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On modular sequence spaces

by

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Abstract. Several results concerning Orlicz sequence spaces are generalized to modular sequence spaces. For instance, a modular sequence space has an unconditional base iff it does not contain l_∞, and every modular sequence space contains l_p for some $p \in [1, \infty)$. A condition for reflexivity is also obtained.

1. Introduction. A convex function $M: \boldsymbol{R} \to \boldsymbol{R}$ such that $M(0) = 0$, $M(x) > 0$ for all $x > 0$ is called an *Orlicz function*. Let l_M be the set of all sequences $\{x_n\}$ such that $\sum M(|x_n|/t) < \infty$ for some $t > 0$. We can norm l_M by

$$\|\{x_n\}\|_M = \inf\Big\{t > 0:\ \sum M(|x_n|/t) \leqslant 1\Big\}.$$

$(l_M, \|\cdot\|_M)$ is a Banach space called an *Orlicz sequence space*.

If we consider a sequence $\{M_n\}$ of Orlicz functions, and define $l\{M_n\}$ to be the set of all sequences $\{x_n\}$ such that $\sum M_n(|x_n|/t) < \infty$ for some $t > 0$ and

$$\|\{x_n\}\| = \inf\{t > 0:\ M_n(|x_n|/t) \leqslant 1\},$$

we obtain a Banach space which is called a *modular sequence space*. If $M_n(x) = x^{p_n}$ for some $p_n \in [1, \infty)$, we obtain the modular sequence spaces considered by Nakano [6].

Recently, J. Lindenstrauss and L. Tzafriri [4] proved that every Orlicz sequence space contains a subspace isomorphic to l_p for some $p \in [1, \infty)$. Moreover, if the Orlicz sequence space l_M is separable, then every subspace of l_M contains a subspace isomorphic to l_p for some $p \in [1, \infty)$. The purpose of this paper is to generalize this result to modular sequence spaces. We are going to prove that every modular sequence space contains a subspace isomorphic to l_p for some $p \in [1, \infty)$ and we have the corresponding result for separable modular sequence spaces.

The techniques we use are very similar to those used in the paper of K. J. Lindberg [3]. The only new concept is the "almost equality" introduced in Section 2 to study equivalence of modular sequence spaces. This concept is the chief difference between Orlicz sequence spaces and modular sequence spaces, and gives us rather wide lattitudes in dealing with the latter.

In Section 3, we study separable modular sequence spaces. We introduce the uniform Δ_2 condition, and prove the main results of this paper:

I. *The following are equivalent:*

(a) *$\{M_n\}$ is equivalent to a sequence $\{N_n\}$ that satisfies the uniform Δ_2 condition.*

(b) *The unit vectors of $l\{M_n\}$ form an unconditional basis.*

(c) *$l\{M_n\}$ is separable.*

(d) *$l\{M_n\}$ does not contain a subspace isomorphic to l_∞.*

II. *Let $\{M_n\}$ be a sequence of Orlicz functions satisfying the uniform Δ_2 condition. Then $l\{M_n\}$ contains a complemented subspace isomorphic to some Orlicz sequence space.*

Combining I, II and the result of Lindenstrauss and Tzafriri, we obtain the generalization of that result.

A consequence of II is that a modular sequence space is an Orlicz sequence space iff it has a symmetric basis. This clarifies the relation between the two kinds of spaces, and show why some of the properties of Orlicz sequence spaces cannot be readily generalized to modular sequence spaces, for instance those in [5].

In Section 4, we study the duals of modular sequence spaces and obtain the conditions for reflexivity. We can apply the results to the spaces X_p, $p > 2$, constructed by H. Rosenthal in [7]. We show that $X_q = X_p^*$, where $p^{-1}+q^{-1} = 1$, is a modular sequence space and hence every subspace of X_q contains some l_r, $r \in [q, 2]$.

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2. Preliminaries. Throughout this paper, we shall follow the notations and terminologies used in [4].

For technical reasons, we shall always assume $M_n(1) = 1$ for all n, unless otherwise mentioned. There is no loss of generality in doing this. For suppose $\{M_n\}$ is any sequence of Orlicz functions. Let $a_n > 0$ satisfies $M_n(a_n) = 1$. Define $N_n(x) = M_n(a_n x)$. Then it is trivial to show that $\{N_n\}$ is a sequence of Orlicz functions and $l\{M_n\}$, $l\{N_n\}$ are isometric.

We now introduce the important concept of equivalence. Recall that two Orlicz functions M and N are said to be equivalent if there exist $\alpha, \beta > 0$, $L \geqslant K > 0$ and $x_0 > 0$ such that $x \in [0, x_0]$ implies $KM(\alpha x) \leqslant N(x) \leqslant LM(\beta x)$. We can define something like that for sequences of Orlicz functions, but that definition would be rather unnatural. So we define equivalence in the following manner:

Definition. Two sequences of Orlicz functions $\{M_n\}$, $\{N_n\}$ are said to be *equivalent* if $l\{M_n\} = l\{N_n\}$ as sets.

