Contents of volume XLVIII, number 3

The journal STUDIA MATHEMATICA prints original papers in English, French, German and Russian, mainly on functional analysis, abstract methods of mathematical analysis and on the theory of probabilities. Usually 3 issues constitute a volume.

The papers submitted should be typed on one side only and accompanied by abstracts, normally not exceeding 200 words in length. The authors are requested to send two copies, one of them being the typed, not Xerox copy. Authors are advised to retain a copy of the paper submitted for publication.

Manuscripts and the correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA
ul. Śniadeckich 8
00-950 Warszawa, Poland

Correspondence concerning exchange should be addressed to:

Institute of Mathematics, Polish Academy of Sciences, Exchange,
ul. Śniadeckich 8, 00-950 Warszawa, Poland

The journal is available at your bookseller's or at

ARS POLONA - RUCH
Krakowskie Przedmieście 7,
00-068 Warszawa, Poland

WROCŁAWSKA DRUKARNIA NAUKOWA

On the orders of the extensions of linear functionals to distributions*

by

P. MANKIEWICZ (Warszawa)

Abstract. The problem considered is that of the orders of the extensions of linear functionals to distributions. It is proved that there exists a closed linear subspace $L \subset D(\Omega)$ such that the orders of the extensions to distributions of linear functionals defined on L do not depend on the orders of initial functionals on L.

1. Introduction. Let $D(\Omega)$ denote the space of all functions differentiable infinitely many times with compact supports contained in Ω, where Ω is an arbitrary fixed open subset of $\boldsymbol{R}^n$, $n \in N$. Let L be a linear subspace of $D(\Omega)$ and let φ be a linear functional defined on L. Assume that for every compact subset $K \subset \Omega$ there exist a non-negative integer $h(K)$ and a positive constant C such that the following estimation holds:

$$(1)\qquad |\varphi(f)| \leqslant C \sum_{p \leqslant h(K)} \sup\{|D^p f(t)|:\ t \in K\}$$

for every function f in L with $\operatorname{supp} f \subset K$. If this is the case, then we say that φ is on L of order less than or equal to $\{h(K)\}_{K\in\mathscr{K}}$ where $\mathscr{K}$ denotes the family of all compact subsets of Ω. Note that it is a well-known fact that every distribution is of order less than or equal to some $\{l(K)\}_{K\in\mathscr{K}}$ on $D(\Omega)$.

In the present note we study the following problem. Given a closed linear subspace $L \subset D(\Omega)$ satisfying the condition

(i) every linear functional defined on L of order less than or equal to $\{h(K)\}_{K\in\mathscr{K}}$ on L for some $\{h(K)\}_{K\in\mathscr{K}}$ admits an extension to a distribution.

Does this imply that the orders of the extensions to distributions depend "continuously" on the orders of the initial functionals (defined

* The note is a part of the author's doctoral thesis, which was prepared in the Institute of Mathematics of the Polish Academy of Sciences under the supervision of Prof. W. Słowikowski in 1969.

on L)? In general the answer is negative. In fact we prove the following result (Theorem 4.1).

There exists a closed linear subspace L of $D(\Omega)$ satisfying the condition (i) *such that for every pair $h(K), l(K) \in N_+^{\mathscr{K}}$, where N_+ denotes the set of all non-negative integers, there exists a linear functional φ defined on L of order less than or equal to $\{h(K)\}_{K\in\mathscr{K}}$ on L which does not admit an extension to a distibution of order less than or equal to $\{l(K)\}_{K\in\mathscr{K}}$ on $D(\Omega)$.*

A similar irregularity was observed by Słowikowski in [5], where he constructed an example of a closed linear subspace L of $D(\Omega)$ which did not satisfy condition (i). The same author in his further investigations [6] gave the necessary and sufficient condition for avoiding this irregularity (Słowikowski's Theorem 3.1, Section 3 in the present note). The main result of the present note was mentioned by the author earlier (without proof) in [3].

In this paper, in order to simplify the notations, we use a slightly modified definition of the order of functional (cf. (1) and (2)). For the same reason we consider only the one-dimensional case. However, our considerations remain valid in the general (n-dimensional) case.

2. Preliminaries and notations. Denote by $C(K)$, for every $K \subset \boldsymbol{R}$, the linear space of all complex-valued continuous functions defined on K. For every compact subset $K \subset \boldsymbol{R}$ satisfying the property $\overline{\operatorname{int} K} = K$ and every positive integer n define the linear space

$$C^n(K) = \{f \in C(K)\colon (D^p f)(t) \text{ exists and is uniformly continuous for } t \in \operatorname{int} K \text{ and } p \leqslant n\},$$

where $D^p f$ denotes the p-fold differentiation operator for $p = 0, 1, 2, \ldots$

Put

$$\|f\|_K = \sup\{|f(t)|\colon t \in K\}$$

for $f \in C(K)$ and

$$\|f\|_K^n = \sum_{p\leqslant n} \|\tilde{D}^p f\|_K$$

for $f \in C^n(K)$ and $n \in \boldsymbol{N}$, where $\tilde{D}^p f$ denotes the extension of $(D^p f)(t)$, $t \in \operatorname{int} K$, onto K. Observe that $(C^n(K), \|\cdot\|_K^n)$ is a Banach space for every $n \in \boldsymbol{N}$. Finally, define the linear space

$$\mathscr{E}(G) = \{f \in C(G)\colon D^p f \text{ exists and } \in C(G) \text{ for every } p \in \boldsymbol{N}\}$$

for every open subset $G \subset \boldsymbol{R}$.

Fix an arbitrary open non-empty subset $\Omega \subset \boldsymbol{R}$. Let

$$D(\Omega) = \{f \in \mathscr{E}(\Omega)\colon \operatorname{supp} f \text{ is a compact subset of } \Omega\}.$$

In the sequel we shall write D instead of $D(\Omega)$.

Choose the family $\{K_n\}_{n\in\boldsymbol{N}}$ of compact intervals satisfying the properties

1° $K_n \subset \Omega$ for $n \in \boldsymbol{N}$,

2° the family of sets $\{K_n\}_{n\in\boldsymbol{N}}$ is a covering of the set Ω,

3° the set $\operatorname{int}(K_m - \bigcup_{\substack{n\in\boldsymbol{N}\\ n\neq m}} K_n)$ is non-empty for every $m \in \boldsymbol{N}$.

Let $\mathfrak{N}$ be the set of all sequences of non-negative integers. We order $\mathfrak{N}$ setting for $\mathfrak{h}, \mathfrak{l} \in \mathfrak{N}$, $\mathfrak{h} = \{h_n\}_{n\in\boldsymbol{N}}$, $\mathfrak{l} = \{l_n\}_{n\in\boldsymbol{N}}$, $\mathfrak{l} \leqslant \mathfrak{h}$ if and only if $l_n \leqslant h_n$ for every positive integer n.

