now we show: For each natural number k there exists an ω-null set N_k such that

$$\lim_{\delta \to 0} D_\delta u(t)_k = 0, \qquad t \in Q - N_k,$$

$k = 1, 2, \ldots$ For all $t \in Q$ with the exception of the ω-null set $\bigcup_{k=1}^{\infty} N_k$ we would then have

$$\lim_{\delta \to 0} D_\delta u(t)_a \leqslant \lim_{\delta \to 0} D_\delta u(t)_{[a]+1} = 0, \qquad a > 0;$$

this would complete the proof of (iv) and with (30) it would follow that $u(t) \in M(E, C)$ for ω-a.e. $t \in Q$. In fact, let $\delta_i \downarrow 0$. The functions $D_{\delta_i} u(\cdot)_k \geqslant 0$ are bounded by $2L_k$ and monotonically decreasing pointwise to an ω-integrable limit $\varphi_k \geqslant 0$ so that we have

$$\lim_{i \to \infty} \int_Q D_{\delta_i} u(t)_k\, d\omega(t) = \int_Q \varphi_k(t)\, d\omega(t).$$

On the other hand,

$$\int_Q D_{\delta_i} u(t)_k\, d\omega(t) = v_{\delta_i}(\mu_k, Q) \to 0 \qquad (i \to \infty)$$

so that $\int_Q \varphi_k(t)\, d\omega(t) = 0$ and hence $\varphi_k(t) = 0$ for all $t \in Q$ with the exception of an ω-null set N_k, $k = 1, 2, \ldots$

We now note that (29) is valid for all $g \in \mathscr{E}_E(\mathscr{A})$ of the form $g = \sum_j x_j \chi_{A_j}$ with disjoint $A_j \in \mathscr{A}$: in fact, because of (30) we have

$$\begin{aligned} \int_Q g(t)\, d\omega(t) &= \sum_j \mu(A_j) x_j = \sum_j \int_{A_j} u(t, x_j)\, d\omega(t) \\ &= \int_Q \sum_j u(t, x_j)\, \chi_{A_j}(t)\, d\omega(t) \\ &= \int_Q u\Big(t, \sum_j x_j \chi_{A_j}(t)\Big)\, d\omega(t) \\ &= \int_Q u\big(t, g(t)\big)\, d\omega(t); \end{aligned}$$

the validity of (29) for all $g \in \mathscr{M}_E(\mathscr{A})$ follows from a simple continuity argument. It is also clear that u is uniquely determined by μ. Conversely, given u with (i)–(iv), one proves (27), (28) which imply all properties of the corresponding μ. Lemma 9 is now completely established.

THEOREM 3. *For any $A \in M_{\mathrm{HP}}\big(C(S, E), C\big)$ there exists a bounded regular nonnegative Borel measure ω and a function $u\colon S \to M(E, C)$ (except for at most an ω-null set) with the properties* (i)–(iv) *of Lemma* 9 *on the measure space $(S, \mathscr{B}, \omega)$ such that*

$$Af = \int_S u\big(t, f(t)\big)\, d\omega(t), \qquad f \in C(S, E).$$

Proof. Let $\mu\colon \mathscr{B} \to M(E, C)$ be the set function according to Theorem 1. The relations (4), (5) show that μ also satisfies the assumption 1) of Lemma 9. Let us define a bounded nonnegative measure ω on $\mathscr{B}$ by

$$\omega(B) = \sum_{k=1}^{\infty} 2^{-k} \frac{v(\mu_k, B)}{1 + v(\mu_k, S)}, \qquad B \in \mathscr{B}.$$

We have $\omega \neq 0$ unless $A = 0$ and assumption 2) is easily verified. Hence Theorem 3 follows from Lemma 9.

In certain cases we claim the function $u(\cdot)_a$ is even Bochner-integrable with respect to ω. This is the assertion of

LEMMA 10. *Let $(Q, \mathscr{A}, \omega)$ be a measure space with a bounded nonnegative measure ω. Let X be a subspace of $M(E, C)$ and $\bar{X}_a$ the closure of the subspace $X_a = \{x_a^\beta\colon x^\beta \in X\}$ in $M_a(E, C)$. Assume that* a) $E = C$ *or* b) $\bar{X}_a$ *possesses the Radon–Nikodým property (with respect to ω). Then there exists an algebraic isomorphism between the additive set functions $\mu\colon \mathscr{A} \to X$ with the properties* 1) *and* 2) *and the functions $u\colon Q \to M(E, C)$ (except for at* *most an ω-null set) such that $u(\cdot)_a \in L^\infty_{\bar{X}_a}(\omega)$. The correspondence is given by*

$$\mu(A)_a = \int_A u(t)_a d\omega(t), \qquad A \in \mathscr{A},\ a > 0, \tag{32}$$

and for corresponding μ and u we have (27), (28) *and* (29).

Proof. Consider first case a). Let μ be given and let u be the function corresponding to μ according to Lemma 9. Let $a > 0$. Firstly, $u(t)_a \in M_a(\boldsymbol{C}, \boldsymbol{C})\,\omega$-a.e., and because $M_a(\boldsymbol{C}, \boldsymbol{C}) \subset C(C_a)$ (with $C_a := \{x \in C: |x| \leqslant a\}$) $u(\cdot)_a$ is ω-a.e. separably-valued. Secondly, if Z is the subspace of $M_a(\boldsymbol{C}, \boldsymbol{C})'$ spanned by the functionals Φ_x, $x \in C_a$, with $\langle \Phi_x, \varphi\rangle := \varphi(x)$ for all $\varphi \in M_a(\boldsymbol{C}, \boldsymbol{C})$, then Z is norming and $u(\cdot)_a$ is Z-weakly ω-measurable because for each $x \in C_a$ the function $t \to u(t, x)$ is ω-measurable. These two facts imply that $u(\cdot)_a$ is ω-measurable ([12], p. 105) and together with its boundedness we have $u(\cdot)_a \in L^\infty_{M_a(\boldsymbol{C},\boldsymbol{C})}(\omega)$. Furthermore, for all $\Phi \in Z$ we have by (26)

$$\langle \Phi, \mu(A)_a\rangle = \int_A \langle \Phi, u(t)_a\rangle d\omega(t) = \langle \Phi, \int_A u(t)_a d\omega(t)\rangle, \qquad A \in \mathscr{A},$$

