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Differentiation through starlike sets in R^m

by

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Abstract. The well known theorems of differentation of a multiple integral with respect to a family of bounded convex sets are generalized to the case of unbounded sets.

0. Introduction. The purpose of this paper is to extend to the case of unbounded sets the well known theorems concerning differentiation of multiple integrals with respect to cubes or bounded convex sets.

The problem can be stated in the following way:

Consider a measurable starlike set B in R^m, that is if $x \in B$, then $\alpha x \in B$ for all α, $(0 \leqslant \alpha \leqslant 1)$, such that $|B| < \infty$. If we denote by $\varphi(x)$ the characteristic function of B, then the B_ε, $\varepsilon > 0$, will be the set whose characteristic function is $\varphi(\varepsilon^{-1}x) = \varphi(\varepsilon^{-1}x_1, \ldots, \varepsilon^{-1}x_m)$. The problem is then to study the limit

$$\lim_{\varepsilon\to 0}(1/|B_\varepsilon|)\int_{x+B_\varepsilon} f\,dt \tag{0.1}$$

for $f \in L^p(R^m)$, $p \geqslant 1$.

This problem has its origin in the Theorems 6 and 7 of [1] and our result implies those results.

The main references are [1] and [3].

In this opportunity we would like to thank Professor A. Zygmund for a useful conversation on the problem while this paper was in preparation.

1. Statement of the main results.

1.1. Theorem A. *Suppose that B_ε are under the conditions stated in 0., then:*

i) $\lim_{\varepsilon\to\infty}(1/|B|_\varepsilon)\int_{x+B_\varepsilon} f(t)\,dt = f(x)$ *a.e. if* $f \in L^p(R^m)$, $1 < p \leqslant \infty$.

Furthermore, we have for

$$f^* = \sup_{\varepsilon>0}\left|(1/|B_\varepsilon|)\int_{x+B_\varepsilon} f(t)\,dt\right|$$

the following inequality

ii) $\|f^*\|_p < C_p \|f\|_p$, $p > 1$.

If f and $|f|\log^+|f|$ belong to L^1 we have also in this case

iii) $\lim\limits_{s\to 0} (1/|B_s|) \int\limits_{x+B_s} f\,dt = f(x)$ *a.e.*

1.2. Definition. Let B be a starlike set. We say that $\varphi(a)$, where $a \in \Sigma$ (unit ball of $\boldsymbol{R}^m$), is the boundary function of B, if $\varphi(a) = \sup\{r/(r, a) \in B\}$. Clearly $|B| < \infty$ is equivalent to

$$\int\limits_{\Sigma} \varphi(a)^m d\sigma < \infty \tag{1.2.1}$$

where Σ denotes the unit ball of $\boldsymbol{R}^m$ and $d\sigma$ the "area" element.

1.3. We are going to use a special type of exterior measure. This, will help us to state results for the L^1 case. Our space will be the unit sphere of $\boldsymbol{R}^m$. Let E be a Lebesgue measurable set, $E \subset \Sigma$ and let us denote by G any σ-elementary set $\bigcup\limits_1^\infty I_k$, where the I_k are "cubes" in Σ. Given a fixed real number β, such that $0 < \beta < 1$, we define:

$$\gamma_\beta(E) = \inf_{G \supset E} \sum_1^\infty |I_k|^\beta. \tag{1.3.1.}$$

It is clear that γ_β is well defined for any subset of Σ and furthermore if $E_1 \subset E_2$, then:

$$\gamma_\beta(E_1) \leqslant \gamma_\beta(E_2). \tag{1.3.2}$$

1.4. We say that $\varphi \in L^p_{\gamma_\beta}(\Sigma)$, where φ is Lebesgue measurable and non-negative, $0 < p < \infty$, if the following integral is convergent

$$-\int\limits_0^\infty y^p d\gamma_\beta\{\varphi > y\}. \tag{1.4.1}$$

The above integral exists in the Riemann–Stieltjes sense because of (1.3.2).