$M_n(1) = 1$ for all n implies $|x_n| \leqslant \|\{x_k\}\|$. Hence by an easy consequence of the Closed Graph Theorem, if $\{M_n\}$ and $\{N_n\}$ are equivalent, then the identity map of $l\{M_n\}$ onto $l\{N_n\}$ is an isomorphism.

Proposition 2.1. *Let $\{M_n\}$ and $\{N_n\}$ be two sequences of Orlicz functions. Suppose there exist real numbers $L \geqslant K > 0$, $a > 0$, $a_n \in [0, a]$ and a positive integer n_0 such that for all $n > n_0$ and $x \in [a_n, a]$,*

$$KM_n(x) \leqslant N_n(x) \leqslant LM_n(x)$$

and

$$\sum_{n=1}^{\infty} \sup\{|M_n(x) - N_n(x)|:\ x \in [0, a_n]\} < \infty.$$

Then $\{M_n\}$ and $\{N_n\}$ are equivalent.

Proof. The proof is straight forward. We can clearly assume $a < 1$. Suppose $\{x_n\} \in l\{M_n\}$. Then there exists $t > 0$ such that $\sum M_n(|x_n|/t) < 1$. As $|x_n|/t < 1$ for all n, we can take t so large that $|x_n|/t < a$ for all n. Let $E = \{n \geqslant n_0:\ |x_n|/t < a_n\}$, $F = \{n \geqslant n_0:\ |x_n|/t \geqslant a_n\}$ and $\beta_n = \sup\{|M_n(x) - N_n(x)|:\ x \in [0, a_n]\}$. Then

$$\begin{aligned}\sum_{n=n_0}^{\infty} N_n(|x_n|/t) &\leqslant \sum_{n\in E}\big(M_n(|x_n|/t)+\beta_n\big) + \sum_{n\in F} LM_n(|x_n|/t)\\ &\leqslant \max\{L, 1\}\sum_{n=n_0}^{\infty} M_n(|x_n|/t) + \sum_{n=1}^{\infty}\beta_n < \infty.\end{aligned}$$

So $\{x_n\} \in l\{N_n\}$ and $l\{M_n\} \subset l\{N_n\}$. By symmetry, $l\{M_n\} = l\{N_n\}$.

We now want to prove a partial converse to Proposition 2.1. Before doing that, we have to introduce certain concepts and notations.

Definition. $\{M_n\}$ and $\{N_n\}$ are said to be *almost equal* if there exist $a_n > 0$ for all $n \in \boldsymbol{Z}^+$ such that $M_n(x) = N_n(x)$ for all $x \geqslant a_n$, and $\sum M_n(a_n) < \infty$.

Obviously, almost equality implies equivalence. It plays a crucial role when theorems on Orlicz sequence spaces are generalized to those on modular sequence spaces.

We also need the following definition:

Definition. $c\{M_n\} = \{\{x_n\} \in l\{M_n\}:\ \sum M_n(|x_n|/t) < \infty \text{ for all } t > 0\}$.

It is easy to show that $c\{M_n\}$ is a closed subspace of $l\{M_n\}$, and that the unit vectors form an unconditional basis of $c\{M_n\}$. If $\{M_n\}$, $\{N_n\}$ are equivalent, then $c\{M_n\} = c\{N_n\}$. The following lemma has a trivial proof.

Lemma 2.2. *Let $\{M_n\}$ be a sequence of Orlicz functions. Then the following are equivalent:*

(a) $l\{M_n\} = c\{M_n\}$.

(b) *The unit vectors form an unconditional basis of* $l\{M_n\}$.

(c) *For all* $\{x_n\} \in l\{M_n\}$, $\sum M_n(|x_n|) < \infty$.

Lemma 2.3. *Let $\{M_n\}$ be a sequence of Orlicz functions.*

Suppose $\sum M_n(|x_n|) < \infty$ does not imply $\lim x_n = 0$. Then $l\{M_n\}$ contains a subspace isomorphic to l_∞.

Proof. If the hypothesis is true, then we have $a > 0$ and a subsequence $\{M_{i_n}\}$ such that $\sum_{n=1}^{\infty} M_{i_n}(a) \leqslant 1$. Define $T: l_\infty \to l\{M_n\}$ by $T(\{\beta_n\}) = \{x_n\}$, where

$$x_n = \begin{cases} \beta_m & \text{if } n = i_m, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to show that T is an isomorphism.

We can now prove the partial converse to Proposition 2.1.

Theorem 2.4. *Suppose the unit vectors form bases of $l\{M_n\}$ and $l\{N_n\}$. Then the following are equivalent:*

(a) *$\{M_n\}$ and $\{N_n\}$ are equivalent, i.e. the two unit vector bases are equivalent.*

(b) *There exist $\{M_n^\#\}$, $\{N_n^\#\}$ almost aqual to $\{M_n\}$, $\{N_n\}$, respectively, such that the following are satisfied:*

There exist $L \geqslant K > 0$, $n_0 > 0$ and $a > 0$ such that

$$KM_n^\#(x) \leqslant N_n^\#(x) \leqslant LM_n^\#(x) \tag{1}$$

for all $n > n_0$ and $x \in [0, a]$.