and (32) follows. This representation implies that the essential range of $u(\cdot)_a$ on Q (the set of all $\varphi \in M_a(\boldsymbol{C}, \boldsymbol{C})$ such that for every $\varepsilon > 0$ the set $\{t \in Q: \|u(t) - \varphi\| < \varepsilon\}$ has strictly positive ω-measure ([25], p. 469)) is contained in the closure of the set $\{\mu(A)_a/\omega(A),\ A \in \mathscr{A},\ \omega(A) > 0\}$, hence in $\bar{X}_a$. On the other hand, the set of all $t \in Q$, for which $u(t)_a$ does not belong to its essential range has ω-measure zero ([25], p. 470) so that $u(\cdot)_a$ is ω-a.e. $\bar{X}_a$-valued and hence $u(\cdot)_a \in L^\infty_{\bar{X}_a}(\omega)$, $a > 0$. The remaining assertions are easily verified. — Now we consider case b). Given μ, for every $n = 1, 2, \ldots$ there exists a function $w^{(n)} \in L^\infty_{\bar{X}_n}(\omega)$ (without loss of generality we can assume $w^{(n)}(t) \in \bar{X}_n$ and $\|w^{(n)}(t)\| \leqslant L_n$ for all $t \in Q$) such that

$$\mu(A)_n = \int_A w^{(n)}(t)\, d\omega(t), \qquad A \in \mathscr{A}.$$

If $m < n$, then $w^{(n)}(\cdot)_m \in L^\infty_{\bar{X}_m}(\omega)$ and

$$\mu(A)_m = \int_A w^{(n)}(t)_m d\omega(t), \qquad A \in \mathscr{A}.$$

Hence $w^{(n)}(t) = w^{(n)}(t)_m$ for all $t \in Q$ with the possible exception of an ω-null set $N_{n,m}$. If $N = \bigcup_{n,m} N_{n,m}$ we can define $u(t, x) = 0$ for $t \in N$ or $x = 0$ and $= w^{(n)}(t)x$ for $t \in Q - N$ and $x \in E_n - E_{n-1}$. Then $u(t, x) = w^{(m)}(t)x = w^{(n)}(t)_m x = w^{(n)}(t)x$ for any $x \in E_m - E_{m-1}$, $m = 1, \ldots, n$, $t \in Q - N$, so that $u(\cdot)_n \in L^\infty_{\bar{X}_n}(\omega)$, $n = 1, 2, \ldots$ and $u(\cdot)_a \in L^\infty_{\bar{X}_a}(\omega)$, $a > 0$. The remaining assertions are then clear.

5. Compact and weakly compact transformations. In this section we investigate the compact and weakly compact transformations in $M_{\mathrm{HP}}(C(S, E), F)$.

THEOREM 4. *Let $T \in M_{\mathrm{HP}}(C(S, E), F)$. Then the following conditions are equivalent.*

(A_1) *For each $a > 0$ there exists a nonnegative bounded regular measure λ_a on $\mathscr{B}$ such that the measures $v(\mu_{y',a})$ are absolutely continuous uniformly with respect to $y' \in \sigma'$.*

(A_2) *$U: \mathscr{B} \to M(E, F)$ and for each $a > 0$ the measures $v(\mu_{y',a})$ are σ-additive uniformly with respect to $y' \in \sigma'$.*

(A_3) *For each compact K in S and $x \in E$ the limit* (20)

$$\lim_{G \in \pi(K)} T\big(xv(K, G')\big) = U(K)x$$

exists in $\sigma(F, F')$ and the series $\sum_{j=1}^{\infty} U(K_j)x_j$ converges (strongly) in F for every sequence $\{K_j\}_{j=1}^\infty$ of disjoint compact sets in S and every bounded sequence $\{x_j\}_{j=1}^\infty$ in E.

In this case $U_a: \mathscr{B} \to M_a(E, F)$ is regular in the norm of $M_a(E, F)$, $a > 0$.

Proof. As in the case of linear T ([3], p. 225) one sees that (A_1) and the second condition of (A_2) are equivalent and that they both imply the regularity of the set function $U_a: \mathscr{B} \to M_a(E, F'')$ given by $U_a(B) = U(B)_a$, $B \in \mathscr{B}$. Now let K be a compact set in S and let $x \in E_a$. We shall show $U(K)x \in F$. If $\varepsilon > 0$ we can determine $\delta > 0$ such that $\lambda_a(C) < \delta$ implies $v(\mu_{y',a}, C) < \varepsilon$, $y' \in \sigma'$. λ_a being regular there exists an open set $G_\varepsilon \supset K$ such that $\lambda_a(G_\varepsilon - K) < \delta$. If follows for all open sets G: $K \subset G \subset G_\varepsilon$ with (19)

$$\begin{aligned}\big\|U(K)x - T\big(xv(K, G')\big)\big\| &= \sup_{y' \in \sigma'} \big|\langle U(K)x, y'\rangle - \langle y', T\big(xv(K, G')\big)\rangle\big| \\ &= \sup_{y' \in \sigma'} \big|\mu_{y',a}(K)x - (T'y')\big(xv(K, G')\big)\big| \\ &\leqslant v(\mu_{y',a}, G - K) \leqslant \varepsilon\end{aligned}$$

so that $U(K)x \in F$. If follows $U_a: \mathscr{B} \to M_a(E, F)$ by regularity. This shows that (A_1) and (A_2) are equivalent. Now assume (A_3). If the measures $v(\mu_{y',a})$ are not σ-additive uniformly with respect to $y' \in \sigma'$ there exist an $\varepsilon > 0$, sequences of integers $N_1 < M_1 < \ldots < N_i < M_i < \ldots$, disjoint compact sets K_j^k, $k = 1, \ldots, r_j$, $j = N_i, \ldots, M_i$ and functionals $y_i' \in \sigma'$ such that

$$\sum_{j=N_i}^{M_i} \sum_{k=1}^{r_j} \|\mu_{y_i',a}(K_j^k)\| > \varepsilon, \qquad i = 1, 2, \ldots$$

In view of (6) and $\mu_{y'}(B)\,0 = 0$, $B \in \mathscr{B}$ this implies the existence of elements $x_j^k \in E_\alpha$ such that

$$\Big\|\sum_{j=N_i}^{M_i}\sum_{k=1}^{r_j} U(K_j^k)x_j^k\Big\| \geqslant \Big|\Big\langle y', \sum_{j=N_i}^{M_i}\sum_{k=1}^{r_j} U(K_j^k)x_j^k\Big\rangle\Big| \geqslant \varepsilon/4, \qquad i = 1, 2, \ldots$$

which is a contradiction to our assumption. Hence (A_3) implies the second part and (as before) the first part of (A_2). Also (A_2) implies (A_3) and the theorem is proven.