1.5. Theorem B. *Suppose that the boundary function φ of the set B belongs to $L^{n\beta}_{\gamma_\beta}(\Sigma)$ for some β, $0 < \beta < 1$. Then if μ is a measure defined on $\boldsymbol{R}^m$ having bounded total variation there, we have:*

i) $\lim\limits_{s\to 0} \dfrac{1}{|B_s|} \int\limits_{x+B_s} d\mu$ *exists a.e and equals a.e the density function of μ with respect to the Lebesgue measure.*

ii) *If* $0 < \beta \leqslant 1/2$ *then* $|\{x; \mu^* > \lambda\}| < \dfrac{C}{\lambda} W(\mu)$ *where* $\mu^*(x) = \sup\limits_{s>0} \dfrac{1}{|B_s|} \int\limits_{+B_s} d\mu|$, *$W(\mu)$ denotes the total variation of μ over $\boldsymbol{R}^m$ and C is a constant independent of μ.*

1.6. Now we are going to deal with another type of conditions concerning the shape and the size of the basic set B.

We shall assume first that the boundary function φ is dominated by a finite sum of functions $\varphi_j(a)$ verifying the following properties

(1.6.1)
(a) $\varphi \leqslant \sum\limits_{Y=1}^N \varphi_j(a)$, $\varphi_j(a) \geqslant 0$, $j = 1, 2, \ldots, N$,
(b) $\varphi_j(a) = \varphi_j(|a - a_j|)$, $j = 1, 2, \ldots N$, where $|a - a_j|$ denotes the "chordal" distance from the point a to the point a_j,
(c) $\varphi_j(t)$ is a non-increasing function of the real variable $t > 0$, $j = 1, 2, \ldots N$.

1.7. Theorem C. *Suppose that the boundary function φ of the set B is under the conditions of 1.6, and suppose further that the functions $\varphi_j(a)$ verify the following estimates*

$$\int\limits_{\Sigma} \varphi_j^n(a) \log^+\log^+\varphi_j(a)\,d\sigma < \infty, \quad j = 1, 2, \ldots N \tag{1.7.1}$$

then if μ is a measure defined on $\boldsymbol{R}^m$ and having bounded total variation there we have

i) $\lim\limits_{s\to 0} \dfrac{1}{|B_s|} \int\limits_{x+B_s} d\mu$ *exists a.e and equals a.e the density function of μ with respect to the Lebesgue measure.*

If instead of the conditions (1.7.1) *we have the stronger following ones*

$$\int\limits_{\Sigma} \varphi_j^n \log^+\varphi_j (\log^+\log^+\varphi_j)^{1+\delta} d\sigma < \infty \tag{1.7.2}$$

for some $\delta > 0$ and $j = 1, 2, \ldots, N$. Then, the following weak type estimate holds truth

ii) $|\{x; \mu^* > \lambda\}| < \dfrac{C}{\lambda} W(\mu)$.

2. Auxiliary Lemmas.

2.1. Lemma. *Let $S \subset \boldsymbol{R}^m$ be a bounded subset of $\boldsymbol{R}^m$ and suppose that for each point $x \in S$ there exists an m-dimensional parallelepiped centered at the point x, such that:* 1) *Their edges are parallel to a fixed mutually orthogonal directions in $\boldsymbol{R}^m$.* 2) *The length of the edges is given respectively by m functions $\psi_1(t), \psi_2(t), \ldots, \psi_m(t)$ of the single parameter t, where each one of the $\psi_j(t)$ is a monotone, continuous and non-decreasing function of the variable t verifying $\psi_j(0) = 0$. The length of the edge parallel to the j-th direction is measured by $\psi_j(t)$. Then it is possible to select an at most denumerable family R_k of such parallelepipeds, such that*

i) $R_j \cap R_i = \varphi$ *if* $i \neq j$,

ii) $U_k 5R_k \supset S$, *where* $5R_k$ *denotes the dilatation five times of* R_k *about its center.*

2.2. LEMMA. *If the parallelepipeds are under the conditions of the preceding lemma, then if* μ *is a* σ*-additive measure defined on the Borel subsets of* $\boldsymbol{R}^m$, *having bounded variation there, we have*

i) $\mu^*(x) = \sup\limits_{t>0} \Big| \dfrac{1}{R(t)} \int\limits_{x+R(t)} d\mu \Big|$ *verifies*

$$|\{\mu^*(x) > \lambda\}| < (5^m+1)/\lambda \cdot W(\boldsymbol{R}^m)$$

where $W(\boldsymbol{R}^m)$ *denotes the total variation of* μ *over* $\boldsymbol{R}^m$.

ii) $\|f^*(x)\|_p < C_p \|f\|_p$, $1 < p \leqslant \infty$, *where* C_p *depends only on* p *and on the dimension.*

Clearly, Lemma 2.1 implies Lemma 2.2 and this is done in [3], Vol. II, p. 309 except for the fact that a particular coordinate system is selected.