Proof. (b) $\Rightarrow$ (a) is a special case of Proposition 2.1.

(a) $\Rightarrow$ (b). Let $m, n \in \mathbf{Z}^+$. Define

$$x_{m,n} = \sup\{x \in [0, m^{-2}]:\ mM_n(x) \leqslant N_n(x)\},$$

$$y_{m,n} = \sup\{x \in [0, m^{-2}]:\ mN_n(x) \leqslant M_n(x)\}.$$

Claim: there exists $m \in \mathbf{Z}^+$ such that $\sum_{n=1}^{\infty} M_n(x_{m,n}) < \infty$. For if not, then $\sum_{n=1}^{\infty} M_n(x_{m,n}) = \infty$ for all $m \in \mathbf{Z}^+$. As $M_n(x_{m,n}) \leqslant x_{m,n} \leqslant m^{-2}$ for all $n \in \mathbf{Z}^+$, by an easy induction we can find positive integers

$$p_1 < p_2 < \ldots < p_m < \ldots$$

such that

$$m^{-2} \leqslant \sum_{n=p_m+1}^{p_{m+1}} M_n(x_{m,n}) < 2m^{-2}.$$

Note that for all m and n, we have

$$mM_n(x_{m,n}) \leqslant N_n(x_{m,n}).$$

Let $\{e_n\}$ be the unit vector basis. Then $\sum_{m=1}^{\infty} \sum_{n=p_m+1}^{p_{m+1}} x_{m,n} e_n$ converges in $l\{M_n\}$, since

$$\sum_{m=1}^{\infty} \sum_{n=p_m+1}^{p_{m+1}} M_n(x_{m,n}) \leqslant \sum_{m=1}^{\infty} 2m^{-2} < \infty.$$

On the other hand,

$$\sum_{m=1}^{\infty} \sum_{n=p_m+1}^{p_{m+1}} N_n(x_{m,n}) \geqslant \sum_{m=1}^{\infty} \sum_{n=p_m+1}^{p_{m+1}} m\, M_n(x_{m,n}) \geqslant \sum_{m=1}^{\infty} m^{-1} = \infty.$$

So by Lemma 2.2 (c), $\sum_{m=1}^{\infty} \sum_{n=p_m+1}^{p_{m+1}} x_{m,n} e_n$ diverges in $l\{N_n\}$, contradicting the equivalence of $\{M_n\}$ and $\{N_n\}$.

Now assume $\sum_{n=1}^{\infty} M_n(x_{m,n}) < \infty$ for some $m \in \mathbf{Z}^+$. We are going to construct $\{M_n^\#\}$ almost equal to $\{M_n\}$ such that $N_n(x) \leqslant mM_n^\#(x)$ for all $x \in [0, m^{-2}]$ and all $n >$ some n_0. Because $l\{M_n\}$ has a basis, it cannot contain l_∞. So by Lemma 2.3, $\lim_{n\to\infty} x_{m,n} = 0$. So there exists $n_0 \in \mathbf{Z}^+$ such that $x_{m,n} < m^{-2}$ for all $n > n_0$. By continuity of N_n/M_n on $(0, \infty)$, and by the definition of $x_{m,n}$, we have

$$\frac{N_n(x_{m,n})}{M_n(x_{m,n})} = m, \quad \text{and}$$

$$\frac{N_n(x)}{M_n(x)} < m \quad \text{for all } x \in (x_{m,n}, m^{-2}].$$

Now define $M_n^\#$ to be M_n if $n \leqslant n_0$, and for $n > n_0$, define

$$M_n^\#(x) = \begin{cases} m^{-1}N_n(x), & x \leqslant x_{m,n}, \\ M_n(x), & x \geqslant x_{m,n}. \end{cases}$$

$M_n^\#$ is continuous. It is also convex. To show this, it is enough to show that $m^{-1}N_n'(x_{m,n}) \leqslant M_n'(x_{m,n})$ for all $n > n_0$. ($M_n'(x_{m,n})$ and $N_n'(x_{m,n})$ are the right derivatives at $x_{m,n}$ if the derivatives do not exist.) This follows from

$$\frac{N_n(x)}{M_n(x)} < \frac{N_n(x_{m,n})}{M_n(x_{m,n})} \quad \text{for all } x \in (x_{m,n}, m^{-2}] \text{ and } n > n_0.$$

Hence $(N_n/M_n)'(x_{m,n}) \leqslant 0$. This means that

$$M_n(x_{m,n})N_n'(x_{m,n}) - N_n(x_{m,n})M_n'(x_{m,n}) \leqslant 0.$$

As $N_n(x_{m,n}) = mM_n(x_{m,n})$, we have $m^{-1}N_n'(x_{m,n}) \leqslant M_n'(x_{m,n})$. So $\{M_n^{\#}\}$ is a sequence of Orlicz functions almost equal to $\{M_n\}$. Moreover, $mM_n^{\#}(x) \geqslant N_n(x)$ for all $x \in [0, m^{-2}]$ and all $n > n_0$.