The following theorem characterizes the compact and weakly compact transformations. Its proof is similar to ([3], p. 229) and is omitted here.

THEOREM 5. *Let $T \in M_{\mathrm{HP}}(C(S, E), F)$. Then the following conditions are equivalent.*

(B_1) *T is (weakly) compact.*

(B_2) *$U\colon \mathscr{B} \to M(E, F)$ and the set of finite sums*

$$P_\alpha := \Big\{\sum_j U(B_j)x_j\colon\ B_j \in \mathscr{B},\ B_j \text{ disjoint},\ x_j \in E_\alpha\Big\}$$

is a conditionally (weakly) compact subset of F for $\alpha > 0$.

(B_3) *The integral*

$$\int_S f\,dU =: \hat{T}f, \qquad f \in \mathscr{M}_E(\mathscr{B})$$

defines a (weakly) compact extension $\hat{T}$ in $M(\mathscr{M}_E(\mathscr{B}), F)$.

In this case the conditions (A_i) are satisfied.

We now want to show the weak compactness of T under various conditions. Let $C(S, E)^\beta$ be the linear space of all complex-valued functions f^β on $C(S, E)$ with the property that the restriction f_α^β to $C(S, E)_\alpha$ is bounded. The linear space of the restrictions:

$$C(S, E)_\alpha^\beta = \{f_\alpha^\beta\colon\ f^\beta \in C(S, E)^\beta\}$$

coincides with the linear space of all bounded complex-valued functions on $C(S, E)_\alpha$. Let $C(S, E)_\alpha^\beta$ be normed by the supremum norm. We have shown in ([5], p. 12) that $T\colon C(S, E) \to F$ is weakly compact if and only if $T_\alpha'\colon F' \to C(S, E)_\alpha^\beta$ is weakly compact, $\alpha > 0$. In practice we shall work with a closed linear subspace $Y(\alpha)$ such that

$$T_\alpha' F' \subset Y(\alpha) \subset C(S, E)_\alpha^\beta. \tag{33}$$

Then an equivalent condition is that the image $T_\alpha'\sigma'$ of the unit ball σ' in F' be conditionally weakly compact in $Y(\alpha)$, $\alpha > 0$. Now define E^β and E_α^β similarly. If $U\colon \mathscr{B} \to M(E, F)$ and if one knows a suitable closed linear subspace $X(\alpha)$ such that

$$U(B)_\alpha' F' \subset X(\alpha) \subset E_\alpha^\beta, \tag{34}$$

then $U(B)$ is weakly compact if and only if $U(B)'\sigma'$ is conditionally weakly compact in $X(\alpha)$, $\alpha > 0$.

THEOREM 6. *Let $T \in M_{\mathrm{HP}}(C(S, E), F)$ satisfy one of the conditions (A_i). If $U(B)\colon E \to F$ is weakly compact for $B \in \mathscr{B}$ and if for each $\alpha > 0$ the spaces*

$$\overline{X}_\alpha := \overline{\operatorname{span}}\{\mu_{y',\alpha}(B)\colon\ y' \in F',\ B \in \mathscr{B}\} \subset M_\alpha(E, C)$$

together with their duals have the Radon–Nikodým property (with respect to λ_α), then T is weakly compact.

Proof. With the notation of our assumption (A_1) define a nonnegative bounded regular measure λ by

$$\lambda(B) = \sum_{n=1}^{\infty} 2^{-n}\frac{\lambda_n(B)}{1+\lambda_n(S)}, \qquad B \in \mathscr{B}, \tag{35}$$

and for each $y' \in F'$ define the nonnegative bounded regular measure $\omega_{y'}$ by

$$\omega_{y'}(B) = \sum_{n=1}^{\infty} 2^{-n}\frac{v(\mu_{y',n}, B)}{1+v(\mu_{y',n}, S)}, \qquad B \in \mathscr{B}.$$

It is easy to see that the $\omega_{y'}$ are absolutely continuous with respect to λ uniformly for $y' \in \sigma'$. Let $\alpha > 0$. We shall use the space $\mathring{L}^1_{\overline{X}_\alpha}(S, \mathscr{B}, \lambda)$ of all $\overline{X}_\alpha$-valued λ-integrable functions φ on S with the norm

$$\|\varphi\|_0 = \sup\sum_{j=1}^{r}\Big|\int_{B_j}\varphi(t, x_j)\,d\lambda(t)\Big| \tag{36}$$

(where the supremum is extended over all $\mathscr{B}$-partitions $\{\mathscr{B}_1, \ldots, B_r\}$ and elements $x_1, \ldots, x_r \in E_\alpha$) as a space $Y(\alpha)$ in (33). Indeed, this space can be considered as a subspace of $C(S, E)_\alpha^\beta$: Every φ defines an element $f_\alpha^\beta \in C(S, E)_\alpha^\beta$ by

$$f_\alpha^\beta f = \int_S \varphi(t, f(t))\,d\lambda(t), \qquad f \in C(S, E)_\alpha;$$

by approximating each $f \in C(S, E)_\alpha$ by functions in $\mathscr{E}_{\overline{X}_\alpha}(\mathscr{B})$ one obtains $\|f_\alpha^\beta\| \leqslant \|\varphi\|_0$; the converse inequality follows from the absolute continuity of the integral $\int\|\varphi(t)\|\,d\lambda(t)$, the regularity of λ, Urysohn's Lemma and $\varphi(t, 0) = 0$. Let $X := \overline{\operatorname{span}}\{\mu_{y'}(B), y' \in F', B \in \mathscr{B}\} \subset M(E, C)$. Because $\overline{X}_\alpha$ has the Radon–Nikodým property we can use Lemma 10. For $y' \in F'$ let $u_{y'}\colon S \to M(E, C)$ correspond to $\mu_{y'}$ and let $h_{y'} \in L^1(S, \mathscr{B}, \lambda)$ be the