Take a linear subspace $L \subset D$. A linear functional φ defined on a subspace $L \subset D$ is said to be of order less than or equal to $\mathfrak{h}$, $\mathfrak{h} \in \mathfrak{N}$, $\mathfrak{h} = \{h_n\}_{n\in\boldsymbol{N}}$, if and only if there exists a sequence $\{C_n\}_{n\in\boldsymbol{N}}$ of positive constants such that the following inequality holds

$$|\varphi(f)| \leqslant C_n \sum_{0\leqslant p\leqslant h_n} \sup\{|D^p f|\colon x \in K_n\} \tag{2}$$

for every $n \in \boldsymbol{N}$ and every function $f \in L$ vanishing off K_n. In other words, φ is of order less than or equal to $\mathfrak{h}$ if and only if

$$|\varphi(f)| \leqslant C_n \|f\|_{K_n}^{h_n}$$

for every positive integer n and every function $f \in L$ vanishing off K_n. Notice that if $L = D$, then the definition above coincides with the classical definition of the order of distribution.

With every $\mathfrak{k} = \{k_n\}_{n\in\boldsymbol{N}} \in \mathfrak{N}$ we associate the space

$$D^{\mathfrak{k}} = \{f \in C(\Omega)\colon \operatorname{supp} f \text{ is a compact subset of } \Omega \text{ and } f/_{K_n} \in C^{k_n}(K_n) \text{ for every } n \in \boldsymbol{N}\},$$

where $f/_K$ denotes the restriction of the function f to the set K. Put

$$\|f\|^{\mathfrak{k}} = \sum_{n=1}^{\infty} \|f\|_{K_n}^{k_n} \tag{3}$$

for every $f \in D^{\mathfrak{k}}$. It can easily be seen that the covering $\{K_n\}_{n\in\boldsymbol{N}}$ of the set Ω is locally finite. Hence the sum on the right side of (3) is finite for every $f \in D^{\mathfrak{k}}$.

Define the spaces

$$D_n^{\mathfrak{k}} = \{f \in D^{\mathfrak{k}}\colon \operatorname{supp} f \subset \bigcup_{i=1}^{n} K_n\}$$

for every $\mathfrak{k} \in \mathfrak{N}$ and $n \in \boldsymbol{N}$, and the norms

$$\|f\|_n^{\mathfrak{k}} = \sum_{i=1}^{n} \|f\|_{K_i}^{k_i} = \|f\|^{\mathfrak{k}}$$

for every $f \in D_n^{\mathfrak{f}}$. In this manner with every $\mathfrak{f} \in \mathfrak{N}$ the sequence $\{(D_n^{\mathfrak{f}}, \|\cdot\|_n^{\mathfrak{f}})\}_{n \in \boldsymbol{N}}$ of Banach spaces is associated. In each $D^{\mathfrak{f}}$, $\mathfrak{f} \in \mathfrak{N}$ we introduce the topology $\tau_{\mathfrak{f}}$, which is the inductive limit topology generated by the identical embeddings of the spaces $D_n^{\mathfrak{f}}$ into $D^{\mathfrak{f}}$ and the topologies induced by the norms $\|\cdot\|_n^{\mathfrak{f}}$ in $D_n^{\mathfrak{f}}$, for $n \in N$. It can easily be proved that $(D^{\mathfrak{h}}, \tau_{\mathfrak{h}})$ is an LF-space for every $\mathfrak{h} \in \mathfrak{N}$. Moreover

$$D = \bigcap_{\mathfrak{h} \in \mathfrak{N}} D^{\mathfrak{h}}.$$

Hence we introduce in D the projective limit topology τ_D generated by the canonical embeddings of D into $D^{\mathfrak{h}}$ and the topologies $\tau_{\mathfrak{h}}$ for every $\mathfrak{h} \in \mathfrak{N}$. It can be proved that the topology τ_D is the standard topology on $D = D(\Omega)$, which means that τ_D is the locally convex topology of the uniform convergence with all derivatives on compact supports. The base of neighbourhoods of the origin for the topology τ_D consists of the sets $U_{\tau_D}(\mathfrak{h}, \varepsilon, \{C_n\}_{n \in \boldsymbol{N}})$ of the form

$$U_{\tau_D}(\mathfrak{h}, \varepsilon, \{C_n\}_{n \in \boldsymbol{N}}) = \Big\{f \in D\colon \sum_{n=1}^{\infty} C_n \sum_{p \leqslant h_n} \sup\{|D^{p+1}f|\colon t \in K_n\} < \varepsilon\Big\}, \tag{4}$$

where $\mathfrak{h} = \{h_n\}_{n \in \boldsymbol{N}}$ is an arbitrary sequence in $\mathfrak{N}$, ε is an arbitrary positive number and $\{C_n\}_{n \in \boldsymbol{N}}$ is an arbitrary sequence of positive numbers (cf. [8]).

It follows from (4) that a linear functional φ defined on D is a continuous linear functional on (D, τ_D), i.e., a distribution if and only if there exists an $\mathfrak{h} \in \mathfrak{N}$ such that φ is a continuous linear functional on $(D, \tau_{\mathfrak{h}}/_D)$, where $\tau_{\mathfrak{h}}/_D$ denotes the topology $\tau_{\mathfrak{h}}$ restricted to D. Observe that $D^{\mathfrak{l}} \subset D^{\mathfrak{h}}$ for every $\mathfrak{l}, \mathfrak{h} \in \mathfrak{N}$ and $\mathfrak{l} \geqslant \mathfrak{h}$. In addition the topology $\tau_{\mathfrak{l}}$ restricted to D is finer than the topology $\tau_{\mathfrak{h}}$ restricted to D. Hence if φ is a continuous linear functional defined on a subspace L of $(D^{\mathfrak{h}}, \tau_{\mathfrak{h}})$, then it is a continuous linear functional in $(D^{\mathfrak{l}}, \tau_{\mathfrak{l}})$ for every $\mathfrak{l} \in \mathfrak{N}$, $\mathfrak{l} \geqslant \mathfrak{h}$.

Given a locally convex linear space (X, τ), a linear functional φ defined on a subspace $L \subset X$ is said to be a sequentially continuous linear functional on L if and only if

$$\varphi(x_n) \to \varphi(x_0)$$

for every sequence $\{x_n\}_{n \geqslant 0} \subset L$ with $x_n \to x_0$ $(n \to \infty)$ in the topology τ.

It is a well-known fact that if (X, τ) is an LF-space then a linear functional φ defined on X is continuous if and only if it is a sequentially continuous linear functional on X. The same statement for linear functionals defined on a subspace of an LF-space in general is not true [5], [4].

The following easily verifiable remark shows us that the notion of the sequentially continuous functionals is a rather useful tool in investigations of orders of linear functionals on D.

Remark 2.1. Given a linear functional φ defined on a linear subspace $L \subset D$, the following statements are equivalent (cf. 5.2 of [6]):

(i) φ is of order less than or equal to $\mathfrak{h}$, $\mathfrak{h} \in \mathfrak{N}$,

(ii) φ is a sequentially continuous linear functional defined on a subspace L of $(D^{\mathfrak{h}}, \tau_{\mathfrak{h}})$.