2.3. LEMMA. *Let* $\{T_k\}$ *be a sequence of sublinear operators mapping* σ*-additive measures defined on the Borel subsets of* $\boldsymbol{R}^m$, *having bounded variation there, into measurable functions of* $\boldsymbol{R}^m \to \boldsymbol{R}$. *Suppose that the following weak type inequality is verified:*

i) $\big|E\big(|T_k(\mu)| > \lambda\big)\big| < (C/\lambda) W(R^m)$ *where as before* $W(\boldsymbol{R}^m)$ *denotes the total variation of* μ *over* $\boldsymbol{R}^m$ *and the constant* C *does not depend on* μ *or* T_k.

Let a_k *be a sequence of real numbers and call* $T = \sum\limits_1^\infty a_k T_k$. *Then if* $\sum |a_k|^{1/2} < \infty$ *we have:*

$$|E\{T(\mu)| > \lambda\}| < \frac{C}{\lambda} \Big(\sum_k a_k^{1/2}\Big)^2 \cdot W(\boldsymbol{R}^m). \tag{2.3.1}$$

ii) *If* $\sum |a_k| \cdot |\log |a_{k'}|| < \infty$ *we have for any measurable set* A *such that* $1 \leqslant |A| < \infty$

$$\big|\{E\big(|T(\mu)| > \lambda\big) \cap A\}\big| < 7C(1+|\log C|)\cdot(1+\log|A|)\big(\lambda/(1+|\log\lambda|)\big)^{-1} \times \tag{2.3.2}$$
$$\times \Big(\sum |a_k|\,\big(1+\big|\log|a_k|\big|\big)\cdot W(\boldsymbol{R}^m)\big(1+|\log W(\boldsymbol{R}^m)|\big)\Big).$$

Proof. Except for the value of the right hand constant in (2.3.1), this is done in [2], pp. 121–122 and its verification is straightforward. So we have to prove inequality (2.3.2) only. Let's take the measure μ and define the following sets:

$$X_k(\lambda) = \{|T_k(\mu)| > \lambda/|a_k|\}, \tag{2.3.3}$$

then

$$|X_k(\lambda)| < (C/\lambda) W(\boldsymbol{R}^m). \tag{2.3.4}$$

If $X(\lambda) = \bigcup\limits_1^\infty X_k(\lambda)$ then:

$$|X(\lambda)| < (C/\lambda)\cdot\Big(\sum_1^\infty |a_k|\Big) W(\boldsymbol{R}^m). \tag{2.3.5}$$

Consider a measurable set A such that $|A| < \infty$. Call $D_k(s)$ to the distribution function of $|T_k(\mu)|$ on $\boldsymbol{R}^m$ $\big(\boldsymbol{R}^m - X(\lambda)\big) \cap A = A_\lambda$. According to i) we have:

$$D_k(s) \leqslant \min\big(|A|, (C/s) W(\boldsymbol{R}^m)\big) \tag{2.3.6}$$

that is

$$\begin{cases} D_k(s) \leqslant |A| \ \text{ if } 0 \leqslant s \leqslant (C/|A|) W(\boldsymbol{R}^m), \\ D_k(s) \leqslant (C/s) W(\boldsymbol{R}^m) \ \text{ if } (C/|A|) W(\boldsymbol{R}^m) < s \leqslant (\lambda/|a_k|), \\ D_k(s) = 0 \ \text{ if } s > (\lambda/|a_k|) \quad \text{since } A_\lambda \subset \boldsymbol{R}^m - X(\lambda). \end{cases} \tag{2.3.7}$$

Now we have the following inequality

$$\int\limits_{A_\lambda} |T(\mu)|\,dx \leqslant \sum_1^\infty |a_k| \int\limits_{A_\lambda} |T_k(\mu)|\,dx = \sum_1^\infty |a_k| \int\limits_0^\infty D_k(s)\,ds. \tag{2.3.8}$$