Radon–Nikodým derivative of $\omega_{y'}$ with respect to λ. Then $u_{y'}(\cdot)_a \in L^\infty_{\bar{X}_a}(S, \mathscr{B}, \lambda)$ and with (29) we have

$$(T'_a y')f = \int_S f\,d\mu_{y'} = \int_S u_{y'}\big(t, f(t)\big)\,d\omega_{y'}(t)$$
$$= \int_S h_{y'}(t)\,u_{y'}\big(t, f(t)\big)\,d\lambda(t)$$
$$= \int_S \varphi_{y'}\big(t, f(t)\big)\,d\lambda(t), \qquad f \in C(S, E)_a,$$

where $\varphi_{y'} = h_{y'} u_{y'} \in L^1_{\bar{X}_a}(S, \mathscr{B}, \lambda)$. We have shown

$$T'_a F' \subset \mathring{L}^1_{\bar{X}_a}(S, \mathscr{B}, \lambda) \subset C(S, E)^\beta_a$$

and we can as well prove the conditional weak compactness of $T'_a \sigma'$ in $L^1_{\bar{X}_a}(S, \mathscr{B}, \lambda)$ because the integral norm is equivalent to the norm (36). We have

(i) $\sup\limits_{y' \in \sigma'} \|\varphi_{y'}\| = \sup\limits_{y' \in \sigma'} \int_S \|u_{y'}(t)_a\|\,d\omega_{y'}(t) = \sup\limits_{y' \in \sigma'} v(\mu_{y',a}, S)$
$\leqslant 4 \sup\limits_{y' \in \sigma'} sv(\mu_{y',a}, S) = 4\|T_a\|$ with (27) and (25),

(ii) $\lim\limits_{\lambda(B) \to 0} \sup\limits_{y' \in \sigma'} \int_B \|\varphi_{y'}(t)\|\,d\lambda(t) = \lim\limits_{\lambda(B) \to 0} \sup\limits_{y' \in \sigma'} v(\mu_{y',a}, B) = 0$ because also $\mu_{y',a}$ is absolutely continuous with respect to λ uniformly for $y' \in \sigma'$,

(iii) the set $\{\int_B \varphi_{y'}(t)\,d\lambda(t)\}_{y' \in \sigma'} = \{\mu_{y',a}(B)\}_{y' \in \sigma'} = \{U(B)'_a y'\}_{y' \in \sigma'}$ is conditionally weakly compact in $\bar{X}_a$ for each $B \in \mathscr{B}$, because by ([5], p. 12) and (34) with $X(a) := \bar{X}_a$ the weak compactness of $U(B)$ implies the weak compactness of $U(B)'_a$.

By our assumption on $\bar{X}_a$ and $\bar{X}'_a$ the assertion now follows from [8], Corollary 2 of Theorem 4.

Remark. In general even a linear bounded transformation T needs not to be weakly compact if only one of the spaces $\bar{X}_a$ and $\bar{X}'_a$ fails to have the Radon–Nikodým property though the remaining conditions of Theorem 6 are satisfied ([8], Examples 2 and 7). Theorem 6 generalizes a result of A. Pełczyński ([24], p. 645, Theorem 1').

Let us turn to the special case $E = \mathbf{C}$. We need the following two lemmas. Let π and π' stand for finite and infinite sets of natural numbers.

LEMMA 11. *Let $\{U_j\}_{j=1}^\infty$ be a sequence of operators in $M(\mathbf{C}, F)$ with the following properties:*

(a) $\sup\limits_{\substack{x_j \in \mathbf{C}_a \\ \pi}} \Big\|\sum\limits_{j \in \pi} U_j x_j\Big\| < \infty,\ a > 0,$

(b) $\lim\limits_{\delta \to 0} \sup\limits_{\substack{x_j, x'_j \in \mathbf{C}_a \\ |x_j - x'_j| \leqslant \delta \\ \pi}} \Big\|\sum\limits_{j \in \pi} (U_j x_j - U_j x'_j)\Big\| = 0,\ a > 0,$

(c) *For each π' there exists $U_{\pi'} \in M(\mathbf{C}, F)$ such that $\sum\limits_{j \in \pi'} U_j x = U_{\pi'} x$, $x \in \mathbf{C}$, in the norm of F.*

Then for each $a > 0$ (considering $y' \circ U_j$ as an element in $M(\mathbf{C}, \mathbf{C})$) the sums $\sum\limits_{j=1}^\infty \|(y' \circ U_j)_a\|$ converge uniformly for $y' \in \sigma'$, the sums $\sum\limits_{j=1}^\infty U_j x_j$ converge uniformly for $\xi = \{x_j\}_{j=1} \in l^\infty_a$ and the transformation $H \in M(l^\infty, F)$ given by $H\xi = \sum\limits_{j=1}^\infty U_j x_j$ is compact.

Proof. For each π let $U_\pi \in M(\mathbf{C}, F)$ be the sum of the U_j with $j \in \pi$. Then (a) and (b) say that the family $\{U_{\pi,a}\}_\pi$ is a bounded set of equicontinuous functions in $M_a(\mathbf{C}, F) \subset C(\mathbf{C}_a, F)$ for each $a > 0$. The assumption (c) means that for each $x \in \mathbf{C}_a$ the series $\sum\limits_{j=1}^\infty U_j x$ is unconditionally convergent, so that ([21], p. 245) for each $x \in \mathbf{C}_a$ the set $\{U_\pi x\}_\pi$ is conditionally compact in F. Hence $\{U_{\pi,a}\}_\pi$ is conditionally compact in $M_a(\mathbf{C}, F)$. Let $\varepsilon > 0$ be given. There exist $\delta > 0$ such that the supremum in (b) is smaller than $\varepsilon/2$ and finite sets of complex numbers $\zeta_1, \ldots, \zeta_N \in \mathbf{C}_a$ with

$$\inf_{i=1,\ldots,N} |x - \zeta_i| < \delta, \qquad x \in \mathbf{C}_a.$$

and of operators $V_1, \ldots, V_M \in M_a(\mathbf{C}, F)$ such that

$$\inf_{k=1,\ldots,M} \|U_\pi - V_k\| < \varepsilon/2N \qquad \text{for all } \pi.$$

The finite set $\{V_{k_1}\zeta_1 + \ldots + V_{k_N}\zeta_N,\ 1 \leqslant k_i \leqslant M,\ i = 1, \ldots, N\}$ is then an ε-net for $\{\sum\limits_{j \in \pi} U_j x_j\colon \pi, x_j \in \mathbf{C}_a\}$. Hence this set is conditionally compact in F and all the series $\sum\limits_{j=1}^\infty U_j x_j$, $\|\xi\| \leqslant a$ converge and (as in ([4], p. 911)) uniformly. Therefore, if $H_n \xi := \sum\limits_{j=1}^n U_j x_j$, then

$$\|H_a - H_{n,a}\| = \sup_{\|\xi\| \leqslant a} \Big\|\sum_{j=n+1}^\infty U_j x_j\Big\| \to 0, \qquad a > 0,$$

and the $H_{n,a}$ being compact, H_a is compact ([5], p. 16), $a > 0$. To show the remaining assertion one argues in a similar way as in the proof of $(\mathrm{A}_3) \to (\mathrm{A}_2)$.

