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Translation invariant subspaces of $L^p(G)$

by

AHARON ATZMON* (Los Angeles, Calif. and Orsay, France)

Abstract. The main result of this paper is that if G is a locally compact abelian group which is not compact and $1 \leqslant p < \frac{4}{3}$ then $L^p(G)$ contains a closed translation invariant subspace which is not the closed span of translates of a single function.

1. Introduction. In what follows G is a locally compact abelian group equipped with a Haar measure dx. For $1 \leqslant p \leqslant \infty$ we denote by $L^p(G)$ the classical Banach spaces associated with the pair (G, dx). For a function $f \in L^p(G)$ and $y \in G$, the y-translate of f is the function f_y defined by $f_y(x) = f(x-y)$, $x \in G$. A subspace $V \subset L^p(G)$ is called *translation invariant*, if $f \in V$ implies that $f_y \in V$ for every $y \in G$. In this paper we are concerned with the following problem:

Is every closed translation invariant subspace of $L^p(G)$, $1 \leqslant p < \infty$, the closed span of translates of a single function?

If G is compact, the structure of the closed translation invariant subspaces of $L^p(G)$ is completely determined ([11], p. 94) and it follows easily from their characterization that if G is compact and metrizable, the answer to our problem is affirmative. The structure of the closed translation invariant subspaces of $L^2(G)$ was also completely determined, by Ditkin ([2], p. 111) for $G = R$ and for general G by L. Schwartz ([9], p. 869), who also proved that the answer to our problem is affirmative for metrizable G and $p = 2$.

On the other hand it has been proved in [1] that if G is not compact then $L^1(G)$ contains a closed translation invariant subspace which is not the closed span of translates of finitely many functions, so that the answer to the problem is negative in this case.

For G which is not compact and $p \neq 2$ very little is known about the closed translation invariant subspaces of $L^p(G)$ (see [3], p. 238), and the determination of their structure seems to be far out of our scope. The main result of this paper is that the answer to the problem posed is negative for G not compact and $1 \leqslant p < \frac{4}{3}$. That is we prove:

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THEOREM 1.1. *If G is not compact and $1 \leqslant p < \frac{4}{3}$ then $L^p(G)$ contains a closed translation invariant subspace which is not the closed span of translates of a single function.*

This gives a partial answer to the question raised in ([1], § 5). For other values of p the problem remains open. In the proof of the correspondent result for $p = 1$ in [1], the theory of Varopoulos on tensor products of Banach algebras has been used. Those methods are not applicable to $L^p(G)$ for $p > 1$, and in the proof of Theorem 1.1 we use the methods introduced by Malliavin in [7] and their variants which are due to Kahane ([4], p. 121), Rudin ([8], p. 181) and Lohoue [6].

2. Preliminaries. If G is a locally compact abelian group with dual Γ and $1 \leqslant p \leqslant 2$ we shall denote by $\mathfrak{F}_p(\Gamma)$ the space $\mathfrak{F}L^p(G)$, that is, the set of all functions on Γ which are Fourier transforms of functions in $L^p(G)$. The norm of a function f in $\mathfrak{F}_p(\Gamma)$ which is the Fourier transform of the function F in $L^p(G)$ will be defined by $\|f\|_{\mathfrak{F}_p(\Gamma)} = \|F\|_{L^p(G)}$. Under this norm $\mathfrak{F}_p(\Gamma)$ is a Banach space, which is isometrically isomorphic to $L^p(G)$. For $\mathfrak{F}_1(\Gamma)$ the usual notation $A(\Gamma)$ will be used. We denote by $PM_p(\Gamma)$ the dual space of $\mathfrak{F}_p(\Gamma)$, which is isometrically isomorphic to $L^q(G)$ where $\dfrac{1}{p}+\dfrac{1}{q} = 1$. It is easy to see that if f is in $A(\Gamma)$ and has compact support, then $f \in \mathfrak{F}_p(\Gamma)$ for all $1 \leqslant p \leqslant 2$, and also that the linear functional defined on $\mathfrak{F}_p(\Gamma)$ by

$$\langle g, f\rangle = \int_{\Gamma} g(y) f(y)\, dy, \qquad g \in \mathfrak{F}_p(\Gamma)$$

(where dy denotes the Haar measure of Γ) is in $PM_p(\Gamma)$ for all $1 \leqslant p \leqslant 2$ We shall use those facts in the sequel, without further mention.