On the other hand, according to (2.3.7) it follows:

$$\int\limits_0^\infty D_k(s)\,ds \leqslant CW(\boldsymbol{R}^m) + C' W(\boldsymbol{R}^m) \int\limits_{C\frac{W(\boldsymbol{R}^m)}{|A|}}^{\lambda/|a_k|} (1/s)\,dx \tag{2.3.9}$$

$$\leqslant CW(\boldsymbol{R}^m) + CW(\boldsymbol{R}^m)\big\{|\log\lambda| + \big|\log|a_{k'}|\big| + |\log C| + |\log W(\boldsymbol{R}^m)| + \big|\log|A|\big|\big\}.$$

Therefore:

$$\int\limits_{A_\lambda} |T(\mu)|\,dx \tag{2.3.10}$$
$$\leqslant \Big(\sum_1^\infty |a_k|\Big)\cdot CW(\boldsymbol{R}^m)\cdot\{1+|\log\lambda| + |\log C| + |\log W(\boldsymbol{R}^m)| + \log|A|\} +$$
$$+ CW(\boldsymbol{R}^m)\Big(\sum_1^\infty |a_k|\big|\log|a_k|\big|\Big)$$
$$\leqslant 6(1+\log|A|)\,C(1+|\log C|)\,(1+|\log\lambda|) W(\boldsymbol{R}^m)\,\big(1+|\log W(\boldsymbol{R}^m)|\big) \times$$
$$\times \Big(\sum_1^\infty |a_k|\,\big(1+\big|\log|a_k|\big|\big)\Big).$$

Thus:

$$\big|\{|T(\mu)| > \lambda\} \cap A\big| \leqslant \big|\{|T(\mu)| > \lambda\} \cap A_\lambda\big| + |X(\lambda)| \tag{2.3.11}$$
$$\leqslant 7(1+\log|A|)\,C(1+|\log C|) \times$$
$$\times \big\{\Sigma |a_k|\,\big(1+\big|\log|a_k|\big|\big)\big\}\big(\lambda/(1+|\log\lambda|)\big)^{-1}\cdot W(\boldsymbol{R}^m)\cdot\big(1+|\log W(\boldsymbol{R}^m)|\big)$$

and the lemma is proved.

2.4. LEMMA. *Let* $\{T_k(t)(f)(x)\}$ *be a sequence of differentiation operators through one parameter parallelepipeds, each one of them under the assumptions of Lemma 2.2. Let* a_k *be a sequence of real numbers such that* $\Sigma |a_k|\,\big|\log|a_{k'}|\big|$

$< \infty$. *Call* $T(t)[f](x) = \Sigma a_k T_k(t)[f](x)$. *If* μ *is a* σ*-additive measure defined on the Borel subsets of* $\boldsymbol{R}^m$*, having bounded total variation there and singular with respect to the Lebesgue measure, then:*

(2.4.1) $$\lim_{t\to 0} T(t)[\mu](x) = 0 \quad \text{a.e. in } \boldsymbol{R}^m.$$

Proof. Without loss of generality we may assume that $\mu \geqslant 0$. Let us take a cube Q such that $|Q| = 1$. Let $C_0 = 14(5^m+1)\cdot(1+\log(5^m+1))$. Let us take $\varepsilon > 0$, $\delta > 0$ and $k_0(\varepsilon, \delta)$ such that

(2.4.2) $$(1+|\log\varepsilon|)\varepsilon^{-1}C_0\mu(R^m)\sum_{k_0}^{\infty}|a_k|\big(1+|\log|_k||\big) < \delta.$$

On the other hand:

(2.4.3) $$\sup_{0<t<\infty}\Big|\sum_{k_0}^{\infty}(a_k)T_k(t)[\mu](x)\Big| \leqslant \Big|\sum_{k_0}^{\infty}|a_k|\sup_{0<t<\infty}[\mu](x)\Big| = \mu_{k_0}^*(x).$$

According to Lemmas (2.2) and (2.3) we have:

(2.4.4)
$$|Q\cap\{|\mu_{k_0}^*(x) > \varepsilon\}| < (1+|\log\varepsilon|)\varepsilon^{-1}C_0\mu(\boldsymbol{R}^m)\cdot\sum_{k_0}^{\infty}|a_k|\big(1+|\log|a_k||\big) < \delta.$$

Now

(2.4.5) $$|T(t)[\mu](x)| \leqslant \Big|\sum_{1}^{k_0}a_kT_k(t)[\mu](x)\Big| + \mu_{k_0}^*(x).$$

Since $\lim_{t\to 0} T_k(t)[\mu](x) = 0$ a.e. in $\boldsymbol{R}^m$ for each k, then

(2.4.6) $$\overline{\lim_{t\to 0}}|T(t)[\mu](x)| \leqslant \mu_k^*(x) \leqslant \varepsilon$$

except for a set whose measure is less than $\delta > 0$ in Q. Since we can select $\varepsilon > 0$ and $\delta > 0$ to be arbitrary, the lemma is proved.