On the other hand, consider the linear space

$$D_n = \Big\{f \in D\colon \operatorname{supp} f \subset \bigcup_{i=1}^{n} K_i\Big\}$$

for every positive integer n, endowed with the topology τ_n of uniform convergence with all derivatives (the topology τ_n is induced by the system of pseudonorms $\{\|\cdot\|_{K_i}^p\colon p \in N \text{ and } i = 1, 2, \ldots, n\}$). Observe that

$$D = \bigcup_{n=1}^{\infty} D_n.$$

It can be proved that the inductive limit topology generated by the identical embeddings of D_n into D and the topologies τ_n, $n \in \boldsymbol{N}$, coincides with the topology τ_D. Since the space (D_n, τ_n) is a Fréchet space, we infer that (D, τ_D) is an LF-space. Hence a linear functional φ defined on D is a distribution (is a τ_D-continuous functional) if and only if $\varphi/_{D_n}$ is a continuous functional on (D_n, τ_n) for every $n \in \boldsymbol{N}$. Finally we observe that a sequence $\{f_n\}_{n \in \boldsymbol{N}}$ of functions in D is convergent to a function $f_0 \in D$ in the topology τ_D if and only if

(α) there exists a compact subset $K \subset \Omega$ such that $\operatorname{supp} f_i \subset K$ for $i = 0, 1, 2, \ldots$,

(β) $D^p f_i \to D^p f_0$ uniformly $(i \to \infty)$ for every $p = 0, 1, \ldots$

In similar manner we can describe the property that a sequence $\{f_n\}_{n \in \boldsymbol{N}}$ of functions in $D^{\mathfrak{h}}$ is convergent to a function $f_0 \in D^{\mathfrak{h}}$ in the topology $\tau_{\mathfrak{h}}$.

We complete this section with a theorem which summarizes the facts given above.

THEOREM 2.2. *Given a linear functional φ defined on D, the following conditions are equivalent:*

(i) *φ is a distribution,*

(ii) *φ is a continuous functional in (D, τ_D),*

(iii) *$\varphi/_{D_n}$ is a continuous functional in (D_n, τ_n) for every $n \in \boldsymbol{N}$,*

(iv) *φ is a sequentially continuous linear functional on (D, τ_D),*

(v) *there exists an $\mathfrak{h} \in \mathfrak{N}$ such that φ is a continuous functional in $(D^{\mathfrak{h}}, \tau_{\mathfrak{h}})$.*

3. Słowikowski's theorem. Given: an open subset $\Omega \subset \boldsymbol{R}$ and a family of compact intervals satisfying the conditions 1°–3° of the previous section. To formulate the theorem of Słowikowski [6] we need the following spaces:

$$D^{\mathfrak{h},m} = \{f\in D^{\mathfrak{h}}\colon f/_{\Omega-\bigcup_{i=1}^{m} K_i}\in\mathscr{E}(\Omega-\bigcup_{i=1}^{m} K_i)\}$$

for $\mathfrak{h}\in\mathfrak{N}$ and $m\in\mathbf{N}$, and

$$D_n^{\mathfrak{h},m} = \{f\in D^{\mathfrak{h},m}\colon \operatorname{supp} f\subset\bigcup_{i=1}^{n} K_i\}$$

for $\mathfrak{h}\in\mathfrak{N}$ and $n, m\in N$. In each space $D_n^{\mathfrak{h},m}$ we introduce the locally convex topology $\tau_n^{\mathfrak{h},m}$ of simultaneous uniform convergence with all derivatives off $\bigcup_{i=1}^{m} K_i$ and convergence with respect to the norm $\|\cdot\|^{\mathfrak{h}}$. Thus we obtain a Fréchet space $(D_n^{\mathfrak{h},m}, \tau_n^{\mathfrak{h},m})$ for every $\mathfrak{h}\in\mathfrak{N}$ and $n, m\in\mathbf{N}$.

Given a locally convex linear vector space (X, τ), following [6] we call a linear subspace $L\subset X$ *well located* in (X, τ) if and only if every sequentially continuous linear functional defined on L admits an extension to a sequentially continuous linear functional defined on (X, τ). In view of Theorem 2.2 we infer that a subspace L of D is well located in (D, τ_D) if and only if every sequentially continuous linear functional on L is continuous in the locally convex topology induced on L by the topology τ_D.

In the sequel we shall use the following theorem (Theorem 2.1 of [6]).

THEOREM 3.1. *A linear subspace $L\subset D$ is well located in (D, τ_D) if and only if to every $\mathfrak{h}_1\in\mathfrak{N}$ there corresponds an $\mathfrak{h}_2\in\mathfrak{N}$ such that to every $k_1\in\mathbf{N}$ there corresponds a $k_2\in\mathbf{N}$ such that for every $p\in\mathbf{N}$ the following inclusion holds:*

$$D_{k_1}^{\mathfrak{h}_1}\cap\mathrm{cl}_{\tau_p^{\mathfrak{h}_2,k_1}}(L\cap D_p)\subset\mathrm{cl}_{\|\cdot\|_{k_2}^{\mathfrak{h}_1}}(L\cap D_{k_2}),$$

where $\mathrm{cl}_{\tau_p^{\mathfrak{h}_2,k_1}}(L\cap D_p)$ denotes the closure of $L\cap D_p$ in the topology $\tau_p^{\mathfrak{h}_2,k_1}$ and $\mathrm{cl}_{\|\cdot\|_{k_2}^{\mathfrak{h}_1}}(L\cap D_{k_2})$ denotes the closure of $L\cap D_{k_2}$ in the topology induced by the norm $\|\cdot\|_{k_2}^{\mathfrak{h}_1}$.

4. Main result. Given: an open subset $\Omega\subset\mathbf{R}$ and a family of compact intervals $\{K_n\}_{n\in\mathbf{N}}$ satisfying the conditions $1^\circ-3^\circ$ of Section 2.

THEOREM 4.1. *There exists a linear subspace $L\subset D(\Omega) = D$ closed in (D, τ_D) such that*

(i) *every linear functional defined on L and sequentially continuous in (D, τ_D) admits an extension on the whole D to a distribution:*

(ii) *for every pair $\mathfrak{k}, \mathfrak{l}\in\mathfrak{N}$, $\mathfrak{k}\geqslant\mathfrak{l}$, there exists a linear functional defined on L and sequentially continuous in $(D^{\mathfrak{l}}, \tau_{\mathfrak{l}})$ which does not admit an extension to a linear functional defined on D which is sequentially continuous in $(D^{\mathfrak{k}}, \tau_{\mathfrak{k}})$.*

Note that point (i) of the thorem states that L is well located in (D, τ_D). Hence Theorem 4.1 can be reformulated as follows:

THEOREM 4.2. *There exists a linear subspace $L\subset D$, closed in (D, τ_D) such that*

(i) *L is well located in (D, τ_D),*

(ii) *if $\mathfrak{l}\in\mathfrak{N}$ then L is not well located in $(D^{\mathfrak{l}}, \tau_{\mathfrak{l}})$. Moreover, for every pair $\mathfrak{l}, \mathfrak{k}\in\mathfrak{N}$, $\mathfrak{l}\leqslant\mathfrak{k}$, there exists a linear functional defined on L of order less than or equal to $\mathfrak{l}$ which does not admit an extension to a distribution of order less than or equal to $\mathfrak{k}$.*

To prove the theorem we start with the

Construction of the linear subspace $L\subset D$. Let $\{K_{1,m}^n\}_{n,m\in\mathbf{N}}$ be a family of closed intervals satisfying the conditions:

(i) $K_{1,m}^n\subset\operatorname{int}(K_1-\bigcup_{i=2}^{\infty} K_i)$ for $n, m = 1, 2, \ldots,$

(ii) $K_{1,m}^n\cap K_{1,q}^p = \emptyset$ for $(n, m)\neq(p, q)$,

(iii) $\operatorname{int}K_{1,m}^n\neq\emptyset$ for $n, m = 1, 2, \ldots$

Next, with every $n\geqslant 2$ we construct a sequence $\{K_{n,m}\}_{m\in\mathbf{N}}$ of closed, mutally disjoint intervals with non-empty interior such that

$$K_{n,m}\subset\operatorname{int}(K_n-\bigcup_{\substack{i\in\mathbf{N}\\ i\neq n}} K_i)\quad\text{for } m\in\mathbf{N} \text{ and } n\geqslant 2.$$

The existence of such families of intervals follows from the conditions $1^\circ-3^\circ$ satisfied by the covering $\{K_n\}_{n\in\mathbf{N}}$.

To every $n\geqslant 2$ and $m\in\mathbf{N}$ we assign a point $t_{n,m}\in\operatorname{int}K_{n,m}$ and a sequence $\{g_{n,m}^k\}_{k\in\mathbf{N}}$ of functions in D such that $\operatorname{supp} g_{n,m}^k\subset K_{n,m}$ for every $k\in\mathbf{N}$ and that the sequence $\{g_{n,m}^k\}_{k\in\mathbf{N}}$ tends to zero uniformly with the derivatives up to the order m while

$$(D^{m+1}g_{n,m}^k)(t_{n,m}) = 1\quad\text{for } k = 1, 2, \ldots$$

Finally, to every $n\geqslant 2$ and $m\in\mathbf{N}$ we assign a function $f_{1,m}^n$ in D with $\operatorname{supp} f_{1,m}^n\subset K_{1,m}^n$, not identically equal to zero.

Put

$$h_{n,m}^k = f_{1,m}^n + g_{n,m}^k\quad\text{for } m, k\in\mathbf{N} \text{ and } n\geqslant 2,$$

and let

$$L' = \operatorname{Lin}\{h_{n,m}^k\colon m, k\in\mathbf{N} \text{ and } n\geqslant 2\}.$$

Define

$$L = \mathrm{cl}_{\tau_D}(L)$$

where $\mathrm{cl}_{\tau_D}(L)$ denotes the closure of L in (D, τ_D). The set L has the required properties. Indeed, by the definition L is a closed subspace of

(D, τ_D). In order to verify the second part of Theorem 4.1 (or Theorem 4.2) we need the following lemmas.

LEMMA 4.3. *If* $f \in L$ *and* $\operatorname{supp} f \subset \bigcup_{i=1}^{n} K_i$, *then* $\operatorname{supp} f \cap K_{1,q}^r = \emptyset$ *for* $n < r$ *and* $q = 1, 2, \ldots$

Proof. Let $f \in L$ and $\operatorname{supp} f \subset \bigcup_{i=1}^{n} K_i$. Suppose that $\operatorname{supp} f \cap K_{1,q}^r \neq \emptyset$ for some r and q, where $r > n$ and $q \in \boldsymbol{N}$. Then

$$\sup\{|f|:\ t \in K_{1,q}^r\} > 0.$$

The set L is the closure in the topology τ_D of the set L'. Hence, for every neighbourhood $U(f)$ of the function f in (D, τ_D) of the form

$$U(f) = \Big\{f' \in D:\ \sum_{i=1}^{\infty} N_i \sum_{p \leqslant p_i} \sup\{|D^p(f-f')|:\ t \in K_i\} < \varepsilon\Big\},$$

where $\{p_i\}_{i \in \boldsymbol{N}}$ is an arbitrary fixed element of $\mathfrak{N}$ and $\{N_i\}_{i \in \boldsymbol{N}}$ is an arbitrary fixed sequence of positive numbers, we have

$$U(f) \cap L' = \emptyset.$$

In particular, the statement above remains true if we put $\mathfrak{p} = \{p_i\}_{i \in \boldsymbol{N}} \in \mathfrak{N}$, $p_i = q+1$ for $i = 1, 2, \ldots$ and $N_i = 1$ for $i \in \boldsymbol{N}$. Denote by $U(f)$ the neighbourhood which corresponds to the $\mathfrak{p}$ so defined and the sequence $\{N_i\}_{i \in \boldsymbol{N}}$. Let $f' \in U(f) \cap L'$. It follows from the definition of the set L' that f' can be represented as a finite linear combination of the functions $\{h_{n,m}^k:\ m, k \in \boldsymbol{N}, n \geqslant 2\}$. Let

$$f' = \sum_{k,i,m} a_{i,m}^k h_{i,m}^k.$$

Since, according to the assumptions, $\operatorname{supp} f \cap K_r = \emptyset$ and $\operatorname{supp} h_{i,m}^k \cap K_r = \emptyset$ for $i \neq r$, $i \in \boldsymbol{N}$, we have

$$\begin{aligned}
\varepsilon &> \sum_{i=1}^{\infty} \sum_{p \leqslant p_i} \sup\Big\{\Big|D^p\Big(f - \sum_{k,i,m} a_{i,m}^k h_{i,m}^k\Big)\Big|:\ t \in K_i\Big\} \\
&\geqslant \sum_{p \leqslant p_r} \sup\Big\{\Big|D^p\Big(f - \sum_{k,i,m} a_{i,m}^k h_{i,m}^k\Big)\Big|:\ t \in K_r\Big\} \\
&\geqslant \sum_{p \leqslant q+1} \sup\Big\{\Big|D^p\Big(\sum_{k,m} a_{r,m}^k h_{r,m}^k\Big)\Big|:\ t \in K_r\Big\}.
\end{aligned}$$

Since $t_{r,q} \in K_r$, we have

$$\varepsilon > \Big|D^{q+1}\Big(\sum_{k,m} a_{r,m}^k h_{r,m}^k\Big)(t_{r,q})\Big|.$$

Next, observe that the functions $h_{r,m}^k$ identically vanish in some neighbourhood of $t_{r,q}$ for $m \neq q$. Hence

$$\varepsilon > \Big|D^{q+1}\Big(\sum_k a_{r,q}^k h_{r,q}^k\Big)(t_{r,q})\Big|.$$