We now want to obtain the other side of the inequality in (1). By substituting $y_{m,n}$ for $x_{m,n}$, N_n for M_n, $M_n^{\#}$ for N_n, and repeating our previous arguments, we are able to obtain m', n_1 and $N_n^{\#}$ such that $m'N_n^{\#}(x) \geqslant M_n^{\#}(x)$ for all $x \in [0, (m')^{-2}]$ and all $n > n_1$, where $N_n^{\#} = N_n$ for $n \leqslant n_1$, and

$$N_n^{\#}(x) = \begin{cases} (m')^{-1}M_n^{\#}(x), & x \leqslant y_{m',n}, \\ N_n(x), & x \geqslant y_{m',n}. \end{cases}$$

We can clearly assume $m = m'$ and $n_0 = n_1$. So we only have to prove $mM_n^{\#}(x) \geqslant N_n^{\#}(x)$ for all $x \in [0, m^{-2}]$ and $n > n_0$ to complete our proof of (1).

Suppose $n > n_0$ and $x \in [0, m^{-2}]$. If $x \in [0, y_{m,n}]$, then $N_n^{\#}(x) = m^{-1}M_n^{\#}(x)$ If $x \in [y_{m,n}, m^{-2}]$, then $N_n^{\#}(x) = N_n(x) \leqslant mM_n^{\#}(x)$. Hence for all $x \in [0, m^{-2}]$ and for all $n > n_0$, we have $N_n^{\#}(x) \leqslant mM_n^{\#}(x)$.

This completes the proof of (1) with $L = m$, $K = m^{-1}$, and $a = m^{-2}$.

We now introduce some concepts which will be studied in greater detail in Sections 3 and 4.

Definition. Let M be an Orlicz function. If $M'(0) = 0$, then M is said to be an *M-function*.

If M is an M-function, then we can define the Young complementary M-function M^* of M, where $(M^*)'(t) = \sup\{s: M'(s) \leqslant t\}$ and $M^*(y) = \int_0^y M^{*\prime}(t)\,dt$. Recall that for Orlicz sequence spaces, we have $(c_M)^* \cong l_{M^*}$. We want to prove a corresponding result for modular sequence spaces. But in order to do that, the sequence must be a sequence of M-functions. So we need our next proposition. Before stating that proposition, we need two more definitions.

Definition. A sequence of Orlicz functions $\{M_n\}$ is said to satisfy the *uniform Δ_2 condition* if there exist $p \geqslant 1$ and n_0 such that for all $x \in (0, 1)$ and $n > n_0$, we have $xM_n'(x)/M_n(x) \leqslant p$.

It is said to satisfy the *uniform Δ_2^* condition* if there exist $q > 1$ and n_0 such that for all $x \in (0, 1)$ and $n > n_0$, we have $xM_n'(x)/M_n(x) \geqslant q$.

Proposition 2.5. *Every sequence $\{M_n\}$ of Orlicz functions is equivalent to a sequence $\{N_n\}$ of M-functions. Moreover, if $\{M_n\}$ satisfies the uniform Δ_2 condition (resp. the uniform Δ_2^* condition), then so does $\{N_n\}$.*

Proof. Let $\{x_n\} \in l\{M_n\}$, $x_n > 0$ and $\sum M_n(x_n) < 1$. Then $x_n < 1$ for all n. Put $p_n = x_nM_n(x_n)/M_n(x_n)$. First assume $p_n > 1$ for all n. Define

$$N_n(x) = \begin{cases} x^{p_n}M_n(x_n)/x_n^{p_n}, & x \in [0, x_n], \\ M_n(x), & x > x_n. \end{cases}$$

Clearly, N_n is an M-functions and $\{M_n\}$, $\{N_n\}$ are almost equal. Suppose $\{M_n\}$ satisfies the uniform Δ_2 condition. Then there exist p, n_0 such that $xM_n'(x)/M_n(x) \leqslant p$ for all $x \in (0, 1)$ and $n > n_0$. So $p_n \leqslant p$ for all $n > n_0$.

$$\frac{xN_n'(x)}{N_n(x)} = \begin{cases} p_n, & x \in (0, x_n), \\ \dfrac{xM_n'(x)}{M_n(x)}, & x \geqslant x_n. \end{cases}$$

Hence $\{N_n\}$ satisfies the uniform Δ_2 condition. The uniform Δ_2^* case is similar.

We now consider the case where we cannot have $p_n > 1$ for all n. So suppose $xM_n'(x)/M_n(x) = 1$ for all $x \in [0, x_n]$. Let E be the set of all such n's. For $n \in E$, define

$$M_n^{\#}(x) = \begin{cases} \dfrac{M_n(x_n)x}{2x_n - x_nM_n(x_n)}, & x \in [0, x_n], \\ \dfrac{M_n(x) - \frac{1}{2}M_n(x_n)}{1 - \frac{1}{2}M_n(x_n)}, & x \geqslant x_n. \end{cases}$$

Define $M_n^{\#}$ to be M_n for $n \notin E$. Then $\{M_n^{\#}\}$ is a sequence of Orlicz functions.