It is well known that $L^p(G)$ is a Banach module over $L^1(G)$ with respect to convolution and for $1 \leqslant p < \infty$, V is a closed translation invariant subspace of $L^p(G)$ if and only if V is a closed submodule of $L^p(G)$. This implies that for $1 \leqslant p \leqslant 2$, $\mathfrak{F}_p(\Gamma)$ is a module over $A(\Gamma)$ with respect to usual multiplication of functions and that the Fourier transform establishes a one-to-one correspondence between the closed translation invariant subspaces of $L^p(G)$ and the closed submodules of $\mathfrak{F}_p(\Gamma)$.

If S is a subset of $\mathfrak{F}_p(\Gamma)$, then the smallest closed submodule of $\mathfrak{F}_p(\Gamma)$ which contains S, will be called the submodule generated by S in $\mathfrak{F}_p(\Gamma)$.

It is now clear that Theorem 1.1 is equivalent to the following:

THEOREM 2.1. *If Γ is not discrete and $1 \leqslant p < \frac{4}{3}$ then $\mathfrak{F}_p(\Gamma)$ contains a closed submodule which is not generated by a single function.*

The following lemma which is a version of Malliavins construction in [7] is essentially due to Kahane, Rudin and Lohoue. It is of fundamental importance in the proof of Theorem 2.1.

LEMMA 2.2. *If Γ is not discrete, there exist in $A(\Gamma)$ real valued functions f_1, f_2 and a positive function φ, all with compact support, such that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \|\varphi \operatorname{Exp}[i(uf_1+vf_2)]\|_{PM_r(\Gamma)} (u^2+v^2)\, du\, dv < \infty \tag{2.1}$$

for all $1 \leqslant r < 2$.

Proof of Lemma 2.2. This is a version of Kahane's theorem in [4], p. 121, and is proved by Rudin ([8], p. 181) for Γ compact and $p = 1$ with $\varphi \equiv 1$ and by Lohoue ([6], p. 126) for Γ compact and $1 \leqslant p < 2$, again with $\varphi \equiv 1$. The general case is deduced from the compact case by standard arguments using the structure theorem ([8], p. 40) and the following two facts:

(1) If Γ is a locally compact group, Λ is an open subgroup of Γ and $1 \leqslant p < 2$, then if f is a continuous function on Γ which is in $\mathfrak{F}_p(\Gamma)$, then the restriction of f to Λ belongs to $\mathfrak{F}_p(\Lambda)$. This is proved by using ([8], 2.7.3).

(2) If H is a compact group, n a positive integer, $1 \leqslant p < 2$, and f is a continuous function in $\boldsymbol{R}^n \times H$ with support in $[-1, 1]^n \times H$, then $f \in \mathfrak{F}_p(\boldsymbol{R}^n \times H)$ if and only if $f \in \mathfrak{F}_p(T^n \times H)$, where T denotes the circle group identified with $[-\pi, \pi)$.

This is proved by the same method as the classical theorem of Wiener on the local isomorphism of $A(\boldsymbol{R})$ and $A(T)$ (see [5], p. 227).

Using those facts the proof of the lemma is straightforward and we omit the details.

Let now $\Gamma, f_1, f_2, \varphi$ be as in Lemma 2.1. By a standard reduction (see [5], p. 232) we may also assume that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big\{\int_{\Gamma} \varphi(y) \operatorname{Exp}[i(uf_1(y)+vf_2(y))]\, dy\Big\}\, du\, dv \neq 0. \tag{2.2}$$

LEMMA 2.2. *Let Γ, f_1, f_2, and φ be as in Lemma 2.1, and assume that (2.2) also holds. Then there exists a positive nonzero Borel measure μ on Γ, with compact support, such that for every $g \in A(\Gamma)$*

$$\int_{\Gamma} g(x)\, d\mu(x) = \int_{\boldsymbol{R}^2} \langle g, \varphi \operatorname{Exp}[i(uf_1+vf_2)]\rangle\, du\, dv. \tag{2.3}$$