Finally, since $(D^{q+1} h_{r,q}^k)(t_{r,q}) = 1$ for every $k \in \boldsymbol{N}$, we obtain

$$\varepsilon > \Big|\sum_k a_{r,q}^k\Big|. \tag{5}$$

On the other hand, since $K_{1,q}^r \subset K_1$ and $h_{i,m}^k$ identically vanish on $K_{1,q}^r$ for $(i, m) \neq (r, q)$ and $h_{r,q}^k$ restricted to $K_{1,q}^r$ is equal to $f_{1,q}^r$ for every $k \in \boldsymbol{N}$, we infer that

$$\begin{aligned}
\varepsilon &> \sum_{i=1}^{\infty} \sum_{p \leqslant p_i} \sup\Big\{\Big|D^p\Big(f - \sum_{k,i,m} a_{i,m}^k h_{i,m}^k\Big)\Big|:\ t \in K_i\Big\} \\
&\geqslant \sup\Big\{\Big|f - \sum_{k,i,m} a_{i,m}^k h_{i,m}^k\Big|:\ t \in K_1\Big\} \\
&\geqslant \sup\Big\{\Big|f - \sum_{k,i,m} a_{i,m}^k h_{i,m}^k\Big|:\ t \in K_{1,q}^r\Big\} \\
&= \sup\Big\{\Big|f - \Big(\sum_k a_{r,q}^k\Big) f_{1,q}^r\Big|:\ t \in K_{1,q}^r\Big\} \\
&\geqslant \sup\{|f|:\ t \in K_{1,q}^r\} - \Big|\sum_k a_{r,q}^k\Big| \cdot \sup\{|f_{1,q}^r|:\ t \in K_{1,q}^r\}.
\end{aligned}$$

Hence we have

$$\varepsilon > \sup\{|f|:\ t \in K_{1,q}^r\} - \Big|\sum_k a_{r,q}^k\Big| \cdot \sup\{|f_{1,q}^r|:\ t \in K_{1,q}^r\}.$$

Thus by (5) we obtain

$$\varepsilon(1 + \sup\{|f_{1,q}^r|:\ t \in K_{1,q}^r\}) > \sup\{|f|:\ t \in K_{1,q}^r\}.$$

With ε tending to zero we prove that f identically vanishes on $K_{1,q}^r$. Hence, if $\operatorname{supp} f \subset \bigcup_{i=1}^{n} K_i$ then $\operatorname{supp} f \cap K_{1,q}^r = \emptyset$ for $r > n$ and $q = 1, 2, \ldots$ which concludes the proof of the lemma.

It follows from Lemma 4.3 that $f_{1,m}^n \notin L$ for $m \in \boldsymbol{N}$ and $n \geqslant 2$.

LEMMA 4.4. *For every positive integer* n *the set* $L' \cap D_n$ *is dense in* $L \cap D_n$ *(with respect to the topology* τ_D*).*

Proof. Let $f \in L \cap D_n$ and let $U_l(f)$, for every positive integer l, denote the neighbourhood of f in (D, τ_D) as in the proof of the previous lemma generated by $\varepsilon = 1/l$ and $\mathfrak{p} = \{p_i\}_{i \in \boldsymbol{N}} \in \mathfrak{N}$, $p_i = l$ for $i \in \boldsymbol{N}$ and $N_i = 1$ for

$i \in N$. Next, let $f_l \in U_l(f) \cap L'$ for $l \in N$. In the same manner as in the proof of Lemma 4.3, for every $l \in N$ f_l can be represented as a finite linear combination of the functions $\{h^k_{r,m}\colon m, k \in N, r \geqslant 2\}$. Let

$$f_l = \sum_{i,m,k} a^{k,l}_{i,m} h^k_{i,m}$$

for $l \in N$. By Lemma 4.3 we have $\operatorname{supp} f \cap \operatorname{supp} h^k_{i,m} = \emptyset$ for $i > n$. Hence $\tilde{f}_l \in U_l(f) \cap L'$, where

$$\tilde{f}_l = \sum_{i \leqslant n} \sum_{m,k} a^{k,l}_{i,m} h^k_{i,m}.$$

Indeed, since $\operatorname{supp} f \subset \bigcup_{i=1}^{n} K_i$, we infer that

$$\begin{aligned}\frac{1}{l} &> \sum_{s=1}^{\infty} \sum_{p \leqslant l} \sup\{|D^p(f-f_l)|\colon t \in K_s\} \\ &\geqslant \sum_{s=1}^{n} \sum_{p \leqslant l} \sup\Big\{\Big|D^p\Big(f - \sum_{i=1}^{n} \sum_{m,k} a^{k,l}_{i,m} h^k_{i,m}\Big)\Big|\colon t \in K_s\Big\} \\ &\geqslant \sum_{s=1}^{n} \sum_{p \leqslant l} \sup\{|D^p(f-\tilde{f}_l)|\colon t \in K_s\}.\end{aligned}$$

Finally,

$$\operatorname{supp} \tilde{f}_l \subset \bigcup_{i=1}^{n} K_i \quad \text{for every } l \in N$$

and

$$\sup\{|D^p(f-\tilde{f}_l)|\colon t \in \bigcup_{i=1}^{n} K_i\} < \frac{1}{l} \quad \text{for } p \leqslant 1.$$

Hence the sequence $\{\tilde{f}_l\}_{l \in N}$ is convergent uniformly with all derivatives to the function f (is convergent in (D, τ_D)), which concludes the proof of the lemma.

LEMMA 4.5. *For every $\mathfrak{k} \in \mathfrak{N}$, $\mathfrak{k} = \{k_i\}_{i \in N}$ and for an arbitrary sequence $\{a_i\}_{i \geqslant 2}$ of complex numbers there exists a uniquely determined linear functional φ defined on L such that φ is of order less than or equal to $\mathfrak{h}$, for every $\mathfrak{h} \in \mathfrak{N}$, and satisfies the conditions*

(i) $\varphi(h^k_{n,k_n}) = a_n$, *for every* $k \in N$, $n \geqslant 2$,

(ii) $\varphi(h^k_{p,q}) = 0$, *for* $q \neq k_p$, $p \geqslant 2$, $k \in N$.

Proof. Given $\mathfrak{k} \in \mathfrak{N}$ and the sequence $\{a_i\}_{i \geqslant 2}$ of complex numbers, we shall prove the lemma by constructing the functional φ satisfying the desired conditions. First we set the required values of the functional φ

on the functions $\{h^k_{n,m}\colon m, k \in N, n \geqslant 2\}$. Next we extend the functional φ on to the whole L', putting for $f \in L'$,

$$f = \sum_{n,m,k} a^k_{n,m} h^k_{n,m},$$

$$\varphi(f) = \sum_{n,m,k} a^k_{n,m} \varphi(h^k_{n,m}) = \sum_{n} \Big(\sum_{k} a^k_{n,k_n}\Big) a_n.$$

Such an extension is well defined. Indeed, let

$$0 = \sum_{n,m,k} a^k_{n,m} h^k_{n,m} = \sum_{n,m} \Big(\sum_{k} a^k_{n,m}(f^n_{1,m} + g^k_{n,m})\Big).$$

Since the supports of the functions $\{f^n_{1,m}\colon m \in N, n \geqslant 2\}$ are disjoint with the supports of the functions $\{g^k_{n,m}\colon m, k \in N, n \geqslant 2\}$, we infer that

$$0 = \sum_{n,m} \Big(\sum_{k} a^k_{n,m}\Big) f^n_{1,m}.$$

Next, since the supports of the functions $\{f^n_{1,m}\colon m \in N, n \geqslant 2\}$ are mutally disjoint, we obtain

$$\sum_{k} a^k_{n,m} = 0 \quad \text{for } m \in N, n \geqslant 2.$$

This implies that for every function $f \in L'$ the corresponding numbers $\sum_k a^k_{n,m}$ are uniquely determined by the function f.