Let $n \in E$. It is not hard to show that $|M_n(x) - M_n^{\#}(x)| \leqslant M_n(x_n)$ for all $x \in [0, 1]$. So $\sum \sup_{x \in [0,1]} |M_n(x) - M_n^{\#}(x)| \leqslant \sum M_n(x_n) < \infty$ and $\{M_n\}$, $\{M_n^{\#}\}$ are equivalent. Also,

$$\frac{x(M_n^{\#})'(x)}{M_n^{\#}(x)} = \begin{cases} 1, & x \in (0, x_n), \\ \dfrac{xM_n'(x)\left(1 - \frac{1}{2}M_n(x_n)\right)}{M_n(x) - \frac{1}{2}M_n(x_n)}, & x \in [x_n, 1]. \end{cases}$$

So $x_n(M_n^{\#})'(x_n)/M_n^{\#}(x_n) = 2 - M_n(x_n) > 1$. This reduces back to the first case.

Suppose $\{M_n\}$ satisfies the uniform Δ_2 condition. Let p, n_0 be as in the definition. For $n > n_0$, $n \in E$ and $x \in [x_n, 1]$,

$$\frac{x(M_n^{\#})'(x)}{M_n^{\#}(x)} = \frac{xM_n'(x)}{M_n(x)} \cdot \frac{M_n(x)\left(1 - \frac{1}{2}M_n(x_n)\right)}{M_n(x) - \frac{1}{2}M_n(x_n)} \leqslant \frac{pM_n(x_n)\left(1 - \frac{1}{2}M_n(x_n)\right)}{M_n(x_n) - \frac{1}{2}M_n(x_n)} \leqslant 2p.$$

So $\{M_n^{\#}\}$ satisfies the uniform Δ_2 condition. Note that $\{M_n\}$ cannot satisfy the uniform Δ_2^* condition if E is infinite.

If M_n' is strictly increasing and continuous, then $(M_n^*)' = (M_n')^{-1}$. This simplifies computations a lot. Hence the following proposition is quite useful for Section 4.

PROPOSITION 2.6. *Let $\{M_n\}$ be a sequence of M-functions satisfying the uniform Δ_2 contition. Then $\{M_n\}$ is equivalent to a sequence $\{N_n\}$ of M-functions satisfying the uniform Δ_2 condition, and N_n' is continuous and strictly increasing for all n. Moreover, if $\{M_n\}$ satisfies the uniform Δ_2^* condition, so does $\{N_n\}$.*

Proof. Let p, n_0 be such that $xM_n'(x)/M_n(x) \leqslant p$ for all $x \in (0,1)$ and $n > n_0$. Define

$$N_n(x) = \frac{\int_0^x \big(M_n(t)/t\big)\,dt}{\int_0^1 \big(M_n(t)/t\big)\,dt}.$$

Clearly, N_n is an Orlicz function, with $N_n(1) = 1$. As $M_n'(x) \geqslant M_n(x)/x$, $\lim_{x\to 0} M_n(x)/x = 0$. So $N_n'(0) = 0$ and N_n is an M-function.

Since $1 \leqslant tM_n'(t)/M_n(t) \leqslant p$ for all $n > n_0$ and $t \in (0,1)$, we have

$$M_n(t)/t \leqslant M_n'(t) \leqslant pM_n(t)/t.$$

Therefore

$$\int_0^x \big(M_n(t)/t\big)\,dt \leqslant M_n(x) \leqslant p\int_0^x \big(M_n(t)/t\big)\,dt.$$

Thus we have

$$N_n(x)/p \leqslant M_n(x) \leqslant pN_n(x)$$

for all $x \in [0,1]$ and $n > n_0$. So $\{M_n\}$, $\{N_n\}$ are equivalent.

For $x \in (0,1)$ and $n > n_0$, it is easy to see that $xN_n'(x)/N_n(x) \leqslant p$. So $\{N_n\}$ satisfies the uniform Δ_2 condition.

$$N_n'(x) = \frac{M_n(x)/x}{\int_0^1 \big(M_n(t)/t\big)\,dt}.$$

Hence N_n' is continuous. As M_n is an M-function, $M_n(x)/x$ is strictly increasing. So N_n' is strictly increasing.

Finally, suppose $\{N_n\}$ satisfies the uniform Δ_2^* condition. Then there exist $q > 1$ and n_0 such that $xM_n'(x)/M_n(x) \geqslant q$ for all $x \in (0,1)$ and $n > n_0$. It is easy to show that $xN_n'(x)/N_n(x) \geqslant q$ for all $x \in (0,1)$ and $n > n_0$. Hence $\{N_n\}$ satisfies the uniform Δ_2^* condition.