LEMMA 12. *Let $(Q, \mathscr{A}, \lambda)$ be a measure space with a bounded nonnegative measure λ. Then a subset M in $L^1_{M_a(\mathbf{C},\mathbf{C})}(Q, \mathscr{A}, \lambda)$ is conditionally weakly compact if*

(i) $\sup\limits_{\varphi \in M} \int_Q \|\varphi(t)\|\,d\lambda(t) < \infty,$

(ii) $\lim\limits_{\lambda(A) \to 0} \sup\limits_{\varphi \in M} \int_A \|\varphi(t)\|\,d\lambda(t) = 0,$

(iii)' $\lim\limits_{\delta \to 0} \sup\limits_{\varphi \in M} \int_Q D_\delta \varphi(t)\,d\lambda(t) = 0.$

Proof. Our result will follow from an application of ([8], Theorem 6). We have to show that the assumptions are satisfied. The space $M_a(\boldsymbol{C}, \boldsymbol{C})$ is the space of all continuous functions on $\boldsymbol{C}_a$ vanishing at 0 and it has a basis. Besides of (i) and (ii) we have (iii): the set $\{\int_A \varphi d\lambda\}_{\varphi \in M}$ is conditionally weakly compact in $M_a(\boldsymbol{C}, \boldsymbol{C})$ for each $A \in \mathscr{A}$. Indeed, a fortiori the set $\{\int_A \varphi d\lambda, \varphi \in M, A \in \mathscr{A}\}$ is bounded by (i) and equicontinuous by (iii)′. We now show (iv): the functions $\varphi \in M$ are uniformly λ-almost compact-valued.

First, for every $\eta > 0$ there exists $L > 0$ such that

$$\lambda(\{\|\varphi(t)\| \geqslant L\}) < \eta, \qquad \varphi \in M.$$

Otherwise namely there exists some $\eta > 0$, a sequence of numbers $\{L_k\}_{k=1}^{\infty}$ with $L_k \uparrow \infty$ for $k \to \infty$ a sequence $\{\varphi_k\}_{k=1}^{\infty} \subset M$ such that

$$\lambda(\{\|\varphi_k(t)\| \geqslant L_k\}) \geqslant \eta, \qquad k = 1, 2, \ldots$$

This implies

$$\int_Q \|\varphi_k(t)\| \, d\lambda(t) \geqslant L_k \cdot \eta, \qquad k = 1, 2, \ldots$$

which contradicts (i). If we put

$$A_\varphi = \{\|\varphi(t)\| \geqslant L\}$$

then $\lambda(A_\varphi) < \eta$, $\varphi \in M$ and the set of functions $\{\varphi(t), t \in A'_\varphi, \varphi \in M\}$ on $\boldsymbol{C}_a$ is bounded.

Second, for every $\eta > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\lambda(\{D_\delta \varphi(t) \geqslant \varepsilon\}) < \eta, \qquad \varphi \in M.$$

Otherwise namely there exist some $\eta, \varepsilon > 0$, a sequence of numbers $\{\delta_k\}_{k=1}^{\infty}$ with $\delta_k \downarrow 0$ for $k \to \infty$ and a sequence $\{\varphi_k\}_{k=1}^{\infty} \subset M$ such that

$$\lambda(\{D_{\delta_k}\varphi_k(t) \geqslant \varepsilon\}) \geqslant \eta, \qquad k = 1, 2, \ldots$$

This implies

$$\int_Q D_{\delta_k}\varphi_k(t) \, d\lambda(t) \geqslant \varepsilon \cdot \eta, \qquad k = 1, 2, \ldots$$

which contradicts (ii). Hence, if we consider a sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ with $\varepsilon_i \downarrow 0$ for $i \to \infty$, for every ε_i and $\eta/2^i$ we can determine $\delta_i > 0$ and sets $B_{\varphi,i} \in \mathscr{A}$ with $\lambda(B_{\varphi,i}) < \eta/2^i$ such that

$$D_{\delta_i}\varphi(t) < \varepsilon_i \quad \text{for all } t \in B'_{\varphi,i}, \varphi \in M, i = 1, 2, \ldots$$

If we let $B_\varphi := \bigcup_{i=1}^{\infty} B_{\varphi,i}$ then $\lambda(B_\varphi) < \eta$ and the set of functions $\{\varphi(t), t \in B_\varphi, \varphi \in M\}$ on $\boldsymbol{C}_a$ is equicontinuous. For given $\eta > 0$ we have constructed a system $\{A_\varphi \cup B_\varphi =: C_\varphi\}_{\varphi \in M}$ of sets in $\mathscr{A}$ such that $\lambda(C_\varphi) < 2\eta$ and the set $\{\varphi(t), t \in C'_\varphi, \varphi \in M\}$ is conditionally compact in $M_a(\boldsymbol{C}, \boldsymbol{C})$. This proves (iv).