Observe that $\varphi/_{L' \cap D_n}$ is a bounded linear functional with respect to the norm $\|\cdot\|^{\mathfrak{l}_0}_n$, for every $n \in N$, where $\mathfrak{l}_0 = \{l_i\}_{i \in N}$, $l_i = 0$ for $i \in N$. Indeed, let $\{M_i\}_{i \in N}$ be a sequence of positive numbers satisfying the inequality

$$0 \leqslant |a_i| \leqslant M_i \sup\{|f^i_{1,k_i}|\colon t \in K_1\}$$

for $i \in N$. Hence, for every $f \in L' \cap D_n$, $f = \sum_{i \leqslant n} \sum_{m,k} a^k_{i,m} h^k_{i,m}$ we have

$$\begin{aligned}|\varphi(f)| &= \Big|\varphi\Big(\sum_{i \leqslant n} \sum_{m,k} a^k_{i,m} h^k_{i,m}\Big)\Big| \\ &= \Big|\sum_{i \leqslant n} \varphi\Big(\sum_{k} a^k_{i,k_i} h^k_{i,k_i}\Big)\Big| \leqslant \sum_{i \leqslant n} \Big|\varphi\Big(\sum_{k} a^k_{i,k_i} h^k_{i,k_i}\Big)\Big| \\ &\leqslant \sum_{i \leqslant n} \Big|\sum_{k} a^k_{i,k_i}\Big| |a_i| \leqslant \sum_{i \leqslant n} \Big|\sum_{k} a^k_{i,k_i}\Big| M_i \sup\{|f^i_{1,k_i}|\colon t \in K_1\} \\ &\leqslant \operatorname{Max}\{M_i\colon i \leqslant n\} \sum_{i \leqslant n} \Big|\sum_{k} a^k_{i,k_i}\Big| \sup\{|f^i_{1,k_i}| \, t \in K_1\} \\ &= \operatorname{Max}\{M_i\colon i \leqslant n\} \sum_{i \leqslant n} \sup\Big\{\Big|\sum_{k} a^k_{i,k_i} f^i_{1,k_i}\Big|\colon t \in K_1\Big\} \\ &\leqslant n \operatorname{Max}\{M_i\colon i \leqslant n\} \sup\{|f|\colon t \in \bigcup_{i \leqslant n} K_i\} \\ &= n \operatorname{Max}\{M_i\colon i \leqslant n\} \|f\|^{\mathfrak{l}_0}_n.\end{aligned}$$

By the Banach–Hahn theorem $\varphi/_{L'\cap D_n}$ admits a norm-preserving extension to a continuous linear functional in $(L\cap D_n, \|\cdot\|_n^{\mathfrak{l}_0})$ defined on the whole $L\cap D_n$. For every $n\in N$, let $\tilde\varphi_n$ be an extension of $\varphi/_{L'\cap D_n}$ to a $\|\cdot\|_n^{\mathfrak{l}_0}$-continuous linear functional on $L\cap D_n$. Since, by the previous lemma, $L'\cap D_n$ is sequentially dense in $L\cap D_n$ in the topology τ_D, which is finer on $L\cap D_n$ than the topology induced on $L\cap D_n$ by the norm $\|\cdot\|_n^{\mathfrak{l}_0}$, we conclude that such an extension is uniquely determined. Moreover, we have

$$\tilde\varphi_n = \tilde\varphi_{m/L\cap D_n} \quad \text{for } n\leqslant m.$$

Finally, we define

$$\varphi(f) = \tilde\varphi_n(f)$$

for $f\in L$ with $\operatorname{supp} f\in \bigcup_{i=1}^{n} K_i$. Observe that $\varphi/_{L\cap D_n} = \tilde\varphi_n$ is a continuous linear functional on $(L\cap D_n, \|\cdot\|_n^{\mathfrak{l}_0})$. Hence φ is of order less than or equal to $\mathfrak{l}_0$. Since $\mathfrak{l}_0$ is the minimal element of $\mathfrak{N}$, we infer that φ is of order less than or equal to $\mathfrak{h}$ for every $\mathfrak{h}\in\mathfrak{N}$, which concludes the proof of the lemma.

Having Lemma 4.5 we can prove that L satisfies the condition (ii) of Theorem 4.2.

Proof. Given $k=\{k_n\}_{n\in N}\in N$, set

$$a_n = n\sum_{p\leqslant n}\sup\{|D^p f^n_{1,k_n}|: t\in K_1\} \quad \text{for } n\in N,$$

and let φ be the functional from the previous lemma. It has been proved that φ is of the order less than or equal to $\mathfrak{l}$ for every $l\in N$. Suppose that φ is extensible to a distribution of order less than or equal to $\mathfrak{k}$. Let $\tilde\varphi$ be such extension. This means that $\tilde\varphi$ is a continuous linear functional on $(D, \tau_{\mathfrak{k}/D})$. It follows from the definition of the functions $\{h^k_{n,k_n}\}_{n,k\in N}$ that

$$h^k_{n,k_n}\to f^n_{1,k_n} \quad (k\to\infty)$$

with respect to the topology $\tau_{\mathfrak{k}}$. It follows from the continuity of the functional $\tilde\varphi$ in the topology $\tau_{\mathfrak{k}}$ that

$$\tilde\varphi(f^n_{1,k_n}) = \lim_{k\to\infty}\tilde\varphi(h^k_{n,k_n}) = n\sum_{p\leqslant n}\sup\{|D^pf^n_{1,k_n}|: t\in K_1\}.$$

But, on the other hand, $\tilde\varphi$ is a distribution and this implies that there exist constants C and q such that

$$|\tilde\varphi(f)|\leqslant C\sum_{p\leqslant q}\sup\{|D^pf|: t\in K_1\}$$

for every $f\in D$ with $\operatorname{supp} f\in K_1$. Thus we get a contradiction with the evaluation of $\tilde\varphi$ obtained above, which concludes the proof of Theorem

4.2 (ii). Nevertheless, φ can be extended to a distribution of order less than or equal to $\mathfrak{k}_1 = \{k_{1,n}\}_{n\in N}$, where $k_{1,n}=k_n+1$ for $n\in N$. Indeed, such extension can be explicitly written as

$$\tilde\varphi = \sum_{n=1}^{\infty}\Big(n\sum_{p\leqslant n}\sup\{|D^pf^n_{1,k_n}|: t\in K_1\}\,\delta^{k_n+1}(t_{n,k_n})\Big),$$

where $\delta^n(t)$ denotes the distribution which is the evalution at the point t of the n-order derivative.

In order to prove the good location of L in (D,τ_D) we need the following sequence of lemmas.