3. The uniform Δ_2 condition. The uniform Δ_2 condition has been introduced in Section 2. We are going to study its consequences in some

detail. The Δ_2 condition (for small x) for an Orlicz function M is usually defined as follows: For all $\omega > 1$, there exist $K(\omega) \geqslant 1$ and $a(\omega) > 0$ such that $M(\omega x) \leqslant K(\omega)M(x)$ for all $x \in [0, a(\omega)]$. We now show that we have a corresponding definition for the uniform Δ_2 condition.

PROPOSITION 3.1. *Let $\{M_n\}$ be a sequence of Orlicz functions. Then the following are equivalent:*

(a) *$\{M_n\}$ satisfies the uniform Δ_2 condition.*

(b) *For all $\omega > 1$, there exist $a(\omega) > 0$, $K(\omega) \geqslant 1$ and n_0 independent of ω, such that*

(i) *$M_n(\omega x) \leqslant K(\omega)M_n(x)$ for all $x \in [0, a(\omega)]$ and $n > n_0$,*

(ii) *$\underline{\lim}_{\omega\to 1} a(\omega) \geqslant 1$,*

(iii) *$\overline{\lim}_{\omega\to 1} \dfrac{K(\omega)-1}{\omega-1} < \infty$.*

Proof. (a) $\Rightarrow$ (b). Suppose there exist $p \geqslant 1$ and n_0 such that

$$xM_n'(x)/M_n(x) \leqslant p \quad \text{for all } x \in (0,1) \text{ and } n > n_0.$$

First consider $1 < \omega \leqslant \omega_0 < p/(p-1)$. Define $K(\omega) = \omega/(p-\omega p+\omega)$ and $a(\omega) = \omega^{-1}$. As $\omega < p/(p-1)$, $p-\omega p+\omega > 0$.

Let $n > n_0$ and $x \in [0, a(\omega)]$. Then $\omega x < 1$. By the Mean Value Theorem for convex functions, there exists $\beta_n \in (x, \omega x) \subset (0,1)$ such that

$$\frac{M_n(\omega x)-M_n(x)}{\omega x - x} \leqslant M_n'(\beta_n) \leqslant \frac{pM_n(\beta_n)}{\beta_n} \leqslant \frac{pM_n(\omega x)}{\omega x}.$$

Hence $(p-\omega p+\omega)M_n(\omega x) \leqslant \omega M_n(x)$ and we have $M_n(\omega x) \leqslant K(\omega)M_n(x)$.

Consider now $\omega > \omega_0$. There exists a smallest $m \in \mathbf{Z}^+$ such that $\omega^m \geqslant \omega$. Define $K(\omega) = K(\omega_0)^m$ and $a(\omega) = a(\omega_0)\omega_0^{-m}$. Then for all $n > n_0$ and $x \in [0, a(\omega)]$,

$$\begin{aligned} M_n(\omega x) &\leqslant M_n(\omega_0^m x) \leqslant K(\omega_0)M_n(\omega_0^{m-1}x) \\ &\leqslant \ldots \leqslant K(\omega_0)^m M_n(x) = K(\omega)M_n(x). \end{aligned}$$

Finally, it is clear that $a(\omega)$, $K(\omega)$ satisfy (ii), (iii) respectively.

(b) $\Rightarrow$ (a). Suppose $x \in (0,1)$. Beacause $\overline{\lim}_{\omega\to 1}\big(K(\omega)-1\big)/(\omega-1) < \infty$, we have p and $\omega_0 > 1$ such that $\big(K(\omega)-1\big)/(\omega-1) \leqslant p$ for all $\omega \in (1, \omega_0)$. Also, $\underline{\lim}_{\omega\to 1} a(\omega) \geqslant 1$ implies that there is some $\omega \in (1, \omega_0)$ such that $x < a(\omega)$. Then for $n > n_0$,

$$\frac{xM_n'(x)}{M_n(x)} \leqslant \frac{x}{M_n(x)}\,\frac{M_n(\omega x)-M_n(x)}{\omega x - x} = \frac{M_n(\omega x)M_n(x)^{-1}-1}{\omega-1} \leqslant \frac{K(\omega)-1}{\omega-1} = p.$$

So M_n satisfies the uniform Δ_2 condition.

Proposition 3.2. *Suppose $\{M_n\}$ satisfies the uniform Δ_2 condition (resp. the uniform Δ_2^* condition). Then there exists p (resp. $q > 1$) and n_0 such that $M_n(x) \geqslant x^p$ (resp. $M_n(x) \leqslant x^q$) for all $x \in [0, 1]$ and $n > n_0$.*

Proof. Suppose $xM_n'(x)/M_n(x) \leqslant p$ for all $x \in (0, 1)$ and $n > n_0$. Then $M_n'(t)/M_n(t) \leqslant p/t$ for all $t \in (0, 1)$. Hence $\int_x^1 M_n'(t)/M_n(t)\,dt \leqslant \int_x^1 p/t\,dt$ for all $x \in [0, 1]$, i.e. $\log M_n(x) \geqslant p \log x$, i.e. $M_n(x) \geqslant x^p$ for all $x \in [0, 1]$. The other case is similar.