Proof. The identity

$$\int_{\boldsymbol{R}^2} \operatorname{Exp}[i(uf_1(y)+vf_2(y))]\, e^{-\delta\pi(u^2+v^2)}\, du\, dv = \delta^{-1} \operatorname{Exp}\Big[-\frac{\pi}{\delta}\big(f_1^2(y)+f_2^2(y)\big)\Big]$$

which holds for every $\delta > 0$ and $y \in \Gamma$, implies by (2.1) and Fubini's theorem, that for every $g \in A(\Gamma)$

$$\int_{R^2} \langle g, \varphi \operatorname{Exp}[i(uf_1+vf_2)]\rangle \, du dv = \lim_{\delta\to 0} \int_\Gamma \varphi(y) g(y) \delta^{-1} \operatorname{Exp}\left[-\frac{\pi}{\delta}(f_1^2(y)+f_2^2(y)\right] dy$$

and since φ is positive this shows that the continuous linear functional on $A(\Gamma)$

$$g \to \int_{R^2} \langle g, \varphi \operatorname{Exp}[i(uf_1+vf_2)]\rangle \, du dv$$

is a positive functional, and therefore by the same argument as in the proof of Theorem V in [10] (p. 29) there exists a positive Borel measure μ on Γ such that (2.3) holds. Clearly the support of μ is contained in the support of φ. Finally using (2.3) with g in $A(\Gamma)$ such that $g(x) = 1$ for $x \in \operatorname{supp} \varphi$, it follows from (2.2) that $\int_\Gamma d\mu \neq 0$, hence $\mu \neq 0$, and the proof is complete.

3. Proof of Theorem 2.1. Let now Γ, f_1, f_2 and φ be as in Lemma 2.2. We shall denote by U the ideal which is (algebraically) generated in $A(\Gamma)$ by f_1, f_2. Since f_1 and f_2 have compact supports, U is contained in $\mathfrak{F}_p(\Gamma)$ for every $1 \leqslant p \leqslant 2$. We denote by V_p the closure of U in $\mathfrak{F}_p(\Gamma)$. V_p is clearly a closed submodule of $\mathfrak{F}_p(\Gamma)$. Theorem 2.1 follows now from the following:

PROPOSITION 3.1. *For $1 \leqslant p < \frac{4}{3}$, V_p is not generated by a single function.*

Proof. Let μ be the measure on Γ which satisfies (2.3). We define the bilinear functional L on $U \times U$ as follows:

If $g = g_1 f_1 + g_2 f_2$ and $h = h_1 f_1 + h_2 f_2$ with $g_j, h_j \in A(\Gamma)$ $(j = 1, 2)$ then

$$L(g, h) = \int_\Gamma (g_1 h_2 - g_2 h_1) \, d\mu. \tag{3.1}$$

We show now that L is well defined and bounded on $U \times U$ with respect to the $\mathfrak{F}_p(\Gamma)$ norm, for every $1 \leqslant p < \frac{4}{3}$. Using (2.1), and integrating by parts, (2.3) implies that if g and g_1 are as in (3.1), then

$$\int_\Gamma |g_1|^2 \, d\mu = -\tfrac{1}{2} \int_{R^2} u^2 \langle g\bar{g}, \varphi \operatorname{Exp}[i(uf_1+vf_2)]\rangle \, du dv. \tag{3.2}$$

Let now $1 \leqslant p < \frac{4}{3}$ be fixed, and define r by $\dfrac{1}{r} = \dfrac{2}{p} - 1$. Young's inequality for convolutions ([3], p. 241) implies that

$$\|g\bar{g}\|_{\mathfrak{F}_r(\Gamma)} \leqslant \|g\|^2_{\mathfrak{F}_p(\Gamma)}. \tag{3.3}$$

Now since $1 \leqslant r < 2$ (here we use the fact that $1 \leqslant p < \frac{4}{3}$!), we infer from (2.1), (3.2) and (3.3) that

$$\int_\Gamma |g_1|^2 \, d\mu \leqslant C_r^2 \|g\|^2_{\mathfrak{F}_p(\Gamma)} \tag{3.4}$$

where C_r denotes the left-hand side of (2.1).

Similarly if g and g_2 are as in (3.1) we get that

$$\int_\Gamma |g_2|^2 \, d\mu \leqslant C_r^2 \|g\|^2_{\mathfrak{F}_p(\Gamma)}. \tag{3.5}$$

Let now g_j, g_j' $(j = 1, 2)$ be functions in $A(\Gamma)$ such that

$$(g_1 - g_1')f_1 + (g_2 - g_2')f_2 = 0.$$

It follows from (3.4) and (3.5) that $\int_\Gamma |g_j - g_j'|^2 \, d\mu = 0$, $(j = 1, 2)$ hence $g_j(x) = g_j'(x)$ $(j = 1, 2)$ for all x in the support of μ, and this shows that L is well defined. Applying Schwartz inequality to both terms in the right-hand side of (3.1) and using (3.4) and (3.5) together with their analogues for h, h_1, h_2, we get that

$$|L(g, h)| \leqslant 2C_r \|g\|_{\mathfrak{F}_p(\Gamma)} \|h\|_{\mathfrak{F}_p(\Gamma)},$$

and this shows that L is bounded on $U \times U$ with respect to the $\mathfrak{F}_p(\Gamma)$ norm. Now using the fact that U is dense in V_p we extend L to a bounded bilinear functional on $V_p \times V_p$ which we continue to denote by L.