As for the last assumption (v) made in ([8], Theorem 6) it is sufficient to show: If for a sequence $\{\varphi_k\}_{k=1}^{\infty}$ in M and a set function μ_a: $\mathscr{B} \to M_a(\boldsymbol{C}, \boldsymbol{C})$ we have

$$\left\langle \Phi, \int_A \varphi_k d\lambda \right\rangle \to \langle \Phi, \mu_a(A)\rangle, \qquad A \in \mathscr{A}, \ \Phi \in M_a(\boldsymbol{C}, \boldsymbol{C})',$$

for $k \to \infty$, then μ_a has a derivative in $L^1_{M_a(C,C)}(Q, \mathscr{A}, \lambda)$ with respect to λ. It follows from ([8], Theorem 3) that μ_a is a measure on $\mathscr{A}$ of bounded variation with values in $M_a(\boldsymbol{C}, \boldsymbol{C})$. Furthermore, μ_a is absolutely continuous with respect to λ and for any $\mathscr{A}$-partition $\{A_1, \ldots, A_r\}$ in Q and elements $x_j, x'_j \in E_a$ with $|x_j - x'_j| \leqslant \delta$, $j = 1, \ldots, r$ we have (Φ_x is again the evaluation functional at x)

$$\begin{aligned}
\Big|\sum_{j=1}^{r}\big(\mu_a(A_j)x_j - \mu_a(A_j)x'_j\big)\Big| &= \Big|\sum_{j=1}^{r}\langle \Phi_{x_j} - \Phi_{x'_j}, \mu_a(A_j)\Big| \\
&= \lim_{k\to\infty}\Big|\sum_{j=1}^{r}\Big\langle \Phi_{x'_j} - \Phi_{x'_j}, \int_{B_j}\varphi_k d\lambda\Big\rangle\Big| \\
&= \lim_{k\to\infty}\Big|\sum_{j=1}^{r}\int_{B_j}\big(\varphi_k(t)x_j - \varphi_k(t)x'_j\big)d\lambda(t)\Big| \\
&\leqslant \limsup_{k\to\infty}\int_Q D_\delta\varphi_k(t)\,d\lambda(t),
\end{aligned}$$

so that $\lim_{\delta\to 0} sv_\delta(\mu_a, Q) = 0$. One can extend each element in $M_a(\boldsymbol{C}, \boldsymbol{C})$ to $M(\boldsymbol{C}, \boldsymbol{C})$ without increasing the norm and modulus of continuity. Hence we can assume that μ_a is the restriction of a set function μ: $\mathscr{A} \to M(\boldsymbol{C}, \boldsymbol{C})$ which satisfies the assumption of Lemma 10 with $\omega = v(\mu_a)$. Let u: $Q \to M(\boldsymbol{C}, \boldsymbol{C})$ be the function corresponding to μ. Then we have $u(\cdot)_a \in L^\infty_{M_a(C,C)}(Q, \mathscr{A}, v(\mu_a))$ and for some $h \in L^1(Q, \mathscr{A}, \lambda)$

$$\mu_a(A) = \mu(A)_a = \int_A u(t)_a dv(\mu_a, t) = \int_A h(t)u(t)_a d\lambda(t), \qquad A \in \mathscr{A}$$

with $h(\cdot)u(\cdot)_a \in L^1_{M_a(C,C)}(Q, \mathscr{A}, \lambda)$. This proves (v).

The following theorem characterizes the weakly compact transformations for $E = \boldsymbol{C}$.

THEOREM 7. *Let $T \in M_{\mathrm{HP}}(C(S), F)$. Then the following conditions are equivalent.*

(C_1) *T is weakly compact.*

(C_2) $U\colon \mathscr{B} \to M(\boldsymbol{C}, F)$.

(A_i), $i = 1, 2, 3$ *hold*.

Proof. The implication $(C_1) \to (C_2)$ follows from Theorem 5. Now assume (C_2). To show that (A_3) holds, consider a sequence $\{K_j\}_{j=1}^{\infty}$ of disjoint compact sets in S and let $U_j := U(K_j)$, $j = 1, 2, \ldots$ It follows from Theorem 2, (i) that the first two assumptions of Lemma 11 are satisfied. The third assumption follows from a straightforward application of Orlicz' Theorem, because for every $y' \in F'$ and $x \in \boldsymbol{C}$

$$\lim_{n\to\infty} \Big\langle y', \sum_{\substack{j=1\\ j\in\pi'}}^{n} U(K_j)x \Big\rangle = \sum_{\substack{j=1\\ j\in\pi'}}^{\infty} \mu_{y',a}(K_j)x = \langle y', U\big(\bigcup_{\substack{j=1\\ j\in\pi'}}^{\infty} K_j\big)x\rangle.$$

Hence (A_3) follows from Lemma 11. It remains to show that e.g. (A_1) implies (C_2). We proceed similarly as in the proof of Theorem 6. We let $X := M(\boldsymbol{C}, \boldsymbol{C})$ and $X_a := M_a(\boldsymbol{C}, \boldsymbol{C}) = \overline{X}_a$ and use the space $\mathring{L}^1_{X_a}(S, \mathscr{B}, \lambda)$ as $Y(a)$ in (33). Because $E = \boldsymbol{C}$ Lemma 10 applies again and we have

$$T'_a F' \subset \mathring{L}^1_{\overline{X}_a}(S, \mathscr{B}, \lambda) \subset C(S, E)^{\beta}_a.$$

To prove the conditional weak compactness of the set $\{\varphi_{y'}\}_{y'\in\sigma'} =: M$ in $L^1_{\overline{X}_a}(S, \mathscr{B}, \lambda)$ we apply Lemma 12. Clearly (i)–(ii) hold as before. As for (iii) note that with (28) and (23) we have

$$\int_S D_\delta \varphi_{y'}(t)\, d\lambda(t) = \int_S D_\delta u_{y'}(t)\, d\omega_{y'}(t) = v_\delta(\mu_{y',a}, S) \leqslant 4 D_\delta T_a$$

uniformly for $y' \in \sigma'$. Hence (A_1) implies (C_1).

LEMMA 13. *If F has no subspace isomorphic to c_0, then any $T \in M_{\mathrm{HP}}(C(S, E), F)$ satisfies the conditions* (A_i).

Proof. We first assume that S is compact and metric with metric d. We show that (A_3) is satisfied. Let K be a compact set in S and $x \in E_a$. Consider the sequence of open sets $G_n := \{d(t, K) < 1/n\}$, $n = 1, 2, \ldots$ and let $u_n := u(\overline{G_{n+1}}, G'_n)$. Then we have according to (21)

$$\begin{aligned}\langle U(K)x, y'\rangle &= \lim_{n\to\infty} \langle y', T(xu_n)\rangle \\ &= \lim_{n\to\infty} \sum_{k=2}^{n} \langle y', T(xu_k) - T(xu_{k-1})\rangle + \langle y', T(xu_1)\rangle.\end{aligned}$$