2.5. Lemma. *Let* $K(x)$ *be a measurable function in* $\boldsymbol{R}^m$*, such that* $K \in L^1(\boldsymbol{R}^m)$ *and* $\int_{\boldsymbol{R}^m} K(x)dx = 1$. *Call* $K_\varepsilon = \varepsilon^{-m}K(\varepsilon^{-1}x)$. *Then*

(2.5.1) $$\lim_{\varepsilon\to 0} K_\varepsilon f = f \quad \text{a.e. in } \boldsymbol{R}^m$$

provided that f *belongs to* L^∞.

Proof. Consider $(K_\varepsilon * f - f)$ and fix $\delta > 0$.

(2.5.2) $$(K*f-f)(x) = \varepsilon^{-m}\int K(\varepsilon^{-1}y)\big(f(x-y)-f(x)\big)dy.$$

Now let us split K into $K_1+K_2 = K$ such that

(2.5.3) $$|K_1| \leqslant M \text{ and } K_1 = 0 \text{ if } |x| > L, \qquad \|K_2\|_1 < (1/2)\delta\|f\|_\infty^{-1}.$$

As it is readily seen, the right hand member of (2.5.2) is dominated by

(2.5.4) $$M\varepsilon^{-m}\int_{|y|<\varepsilon L}|f(x-y)-f(x)|dy + 2\|f\|_\infty\varepsilon^{-m}\int|K_2(\varepsilon^{-1}y)|dy.$$

Therefore

$$\overline{\lim_{\varepsilon\to 0}}|K_\varepsilon * f - f| < \delta$$

provided that x is a strong Lebesgue point of $f(y)$. Since $\delta > 0$ is chosen to be arbitrary, (2.5.1) holds.

3. Proof of Theorem A.

3.1. The condition $|B| < \infty$ is equivalent to the following condition on the boundary function $\varphi(\alpha)$

(3.1.1) $$\int_{\Sigma}\varphi^m(\alpha)d\sigma < \infty.$$

It is possible to find a family $\{Q_k\}$ of "cubes" on Σ and $\Psi_k(\alpha)$ the corresponding characteristic functions such that

(3.1.2) $$\varphi^m(\alpha) \leqslant \sum_{k=1}^{\infty}C_k^m\cdot\Psi_k(\alpha) \quad \text{a.e. in } \Sigma.$$

Denoting by $A(Q_k)$ the "area" of the cube on Σ verifying the following further condition

(3.1.3) $$\sum_k C_k^m A(Q_k) \leqslant 2\int_{\Sigma}\varphi^m(\alpha)d\sigma.$$

Consider now the parallelepiped R_k in $\boldsymbol{R}^m$, whose main "axis" goes from the origin 0 to the center C_k of the "cube" Q_k and whose intersection with Σ is Q_k. The length of the main axis is $2C_k$, where R_k is symmetric about the origin.

Clearly, the following relation is fulfilled:

(3.1.4) $\{(\alpha, \varrho)\in\boldsymbol{R}^m$ such that $\alpha\in Q_k,\ 0 \leqslant \varrho \leqslant C_k\} = B_k \subset R_k$ for all k

and furthermore there exists a constant C_0 depending only on the dimension such that:

(3.1.5) $$|R_k| \leqslant C_0|B_k|.$$

From conditions (3.1.2) and (3.1.3)

(3.1.6) $$B \subset \bigcup_1^\infty B_k \text{ a.e. and } \sum_1^\infty|B_k| < 2|B|.$$

Call g to the characteristic function of B, g_k and $\tilde{g}_k$ to the characteristic functions of B_k and R_k respectively. Take now a given measurable and locally integrable function f and consider