It follows from (3.1) that for every $f \in U$ and for all $\varphi_1, \varphi_2 \in A(\Gamma)$

$$L(\varphi_1 f, \varphi_2 f) = 0 \tag{3.6}$$

and since U is dense in V_p this also holds for every $f \in V_p$. Hence if f is any element in V_p and $V(f)$ is the closed module generated by f in $\mathfrak{F}_p(\Gamma)$, then (3.6) implies that L annihilates $V(f) \times V(f)$. But from (3.1) it follows that $L(f_1, f_2) = \int d\mu \neq 0$, and therefore L does not annihilate $V_p \times V_p$, thus $V(f) \neq V_p$. This proves Proposition 3.1, and therefore also Theorems 2.1 and 1.1.

4. Generalizations and related results. Using the same methods as in the proof of Theorem 1.1 the following generalization can be proved:

THEOREM 4.1. *If G is not compact and $1 \leqslant p < \dfrac{2(k+1)}{2k+1}$, then $L^p(G)$ contains a closed translation invariant subspace which is not generated by the translates of k functions.*

The proof of the next theorem is also by the same methods.

THEOREM 4.2. *Let B be a semi-simple self-adjoint Banach algebra, which is represented as an algebra of continuous functions on its regular* *maximal ideal space* $\mathfrak{M}$. *If there exist two real functions* f_1, f_2 *in* B *and a positive non-zero Borel measure* ν *on* $\mathfrak{M}$, *with compact support, such that*

$$\int_{R^2} (u^2+v^2) \|\nu \operatorname{Exp}[i(uf_1+vf_2)]\|_{B^*} du dv < \infty \tag{4.1}$$

then B *contains a closed ideal which is not generated by a single function.*

COROLLARY. *Let* $A_p(G)$ *be the Banach algebras defined as in* [6]. *If* $1 < p < \infty$ *and* G *is not discrete,* $A_p(G)$ *contains a closed ideal which is not singly generated.*

Proof. It follows from Lemma 2.2 that condition (4.1) is satisfied for $A_p(G)$.

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UNIVERSITY OF CALIFORNIA, LOS ANGELES
FACULTÉ DES SCIENCES, MATHÉMATIQUE
ORSAY, FRANCE

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A remark on the d-characteristic and the d_Ξ-characteristic of linear operators in a Banach space

by

ŠTEFAN SCHWABIK (Praha)

Abstract. Let X be a Banach space, and Ξ a total space of continuous linear functionals on X which is also a Banach space. It is proved that $I+T$ is a Φ_Ξ-operator provided $T: X \to X$ is compact and Ξ is preserved by the conjugate operator T'. The paper is closely related to the work of D. Przeworska-Rolewicz and S. Rolewicz.

Let X be a linear space (over the field of real or complex numbers) and let A be a linear operator mapping X into itself and such that Ax is defined for all $x \in X$ ($D_A = X$). Let the set of all such operators be denoted by $L_0(X)$.

We denote by

$$N(A) = \{x \in X;\ Ax = 0\}$$

the kernel of the operator A, and by

$$R(A) = \{y \in X;\ y = Ax,\ x \in X\}$$

the range of the operator A, and define

$$\alpha_A = \dim N(A), \quad \beta_A = \dim X/R(A)$$

(dim denotes the dimension of a linear set and $X/R(A)$ means the quotient space). The ordered pair (α_A, β_A) is called the *d-characteristic of the operator A*. The index of the operator A is the number

$$\operatorname{ind} A = \beta_A - \alpha_A.$$

By X' the space of all linear functionals on X is denoted. Let $\Xi \subset X'$ be a total space of linear functionals on X, i.e. if $\xi(x) = 0$ for all $\xi \in \Xi$ then $x = 0$. We write

$$N_\Xi(A') = \{\xi \in \Xi;\ \xi(Ax) = 0 \quad \text{for all } x \in X\}$$

and define

$$\beta_A^\Xi = \dim N_\Xi(A').$$

The ordered pair (α_A, β_A^Ξ) is called the *d_Ξ-characteristic of A*.