The series converges abolutely: We namely have

$$\{u_k(t) \neq u_{k-1}(t)\} \subset G_{k-1} - \overline{G_{k+1}}, \qquad k = 2, 3, \ldots$$

and if we denote by $\lambda_{y',a}$ the measure corresponding to $\mu_{y'}$ according to (12) then we have in view of the corollary of Lemma 8

$$\sum_{k=2}^{n} |\langle y', T(xu_k) - T(xu_{k-1})\rangle| \leqslant 2 \sum_{k=2}^{n} \lambda_{y',a}(G_{k-1} - \overline{G_{k+1}}) \leqslant 4\lambda_{y',a}(G_1 - K).$$

Because the spaces F with no subspace isomorphic to c_0 are characterized by the property that for every sequence $\{y_n\}_{n=1}^{\infty}$ of elements in F such that $\sum_{n=1}^{\infty} |\langle y', y_n\rangle| < \infty$ for all $y' \in F'$ there exists $y \in F$ such that $\langle y', y\rangle = \sum_{n=1}^{\infty} \langle y', y_n\rangle$, it follows that $U(K)x \in F$, $x \in E$ and that also, by Orlicz's Theorem, the second condition of (A_3) is satisfied. Now assume S is compact and Hausdorff. For $x \in E$ define

$$T_x f := T(xf), \qquad f \in C(S).$$

We have $T_x \in M_{\mathrm{HP}}(C(S), F)$. We show that T_x is weakly compact. Consider a sequence $\{f_n\}_{n=1}^{\infty}$ in $C(S)_a$. There exists a compact metric space $\hat{S}$, a sequence $\{\hat{f}_n\}_{n=1}^{\infty}$ in $C(\hat{S})_a$ and a transformation $\hat{T} \in M_{\mathrm{HP}}(C(\hat{S}), F)$ such that

$$T_x f_n = \hat{T}_x \hat{f}_n, \qquad n = 1, 2, \ldots$$

(the details are similar to the linear case ([15], p. 496)). By the part we have proven, $\hat{T}_x$ satisfies the condition (A_i) and is hence weakly compact by Theorem 7. Therefore T_x is weakly compact, for each K in S the net $\{T(xv(K, G')) = T_x(v(K, G')), G \in \pi(K)\}$ has a $\sigma(F, F')$-limit in F and $U(K)x \in F$. The second condition of (A_3) is satisfied as before, and so (A_3) holds also in this case.

Because (A_i) characterizes the weakly compact transformations in the case $E = \boldsymbol{C}$, we have

THEOREM 8. *If F has no subspace isomorphic to c_0, every transformation $T \in M_{\mathrm{HP}}(C(S), F)$ is weakly compact.*

THEOREM 9. *$C(S)$ has the strict Dunford–Pettis property also with respect to all transformations $T \in M_{\mathrm{HP}}(C(S), F)$.*

Proof. Our assertion says that any T of the class considered takes a weak Cauchy sequence $\{f_n\}_{n=1}^{\infty}$ in a strong Cauchy sequence. Indeed, the limits $\lim_{n\to\infty} f_n(t)$ exist for every $t \in S$ and there exists $a > 0$ with $\|f_n\| \leqslant a$, $n = 1, 2, \ldots$ (A_1) means that given $\varepsilon > 0$ there exists $\delta > 0$ such that $\lambda_a(B) < \delta$ implies $\sup_{y'\in\sigma'} v(\mu_{y',a}, B) < \varepsilon$. There exists $\eta > 0$ such that $sv_\eta(U_a, S) < \varepsilon$. According to Egoroff's Theorem there exists an index n_0 and a set $B_0 \in \mathscr{B}$ with $\lambda_a(B_0) < \delta$ such that

$$|f_n(t) - f_m(t)| < \eta, \qquad n, m \geqslant n_0,\ t \in B'_0.$$

It follows with (3)

$$\|Tf_n - Tf_m\| = \Big\| \int_{B_0} f_n dU - \int_{B_0} f_m dU + \Big(\int_{B_0'} f_n dU - \int_{B_0'} f_m dU\Big)\Big\| \leqslant 3\varepsilon$$

for $n, m \geqslant n_0$.

COROLLARY. *If R is an arbitrary weakly compact transformation from a Banach space E into $C(S)$ and T is a weakly compact transformation in $M_{\mathrm{HP}}(C(S), F)$ then $TR\colon E \to F$ is compact.*

The following theorem characterizes the compact transformations for $E = \boldsymbol{C}$.

THEOREM 10. *A transformation $T \in M_{\mathrm{HP}}(C(S), F)$ is a compact if and only if for every $\alpha > 0$ U_α takes its values in a compact set of $M_\alpha(\boldsymbol{C}, F)$.*

Proof. Let T be compact and $\alpha > 0$. The set $\{U(B)_\alpha\}_{B \in \mathscr{B}}$ is bounded and equicontinuous and by Theorem 5, the set $\{U(B)x, x \in E_\alpha\}$ is conditionally compact in F for each $B \in \mathscr{B}$. Hence $\{U(B_\alpha)\}_{B \in \mathscr{B}}$ is conditionally compact in $M_\alpha(\boldsymbol{C}, F)$, $\alpha > 0$. Conversely, given $\alpha > 0$, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $sv_\delta(U_\alpha, S) < \varepsilon/2$ and there are complex numbers $\zeta_1, \ldots, \zeta_N \in \boldsymbol{C}_\alpha$ with

$$\inf_{j=1,\ldots,N} |x - \zeta_j| < \delta, \qquad x \in \boldsymbol{C}_\alpha.$$

Our assumption implies that there are sets $A_1, \ldots, A_M \in \mathscr{B}$ such that

$$\inf_{k=1,\ldots,M} \|U(B)_\alpha - U(A_k)_\alpha\| < \varepsilon/2N, \qquad B \in \mathscr{B}$$

and it follows that the set $\{U(A_{k_1})\zeta_1 + \ldots + U(A_{k_N})\zeta_N, 1 \leqslant k_j \leqslant M, 1 \leqslant k_j \leqslant M, j = 1, \ldots, N\}$ is an ε-net for P_α, so that T is compact by Theorem 5.

6. Transformations $T\colon C(S) \to C(Q)$. We now specialize our results to the case that $E = \boldsymbol{C}$ and $F = C(Q)$ for a compact Hausdorff space Q.