Corollary 3.3. *Suppose $\{M_n\}$ satisfies the uniform Δ_2 condition. Then $\sum M_n(|x_n|) < \infty$ implies $\lim x_n = 0$.*

Proposition 3.4. *Suppose $\{M_n\}$ is a sequence of Orlicz functions satisfying the following condition:*

$$(2)\quad \begin{cases} \text{There exist } p, n_0 \text{ and } a \in (0, 1) \text{ such that} \\ \text{(i) } \inf M_n(a) > 0, \\ \text{(ii) } xM_n'(x)/M_n(x) \leqslant p \text{ for all } x \in [0, a] \text{ and } n > n_0. \end{cases}$$

Then $\{M_n\}$ is equivalent to a sequence $\{N_n\}$ satisfying the uniform Δ_2 condition.

Proof. Let $p_n = aM_n'(a)/M_n(a)$ and define

$$N_n(x) = \begin{cases} a^{p_n} M_n(x)/M_n(a), & x \in [0, a], \\ x^{p_n}, & x \geqslant a. \end{cases}$$

Then N_n is an Orlicz function and $\{N_n\}$ satisfies the uniform Δ_2 condition.

Let $c = \inf M_n(a) > 0$. As $p_n \leqslant p$ for $n > n_0$, and as $a \leqslant 1$, we have $c^{-1} \geqslant a^{p_n}/M_n(a) \geqslant a^p > 0$ for all $n > n_0$. So for $n > n_0$ and $x \in [0, a]$, we have $c^{-1}M_n(x) \geqslant N_n(x) \geqslant a^p M_n(x)$ and $\{M_n\}$, $\{N_n\}$ are equivalent.

Remarks. (a) In (2), (ii) can be replaced by

(ii)′ *For all $\omega > 0$, there exist $a(\omega) > 0$, $K(\omega) \geqslant 1$, and n_0 independent of ω, such that*

$$M_n(\omega x) \leqslant K(\omega) M_n(x)$$

for all $x \in [0, a(\omega)]$ and $n > n_0$.

The proof is similar to that of Proposition 3.1.

(b) (2) is a more natural analogue to the Δ_2 condition (for small x) of an Orlicz function. However, if we use this more general condition as our definition of the uniform Δ_2 condition, we would encounter technical difficulties when considering duals. In particular, (3), (i) is hard to verify for the complementary functions $\{M_n^*\}$.

We are now ready to prove the main result of this paper.

Theorem 3.5. *Let $\{M_n\}$ be a sequence of Orlicz functions. Then the following are equivalent:*

(a) $\{M_n\}$ is almost equal to a sequence $\{N_n\}$ satisfying condition (2) in Proposition 3.4.

(b) $\{M_n\}$ is equivalent to a sequence $\{N_n\}$ satisfying the uniform Δ_2 condition.

(c) $l\{M_n\} = c\{M_n\}$.

(d) $\sum M_n(|x_n|) < \infty$ for all $\{x_n\} \in l\{M_n\}$.

(e) The unit vectors form an unconditional basis of $l\{M_n\}$.

(f) $l\{M_n\}$ has an (unconditional) basis.

(g) $l\{M_n\}$ is separable.

(h) $l\{M_n\}$ has no subspace isomorphic to l_∞.

(i) $c\{M_n\}$ has no subspace isomorphic to c_0.

Proof. (a) ⇒ (b) is Proposition 3.4, (c) ⇔ (d) ⇔ (e) is Lemma 2.2, and (e) ⇒ (f) ⇒ (g) ⇒ (h) are trivial.

(b) ⇒ (c). We first show that $c\{N_n\} = l\{N_n\}$. Suppose $\{x_n\} \in l\{N_n\}$. Then there exists $t > 0$ such that $\sum N_n(|x_n|/t) < \infty$. Let $s > 0$ be arbitrary. If $s \geqslant t$, then clearly $\sum N_n(|x_n|/s) < \infty$. If $s < t$, then $t/s > 1$. By Proposition 3.1, we have n_1, $a(t/s)$ and $K(t/s)$ such that $N_n(tx/s) \leqslant K(t/s) N_n(x)$ for all $x \in [0, a(t/s)]$ and $n > n_1$. By Corollary 3.3, $\lim |x_n|/t = 0$. So there exists n_2 such that $|x_n|/s < a(t/s)$ for all $n > n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then

$$\sum_{n=1}^{\infty} N_n\left(\frac{|x_n|}{s}\right) < \sum_{n=1}^{n_0} N_n\left(\frac{|x_n|}{s}\right) + K\left(\frac{t}{s}\right) \sum_{n=n_0+1}^{\infty} N_n\left(\frac{|x_n|}{t}\right) < \infty.$$

Hence $\{x_n\} \in c\{N_n\}$ and $c\{N_n\} = l\{N_n\}$.

Since $\{M_n\}$, $\{N_n\}$ are equivalent, $c\{M_n\} = c\{N_n\}$. So $c\{M_n\} = c\{N_n\} = l\{N_n\} = l\{M_n\}$.

(c) ⇔ (i) follows from the fact that the unit vector basis in $c\{M_n\}$ is boundedly complete iff $l\{M_n\} = c\{M_n\}$. So the equivalence follows from James′ Theorem.