LEMMA 4.6. *If $f\in D^{\mathfrak{l}}_k\cap \operatorname{cl}_{\tau_p^{\mathfrak{l},k}}(L\cap D_p)$ for $\mathfrak{l}\in\mathfrak{N}$, $p,k\in N$, then*

$$\operatorname{supp} f\cap K^r_{1,q} = \emptyset \quad \textit{for } r>k \textit{ and } q\in N.$$

Proof. Let $f\in D^{\mathfrak{l}}_k\cap\operatorname{cl}_{\tau_p^{\mathfrak{l},k}}(L\cap D_p)$. First we consider the case where $r>p$. By Lemma 4.3 we infer that $\operatorname{supp} f'\cap K^r_{1,q}=\emptyset$ for every $f'\in L\cap D_p$ and $r>p$, $q\in N$. Hence

$$0=\inf\big\{\sup\{|f-f'|: t\in K^r_{1,q}\}: f'\in L\cap D_p\big\} = \sup\{|f|: t\in K^r_{1,q}\}.$$

If this is not the case, then $p\geqslant r>k$ and we proceed as follows. Given $\varepsilon>0$, let $U(0)$ be a neighbourhood of the origin in the topology $\tau^{\mathfrak{l},k}_p$ of the form

$$U(0)=\Big\{f\in D^{\mathfrak{l},k}_p: \sup\Big\{|D^{q+1}f|: t\in\Omega-\bigcup_{i=1}^{k} K_i\Big\}<\varepsilon \text{ and } \sup\{|f|: t\in\Omega\}<\varepsilon\Big\}.$$

If $f\in D^{\mathfrak{l}}_k\cap \operatorname{cl}_{\tau_p^{\mathfrak{l},k}}(L\cap D_p)$, then there exists an $f'\in L\cap D_p$ such that $(f-f')\in U(0)$. By Lemma 4.4 $L'\cap D_p$ is dense in $L\cap D_p$ in the topology τ_D. This implies that $L'\cap D_p$ is dense in $L\cap D_p$ in the topology $\tau^{\mathfrak{l},k}_p$. Hence without any loss of generality we can assume that $f'\in L'\cap D_p$ and

$$f'=\sum_{n\leqslant p}\sum_{m,s}a^s_{n,m}h^s_{n,m}.$$

Thus we have

$$\sup\Big\{\Big|D^{q+1}\Big(f-\sum_{n\leqslant p}\sum_{m,s}a^s_{n,m}h^s_{n,m}\Big)\Big|: t\in\Omega-\bigcup_{i=1}^{k} K_i\Big\}<\varepsilon.$$

Since f vanish identically on $\Omega-\bigcup_{i=1}^{k} K_i$ and $t_{r,q}\in\Omega-\bigcup_{i=1}^{k} K_i$, we have

$$\sup\Big\{\Big|D^{q+1}\Big(\sum_{n\leqslant p}\sum_{m,s}a^s_{n,m}h^s_{n,m}\Big)\Big|: t\in\Omega-\bigcup_{i=1}^{k} K_i\Big\} \geqslant \Big|\Big(D^{q+1}\Big(\sum_{n\leqslant p}\sum_{m,s}a^s_{n,m}h^s_{n,m}\Big)\Big)(t_{r,q})\Big|.$$

Next, in the same manner as in the proof of Lemma 4.3, since $(D^{q+1}h^s_{r,q})(r_{r,q}) = 1$ for every $s \in \boldsymbol{N}$ and $(D^{q+1}h^s_{n,m})(t_{r,q}) = 0$ for $(n, m) \neq (r, q)$ and every $s \in \boldsymbol{N}$, we infer that

$$\Big|\sum_s a^s_{r,q}\Big| < \varepsilon. \tag{6}$$

On the other hand,

$$\varepsilon > \sup\{|f-f'|:\ t \in \Omega\} \geqslant \sup\{|f-f'|:\ t \in K^r_{1,q}\}.$$

In the same manner as in the proof of Lemma 4.3, we obtain

$$\varepsilon > \sup\{|f|:\ t \in K^r_{1,q}\} - \Big|\sum_s a^s_{r,q}\Big| \sup\{|f^r_{1,q}|:\ t \in K^r_{1,q}\}. \tag{7}$$

The inequalities (6) and (7) imply

$$\varepsilon(1+\sup\{|f^r_{1,q}|:\ t \in K^r_{1,q}\}) > \sup\{|f|:\ t \in K^r_{1,q}\}.$$

Since ε has been an arbitrary positive number, the last inequality implies that $\sup\{|f|:\ t \in K^r_{1,q}\} = 0$. Hence $\operatorname{supp} f \cap K^r_{1,q} = \emptyset$, which concludes the proof of the lemma.

LEMMA 4.7. *$L' \cap D_k$ is a dense subset of $D^{\mathfrak{l}}_k \cap \mathrm{cl}_{\tau^{\mathfrak{l},k}_p}(L \cap D_p)$ in the topology $\tau^{\mathfrak{l},k}_p$ for every $\mathfrak{l} \in \mathfrak{N}$ and $k, p \in \boldsymbol{N}$.*

Proof. Let $f \in D^{\mathfrak{l}}_k \cap \mathrm{cl}_{\tau^{\mathfrak{l},k}_p}(L \cap D_p)$ and let $U_n(f)$ be a neighbourhood of f in the topology $\tau^{\mathfrak{l},k}_p$ of the form

$$U_n(f) = \Big\{f' \in D^{\mathfrak{l},k}_p:\ \|f-f'\|^{\mathfrak{l}} < \frac{1}{n}\Big\}$$

for every $n \in \boldsymbol{N}$. By Lemma 4.4 for every $n \in \boldsymbol{N}$ there exists an $f_n \in L' \cap D_p$ such that $f_n \in U_n(f)$. Let

$$f_n = \sum_{r \leqslant p} \sum_{m,s} a^{s,n}_{r,m} h^s_{r,m}$$

for every $n \in \boldsymbol{N}$. By the previous lemma since $\operatorname{supp} f \subset \bigcup_{i=1}^{k} K_i$ we have $\operatorname{supp} f \cap K^r_{1,q} = \emptyset$ for $r > k$. Hence the support of the function f is disjoint with the supports of the functions $\{h^s_{r,m}:\ m, s \in \boldsymbol{N},\ r > k\}$. This implies that

$$\frac{1}{n} > \|f-f_n\|^{\mathfrak{l}} = \Big\|f - \sum_{r \leqslant p} \sum_{m,s} a^{s,n}_{r,m} h^s_{r,m}\Big\|^{\mathfrak{l}} \geqslant \Big\|f - \sum_{r \leqslant k} \sum_{m,s} a^{s,n}_{r,m} h^s_{r,m}\Big\|^{\mathfrak{l}}.$$

Put $\tilde{f}_n = \sum_{r \leqslant k} \sum_{m,s} a^{s,n}_{r,m} h^s_{r,m}$ (in the case where $p < k$ we set, in addition, $a^{s,n}_{r,m} = 0$ for $n, s, m \in \boldsymbol{N}$ and $p < r \leqslant k$). Observe that $\operatorname{supp} \tilde{f}_n \subset \bigcup_{i=1}^{k} K_i$. Hence $\tilde{f}_n \in L' \cap D^{\mathfrak{l}}_k$ for every $n \in \boldsymbol{N}$ and

$$\frac{1}{n} > \|f-\tilde{f}_n\|^{\mathfrak{l}} = \|f-\tilde{f}_n\|^{\mathfrak{l}}_k.$$

This means that the sequence $\{\tilde{f}_n\}_{n \in \boldsymbol{N}}$ converges to f in $(D^{\mathfrak{l}}_k, \|\cdot\|^{\mathfrak{l}}_k)$. Hence $\{\tilde{f}_n\}_{n \in \boldsymbol{N}}$ converges to f in $(D^{\mathfrak{l},k}_p, \tau^{\mathfrak{l},k}_p)$. This follows from the fact that all the function $\{\tilde{f}_n\}_{n \in \boldsymbol{N}}$ vanish off $\bigcup_{i=1}^{k} K_i$.