(3.1.7)
$$\Big|(1/|B_\varepsilon|)\int_{x+B_\varepsilon} f(y)\,dy\Big| = (\varepsilon^{-m}/|B|)\int_{\boldsymbol{R}^m} g(\varepsilon^{-1}y)\,|f(x-y)|\,dy$$
$$\leqslant (1/|B|)\cdot\sum_1^\infty\cdot\int_{\boldsymbol{R}^m}\varepsilon^{-m}g_k(\varepsilon^{-1}y)\,|f(x-y)|\,dy$$
$$\leqslant (1/|B|)\cdot\sum_1^\infty\cdot\int_{\boldsymbol{R}^m}\varepsilon^{-m}\tilde{g}(\varepsilon^{-1}y)\,|f(x-y)|\,dy$$
$$= (1/|B|)\cdot\sum_1^\infty|R_k|\big(1/|R_k(\varepsilon)|\big)\int_{R_k(\varepsilon)}|f(x-y)|\,dy$$

where $R_k(\varepsilon)$ means an homotetic contraction of R_k of ratio $\varepsilon>0$. Call
$$f_k^*(x) = \sup_{\varepsilon>0}\big(1/R_k(\varepsilon)\big)\int_{R_k(\varepsilon)}|f(x-y)|\,dy.$$

Let us observe that each $f_k^*(x)$ is under the conditions of Lemma 2.2. Then

(3.1.8)
$$f^*(x)\leqslant(1/|B|)\sum_{k=1}^\infty|R_k|f_k^*(x).$$

Taking L^p norm over $\boldsymbol{R}^m$ and recalling (3.1.6) and (3.1.5), we obtain

(3.1.9)
$$\|f^*\|_p\leqslant C_0C(m,p)\|f\|_p,\ p>1.$$

If A has finite Lebesgue measure and if f and $|f|\log^+|f|$ belong to L^1 it follows that

(3.1.10)
$$\int_A f^*\,dx\leqslant C_0\Big(C_1|A|+C_2\int_{\boldsymbol{R}^m}|f|\,(1+\log^+|f|)\,dy\Big).$$

Finally, the pointwise convergence follows from the maximal inequalities already proved and from Lemma (2.5).

4. Proof of Theorem B.

4.1. We shall start by expliciting some elementary estimates. Let us consider the integral:

(4.1.1)
$$-\int_0^\infty y^{n\beta}\,d\gamma_\beta\{\varphi>y\}.$$

An integration by parts yields

(4.1.2)
$$n\beta\int_0^\infty\gamma_\beta\{\varphi>y\}\,y^{n\beta-1}\,dy$$

therefore we have:

(4.1.3)
$$n\beta\int_0^1\gamma_\beta\{\varphi>y\}\,y^{n\beta-1}\,dy+\sum_{k=0}^\infty n\beta\int_{2^k}^{2^{k+1}}\gamma_\beta\{\varphi>y\}\,y^{n\beta-1}\,dy$$
$$\geqslant C_1+C_2\sum_{k=1}^\infty\gamma_\beta\{\varphi>2^{k+1}\}2^{kn\beta}.$$

This estimate shows that:

(4.1.4)
$$\sum_{k=1}^\infty 2^{kn\beta}\gamma_\beta\{\varphi>2k\}\leqslant k_1+k_2\Big(-\int_0^\infty y^{n\beta}\,d\gamma_\beta\{\varphi>y\}\Big).$$

Now, for each k, we are going to select a family $I_{k,j}$ of "cubes" on Σ such that:

(4.1.5)
$$\{\varphi>2^k\}\subset\bigcup_{j=1}^\infty I_{k,j},\qquad \sum_{j=1}^\infty A^\beta(I_{k,j})\leqslant 2\gamma_\beta\{\varphi>2^k\},$$

$A(I_{k,j})$ means as before the "volume" of the cube on Σ (4.1.5) together with (4.1.4) shows:

(4.1.6)
$$\sum_{k,j}2^{kn\beta}\{A(I_{k,j})\}^\beta\leqslant\bar{K}_1+\bar{K}_2\Big(-\int_0^\infty y^{n\beta}\,d\nu_\beta\{\varphi>y\}\Big)$$

since $0<\beta<1$ and the terms $2^{kn\beta}\{A(I_{k,j})\}^\beta$ are uniformly bounded we have also:

(4.1.7)
$$\sum_{k,j}2^{nk}A(I_{k,j})<\infty.$$

Calling E_k to the set of points on Σ where $2^k<\varphi\leqslant 2^{k+1}$ we have immediately

(4.1.8)
$$E_k\subset\bigcup_{j=1}^\infty I_{k,j}.$$

Now, we are going to define $B_{k,j}$ to be the following set:

(4.1.9)
$$B_{k,j}=\{(\alpha,\varrho)\in\boldsymbol{R}^m;\ \alpha\in I_{k,j},\ 0<\varrho\leqslant 2^{k+1}\}.$$

Correspondingly, we shall define $R_{k,j}$ as we did before in Theorem A. As before, we shall have:

(4.1.10)
$$|R_{k,j}|<C_1|B_{k,j}|<C_2 2^{nk}A(I_{k,j}).$$

Calling B_0 to the set of (α,ϱ) where $\varrho\leqslant 4$, it follows also

(4.1.11)
$$B\subset B_0\cup\Big(\bigcup_{k,j}B_{k,j}\Big)\subset B_0\cup\Big(\bigcup_{k,j}R_{k,j}\Big).$$

If μ is a non-negative measure having finite total mass on $\boldsymbol{R}^m$ we have

$$(4.1.12)\qquad \frac{1}{|B_\varepsilon|}\int_{x+B_\varepsilon} d\mu \leqslant C\left(\frac{1}{B_0(\varepsilon)}\int_{x+B_0\varepsilon} d\mu + \sum_{k,j}|R_{k,j}|\frac{1}{|R_{k,j}(\varepsilon)|}\int_{x+R_{k,j}(\varepsilon)} d\mu\right).$$

From (4.1.6) it follows that

$$(4.13)\qquad \sum_{k,j}|R_{k,j}|^\beta < \infty$$

therefore:

$$(4.1.14)\qquad \sum_{k,j}|R_{k,j}|\,\big|\log|R_{k,j}|\big| < \infty.$$

According to the above condition i) follows from lemmas (2.4) and (2.5). If $0<\beta\leqslant 1/2$, ii) follows from Lemma 2.3.

5. Proof of Theorem C.

5.1. We shall deal first with the point ii). We are going to reduce it to a particular case of Theorem B. Since we have a finite number of dominating functions (φ_j, $j=1,2,\ldots,N$), it can be reduced to prove the result for the case of only one dominating function. This is done in both cases i) and ii).

Without loss of generality we may assume that the boundary function is coincident with its dominating function. We have the following:

$$(5.1.1)\qquad |B| = -\int_0^\infty y^n dA\{\varphi>y\} = n\int_0^\infty A\{\varphi>y\}y^{n-1}dy,$$

this last integral is bigger than:

$$(5.1.2)\qquad C\sum_1^\infty A\{\varphi>2^k\}2^{kn}.$$

The same type of argument gives

$$(5.1.3)\qquad \sum_3^\infty 2^{kn}k\log^{1+\delta}kA\{\varphi>2^k\}<\infty$$

provided that the following integral is finite:

$$(5.1.4)\qquad \int_\Sigma \varphi^n\log^+\varphi(\log^+\log^+\varphi)^{1+\delta}d\sigma.$$

Notice that $\{\varphi>2^k\}$ is a "sphere" on Σ; therefore, for each k we can find a cube Q_k on Σ, such that:

$$(5.1.5)\qquad Q_k\supset\{\varphi>2^k\}\qquad A(Q_k)<CA\{\varphi>2^k\}.$$

Notice also that $A^{1/2}(Q_k)\geqslant\gamma_{1/2}\{\varphi>2^k\}$; therefore, we have the following chain of estimates:

$$(5.1.6)\qquad -\int_0^\infty y^{n/2}d\gamma_{1/2}\{\varphi>y\}\leqslant K_1+K_2\sum_1^\infty 2^{kn/2}A^{1/2}(Q_k)$$
$$\leqslant K_1+K_2C\sum_1^\infty 2^{kn/2}A^{1/2}\{\varphi>2^k\}$$
$$\leqslant \overline{K}_1+K_2C\sum_3^\infty 2^{kn/2}k^{1/2}\log^{1/2(1+\delta)}kA^{1/2}\{\varphi>2^k\}\frac{1}{k^{1/2}\log^{1/2(1+\delta)}k}.$$

By using Schwartz's inequality we get:

$$(5.1.7)\qquad -\int_0^\infty y^{n/2}d\gamma_{1/2}\{\varphi>y\}\leqslant\overline{K}_1+K_2C\left(\sum_3^\infty\frac{1}{3k\log^{1+\delta}k}\right)^{1/2}\times$$
$$\times\Big(\sum_3^\infty 2^{kn}k\log^{1+\delta}kA\{\varphi>2^ky\}^{1/2}.$$

That is $\varphi\in L^{n/2}_{\gamma_{1/2}}(\Sigma)$. This finishes part ii).