THEOREM 11. A) *The class $M_{\mathrm{HP}}(C(S), C(Q))$ coincides with the class of all operators $T\colon C(S) \to C(Q)$ of the form*

(37) $$(Tf)(q) = \int_S f d\mu_q, \qquad f \in C(Q),\ q \in Q,$$

where

(a_1) *$\mu_q\colon \mathscr{B} \to M(\boldsymbol{C}, \boldsymbol{C})$ is additive for $q \in Q$,*

(a_2) *$\mu_{q,\alpha}\colon \mathscr{B} \to M_\alpha(C, C)$ is regular (and hence σ-additive) for $\alpha > 0$, $q \in Q$,*

(a_3) $\sup_{q \in Q} sv(\mu_{q,\alpha}, S) < \infty$ *and* $\lim_{\delta \to 0} \sup_{q \in Q} sv_\delta(\mu_{q,\alpha}, S) = 0$, $\alpha > 0$,

(a_4) *the integral in* (37) *is a continuous function in $q \in Q$ for each $f \in C(S)$.*

For each T it then follows that

$$\|T_\alpha\| = \sup_{q \in Q} sv(\mu_{q,\alpha}, S),$$

$$D_\delta T_\alpha = \sup_{q \in Q} sv_\delta(\mu_{q,\alpha}, S), \qquad \alpha, \delta > 0.$$

B) *The class of all weakly compact transformations in $M_{\mathrm{HP}}(C(S), C(Q))$ coincides with the class of all operators $T\colon C(S) \to C(Q)$ of the form* (37) *with* (a_1)–(a_3) *and*

(a_4)′ *$\mu_q(B)x$ is continuous in $q \in Q$ for each $x \in E$ and $B \in \mathscr{B}$, or of the form*

(38) $$(Tf)(q) = \int_S K_q(t, f(t))\, d\lambda(t), \qquad f \in C(S),$$

where

(b_1) *λ is a nonnegative regular Borel measure on S with $\lambda(S) < \infty$,*

(b_2) *for every $q \in Q$ we have $K_q(t)\colon x \to K_q(t, x)$ is an element of $M(\boldsymbol{C}, \boldsymbol{C})$ and $K_q(\cdot)_\alpha \in L^1_{M_\alpha(\boldsymbol{C},\boldsymbol{C})}(S, \mathscr{B}, \lambda)$ (except for at most a λ-null set), $\alpha > 0$,*

(b_3) $\sup_{q \in Q} \int_S \|K_q(t)_\alpha\|\, d\lambda(t) < \infty$ *and* $\lim_{\delta \to 0} \sup_{q \in Q} \int_S D_\delta K_q(t)_\alpha\, d\lambda(t) = 0$, $\alpha > 0$,

(b_4) $\int_B K_q(t, x)\, d\lambda(t)$ *is continuous in $q \in Q$ for all $B \in \mathscr{B}$, $x \in E$.*

C) The class of all compact transformations in $M_{\mathrm{HP}}(C(S), C(Q))$ coincides with the class of all operators $T\colon C(S) \to C(Q)$ of the form (37) with (a_1)–(a_3) and

(a_4)″ $\lim_{q' \to q} v(\mu_{q',\alpha} - \mu_{q,\alpha}, S) = 0$, $q \in Q$, $\alpha > 0$,

or of the form (38) *with* (b_1)–(b_3) *and*

(b_4)′ $\lim_{q' \to q} \int_S \|K_{q'}(t)_\alpha - K_q(t)_\alpha\|\, d\lambda(t) = 0$, $q \in Q$, $\alpha > 0$.

Proof. Let y_q' be the evaluation functional on $C(Q)$ at the point $q \in Q$.

A) For $T \in M_{\mathrm{HP}}(C(S), C(Q))$ the set function $\mu_q := \mu_{y_q'}$ has the desired properties by Theorem 2, the converse part follows from Theorem 1.

B) If $T \in M_{\mathrm{HP}}(C(S), C(Q))$ is weakly compact let U be the set function of Theorem 2 which represents T. According to Theorem 7, $U(B)x \in C(Q)$ for $B \in \mathscr{B}$, $x \in \boldsymbol{C}$, so that $\mu_q(B)x = \langle y_q', U(B)x\rangle = (U(B)x)(q)$ is a continuous function of $q \in Q$. Conversely, let T be given by (37) with (a_1)–(a_3) and (a_4). Define an additive set function $V\colon \mathscr{B} \to M(\boldsymbol{C}, C(Q))$ by $(V(B)x)(q) = \mu_q(B)x$ for $B \in \mathscr{B}$, $x \in \boldsymbol{C}$ and $q \in Q$. It follows from (a_3) that V is of bounded semivariation on S. Furthermore, the set function $V_{[x]}\colon \mathscr{B} \to C(Q)$ given by $V_{[x]}(B) = V(B)x$, $B \in \mathscr{B}$ for $x \in \boldsymbol{C}$ is (strongly)

σ-additive. In fact, for any sequence $\{B_j\}_{j=1}^{\infty}$ of disjoint sets in $\mathscr{B}$ we have by (a_2)

$$\lim_{n\to\infty}\sum_{j=1}^{n}\big(V_{[x]}(B_j)\big)(q) = \lim_{n\to\infty}\sum_{j=1}^{n}\mu_q(B_j)x = \mu_q\Big(\bigcup_{j=1}^{\infty}B_j\Big)x = V_{[x]}\Big(\bigcup_{j=1}^{\infty}B_j\Big)(q), \quad q\in Q,$$

which is equivalent to the weak σ-additivity of $V_{[x]}$. Therefore for any such sequence $\{B_j\}_{j=1}^{\infty}$ the sequence of operators $\{V(B_j)\}_{j=1}^{\infty}$ satisfies the assumptions of Lemma 11. It follows that

$$\lim_{n\to\infty}\sup_{y'\in\sigma'}\sum_{j=1}^{n}\big\|\big(y'\circ V(B_j)\big)_\alpha\big\| = 0, \quad \alpha > 0.$$

Hence for each $\alpha > 0$ the variations of the regular measures $\nu_{y'_q,\alpha}$ given by $\nu_{y'_q}(B)x = \langle y'_q, V(B)x\rangle = \mu_q(B)x$, $B\in\mathscr{B}$, $x\in C$ are σ-additive uniformly for $q\in Q$ and therefore absolutely continuous with respect to a nonnegative bounded regular measure on S. Therefore V_α is regular in the norm of $M_\alpha(C, C(Q))$ and V coincides with U of Theorem 2. Because U takes values in $M(C, C(Q))$ T is weakly compact by Theorem 7. The second part of B) follows from Lemma 10 and Theorem 7; λ is constructed as in (35). C) follows from a direct application of the Arzelá–Ascoli Theorem.

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