We proved that if $f \in D^{\mathfrak{l}}_k \cap \mathrm{cl}_{\tau^{\mathfrak{l},k}_p}(L \cup D_p)$ then there exists a sequence of functions $\{\tilde{f}_n\}_{n \in \boldsymbol{N}}$ in $L' \cap D_k$ which converges to f in the topology $\tau^{\mathfrak{l},k}_p$, which concludes the proof of the lemma.

LEMMA 4.8. *The following inclusion holds:*

$$D^{\mathfrak{l}}_k \cap \mathrm{cl}_{\tau^{\mathfrak{l},k}_p}(L \cap D_p) \subset \mathrm{cl}_{\|\cdot\|^{\mathfrak{l}}_k}(L \cap D_k)$$

for every $\mathfrak{l} \in \mathfrak{N}$ and $k, p \in \boldsymbol{N}$.

Proof. By Lemma 4.7, $L' \cap D_k$ is a dense subset of $D^{\mathfrak{l}}_k \cap \mathrm{cl}_{\tau^{\mathfrak{l},k}_p}(L \cap D_p)$ in the topology $\tau^{\mathfrak{l},k}_p$. Since the topology $\tau^{\mathfrak{l},k}_p$ coincides on $D^{\mathfrak{l}}_k$ with the topology induced by the norm $\|\ \|^{\mathfrak{l}}_k$, we have

$$D^{\mathfrak{l}}_k \cap \mathrm{cl}_{\tau^{\mathfrak{l},k}_p}(L \cap D_p) \subset \mathrm{cl}_{\tau^{\mathfrak{l},k}_p}(L' \cap D_k) = \mathrm{cl}_{\|\cdot\|^{\mathfrak{l}}_k}(L' \cap D_k) \subset \mathrm{cl}_{\|\cdot\|^{\mathfrak{l}}_k}(L \cap D_k),$$

which concludes the proof of the lemma.

Proof that L satisfies condition (i) of Theorem 4.2. In order to prove the good location of L it is enough to observe that by the previous lemma the assumptions of Słowikowski's Theorem are fulfiled. Indeed, given $\mathfrak{h}_1 \in \mathfrak{N}$ and $k_1 \in \boldsymbol{N}$, it suffices, by the previous lemma, to set $\mathfrak{h}_2 = \mathfrak{h}_1$ and $k_2 = k_1$ and apply Słowikowski's Theorem. Hence L is well located in (D, τ_D), which concludes the proof of Theorem 4.2 (i).

Remark 4.9. Note that Theorem 4.2 (and Theorem 4.1) remains true if we change the definition of the order of linear functional and define the order as a natural-valued function defined on the family of all compact subsets of Ω (as has been done in the Introduction).

References

[1] M. De Wilde, *Quelques théoremes d'extension de fontionelles lineaires*, Bull. Soc. R. D. Liége, 35 (1966), pp. 552–557.

[2] L. Hörmander, *Linear partial differential operators*, Berlin 1963.

[3] P. Mankiewicz, *On orders of solutions of linear equations in distributions*, Bull. de l'Acad. Pol. des Sci., 17 (1969), pp. 25–28.

[4] P. Mankiewicz, *On the extension of sequentially continuous functionals in LF-spaces*, to appear in Bull. de l'Acad. Pol. des Sci.
[5] W. Słowikowski, *Fontionelles linéaires dans des réunions dénomerables d'espaces de Banach reflexifs*, C. R. Acad. Sc. Paris, 262 (1966), A 870–A 872.
[6] — *Extensions of sequentially continuous linear functionals in inductive sequences of F-spaces*, Studia Math., 26 (1966), pp. 193–221.
[7] — *Epimorfism of adjoint to generalized LF-spaces*, Lecture Notes, 1966, Aarhus University.
[8] J. M. Horváth, *Topological vector spaces and distributions*, I, 1966.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
INSTYTUT MATEMATYCZNY, POLSKA AKADEMIA NAUK

Received April 4, 1972 (516)

STUDIA MATHEMATICA T. XLVIII. (1973)

Eigenschaften von Schauder Basen und Reflexivität

von

ULRICH MERTINS (Karlsruhe)

Zusammenfassung. Für lokalkonvexe Räume E mit einer Schauder Basis werden Sätze vom folgenden Typ bewiesen: E ist reflexiv genau dann, wenn jede Schauder Basis von E eine gewisse Eigenschaft erfüllt.

1. Einleitung. Es bezeichne E einen lokalkonvexen Raum mit einer Schauder Basis $\{x_i\}$. Ist $\{f_i\} \dot{\subset} E'$ die Folge der zugehörigen Koeffizientenfunktionale, so sei die Basis auch durch $\{x_i, f_i\}$ gekennzeichnet.

Der Zusammenhang von Reflexivität des Raumes E und Eigenschaften der Basis $\{x_i, f_i\}$ ist in zahlreichen Untersuchungen (etwa in [1]–[6], [12], [16], [19] und [21]) erörtert. Zwei Basiseigenschaften spielen hierbei eine Hauptrolle: Eine Schauder Basis $\{x_i, f_i\}$ heißt fallend (engl. shrinking), wenn $\{f_i\}$ eine Schauder Basis für E'_b (Dual E' versehen mit der starken Topologie $\beta(E', E)$) ist. Die Basis heißt beschränkt vollständig (engl. boundedly complete), wenn für eine skalare Folge $\{a_i\}$ aus der Beschränktheit der Folge $\{\sum_{i\leqslant n} a_i x_i\}$ ihre Konvergenz in E folgt.

Zunächst zeigte James [5] für (B)-Räume, dann Retherford [16] für tonnelierte und schließlich Cook [1] für beliebige lokalkonvexe Räume, daß E genau dann halb-reflexiv ist, wenn die Basis $\{x_i, f_i\}$ fallend und beschränkt vollständig ist.

Die vorliegende Arbeit befaßt sich mit der Frage, ob ein tonnelierter Raum reflexiv ist, wenn alle seine Schauder Basen fallend bzw. beschränkt vollständig sind. Dabei werden Ergebnisse von Kalton [6] verbessert, der seinerseits Untersuchungen von Singer [19], Zippin [21] und Retherford [16] weitergeführt hat. Dies wird erreicht analog dem Vorgehen von Kalton durch eine Abschwächung der Begriffe "fallend" und "beschränkt vollständig" (Definitionen 2 und 3 in Abschnitt 2). Die hier gewählte Form der Abschwächung ist jedoch der Fragestellung und den Gegebenheiten von Basen in lokalkonvexen Räumen angepaßter als die von Kalton und liefert daher auch weiter reichende Ergebnisse. Zwar ist auch hier die Frage nicht für den allgemeinsten Fall beantwortet; jedoch lassen die offen bleibenden Probleme (A) und (B) in Abschnitt 2 vermuten,