5.2. For the case i) we are going to assume also that φ is coincident with its dominating function.

The fact that $\int_\Sigma\varphi^n\log^+\log^+\varphi\,d\sigma<\infty$ implies that:

$$(5.2.1)\qquad \sum_3^\infty 2^{kn}\log kA\{\varphi>2^k\}<\infty.$$

Since $\{\varphi>2^k\}$ is a "sphere" on Σ, it is possible for each k to find a "cube" Q_k on Σ such that:

$$(5.2.2)\qquad \{\varphi>2^k\}\subset Q_k,\qquad A(Q_k)<2A\{\varphi>2^k\}.$$

If we call E_k to the set of points of Σ where $2^k<\varphi\leqslant 2^{k+1}$, we have clearly $\Sigma_k\subset Q_k$.

Now, as we did before, we define the set B_k to be

$$(5.2.3)\qquad B_k=\{(a,\varrho);\,a\in Q_k,\,0<\varrho\leqslant 2^{k+1}\},\qquad B_0=\{(a,\varrho);\,0<\varrho\leqslant 8\}.$$

The R_k are defined as previously.

We have then the inequalities:

$$(5.2.4)\qquad |R_k|<C_1|B_k|<C_22^{kn}A(Q_k).$$

Also the following holds truth:

(5.2.5) $$B \subset B_0 \cup \{\bigcup_k B_k\} \subset B_0 \cup \{\bigcup_k R_k\}.$$

So, if μ is a non-negative measure having finite total mass on $\boldsymbol{R}^m$ we have:

(5.2.6)
$$\frac{1}{|B_\varepsilon|}\int_{x+B_\varepsilon} d\mu \leqslant C\left(\frac{1}{|B_0(\varepsilon)|}\int_{x+B_0(\varepsilon)} d\mu + \sum_1^\infty 2^{kn} A(Q_k)\frac{1}{|R_k(\varepsilon)|}\int_{x+R_k(\varepsilon)} d\mu\right).$$

Our next step will be to decompose the sequence $\{k\}$ of indices into two non-overlapping subsequences Z_1 and Z_2 according to the following conditions:

(5.2.7) $$\begin{aligned} &k \in Z_1 \text{ if } A(Q_k) < \frac{1}{k^{1+\varepsilon}2^{kn}}, \\ &k \in Z_2 \text{ if } A(Q_k) \geqslant \frac{1}{k^{1+\varepsilon}2^{kn}}, \end{aligned} \qquad \varepsilon > 0.$$

In the first case, that is for those k belonging to Z_1 we have the domination

(5.2.8) $$2^{kn}A(Q_k) < \frac{1}{k^{1+\varepsilon}}$$

and Lemma (2.3) applies for the dominating coefficients $\frac{1}{k^{1+\varepsilon}}$ since

(5.2.9) $$\sum_3^\infty \frac{1}{k^{1+\varepsilon}}\log k < \infty.$$

In the second case, that is for Z_2, (we may assume without loss of generality that $2^{kn}\cdot A(Q_k) \leqslant 1$) the domination $A(Q_k) \geqslant \frac{1}{k^{1+\varepsilon}2^{kn}}$ implies:

(5.2.10) $$\log\frac{1}{2^{kn}A(Q_k)} < (1+\varepsilon)\log k.$$

Consequently:

(5.2.11) $$\begin{aligned} \sum_{k\in Z_2} 2^{kn}A(Q_k)|\log 2^{kn}A(Q_k)| &\leqslant (1+\varepsilon)\sum_3^\infty 2^{kn}\log k A(Q_k) \\ &\leqslant K_1 + K_2\int_\Sigma \varphi^n \log^+\log^+\varphi\, d\sigma. \end{aligned}$$

This finishes the proof of the theorem.

References

[1] A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math., April, 1956, pp. 289–309.

[2] C. P. Calderón, *Some remarks on the multiple Weierstrass transform and Abel summability of multiple Fourier–Hermite series*, Studia Math. 32 (1969), pp. 119–148.

[3] A. Zygmund, *Trigonometrical Series*, Vols. I and II, 1959.

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