(h) ⇒ (a). For each $m, n \in Z^+$, define

$$x_{m,n} = \sup\{x \in [0, 2^{-m}]:\ xM_n'(x) \geqslant 2^m M_n(x)\}.$$

Because M_n is continuous and M_n' is increasing, we have

$$x_{m,n} M_n'(x_{m,n}) \geqslant 2^m M_n(x_{m,n}).$$

Claim. There exists some $m \in Z^+$ such that $\sum_{n=1}^{\infty} M_n(x_{m,n}) < \infty$. Otherwise, by induction, we can choose a sequence of positive integers $p_1 < p_2 < \ldots < p_m < \ldots$ such that

$$2^{-m} \leqslant \sum_{n=p_m+1}^{p_{m+1}} M_n(x_{m,n}) \leqslant 2^{-m+1}.$$

This follows from $M_n(x_{m,n}) \leqslant x_{m,n} \leqslant 2^{-m}$ and $\sum_{n=1}^{\infty} M_n(x_{m,n}) = \infty$ for all m. Note that $x_{m,n}M'_n(x_{m,n}) \geqslant 2^m M_n(x_{m,n})$ implies

$$M_n(2x_{m,n}) \geqslant 2^m M_n(x_{m,n}).$$

For if $x_{m,n} = 0$, there is nothing to prove. If $x_{m,n} > 0$, then

$$\frac{M_n(2x_{m,n})}{M_n(x_{m,n})} \geqslant \frac{M_n(2x_{m,n})-M_n(x_{m,n})}{x_{m,n}}\,\frac{x_{m,n}}{M_n(x_{m,n})} \geqslant \frac{M'_n(x_{m,n})x_{m,n}}{M_n(x_{m,n})} \geqslant 2^m.$$

Now put $w_m = \sum_{n=p_m+1}^{p_{m+1}} x_{m,n}e_n$, where $\{e_n\}$ are the unit vectors. Define $T\colon l_\infty \to l\{M_n\}$ by $T(\{a_m\}) = \sum a_m w_m$. T is well defined because

$$\sum_{m=1}^{\infty}\sum_{n=p_m+1}^{p_{m+1}} M_n\left(\frac{|a_m|x_{m,n}}{2\,\|\{a_i\}\|}\right) \leqslant \sum_{m=1}^{\infty}\sum_{n=p_m+1}^{p_{m+1}} \frac{1}{2} M_n(x_{m,n}) \leqslant \sum_{m=1}^{\infty} 2^{-m} = 1.$$

So $T(\{a_m\}) \in l\{M_n\}$, and $\|T\{a_m\}\| \leqslant 2\,\|\{a_m\}\|$. T is clearly linear, bounded and injective. It remains to prove that T^{-1} is bounded. Suppose $\|T(\{a_m\})\| = 1$. Claim: $|a_m| \leqslant 2$ for all m. Otherwise, we have $|a_k| > 2$ for some k. Then

$$\sum_{m=1}^{\infty}\sum_{n=p_m+1}^{p_{m+1}} M_n(|a_m|x_{m,n}) \geqslant \sum_{n=p_k+1}^{p_{k+1}} M_n(|a_k|x_{k,n})$$
$$> \sum_{n=p_k+1}^{p_{k+1}} M_n(2x_{k,n}) \geqslant \sum_{n=p_k+1}^{p_{n+1}} 2^k M_n(x_{k,n}) \geqslant 1.$$

Hence $\|T(\{a_m\})\| > 1$, a contradiction. So $\|T^{-1}\| \leqslant 2$. This implies $l\{M_n\}$ contains a subspace isomorphic to l_∞, which is impossible.

So there exists some m such that $\sum_{n=1}^{\infty} M_n(x_{m,n}) < \infty$. Take $y_n > x_{m,n}$ such that $M_n(y_n) = M_n(x_{m,n}) + 2^{-n}$. Define

$$N_n(x) = \begin{cases} M_n(x), & x \geqslant y_n, \\ xM_n(y_n)/y_n, & x \leqslant y_n. \end{cases}$$

Clearly $\{N_n\}$ is a sequence of Orlicz functions almost equal to $\{M_n\}$ and satisfying $xN'_n(x)/N_n(x) \leqslant 2^m$ for all n and $x \in [0, 2^{-m}]$. Moreover, $\inf M_n(2^{-m}) > 0$. Otherwise we have $\{M_{n_i}\}$ such that $\sum_{i=1}^{\infty} M_{n_i}(2^{-m}) < \infty$, and by Lemma 2.3, $l\{M_n\}$ contains l_∞, contradicting (h). Hence $\{N_n\}$ satisfies (2), and (a) is proved.

The next theorem is essentially Lemma 3.6. of [3].

THEOREM 3.6. *Suppose $\{M_n\}$ satisfies the uniform Δ_2 condition. Then $l\{M_n\}$ contains a complemented subspace isomorphic to l_M for some Orlicz function M.*

Proof. Let n_0 and p be such that $xM'_n(x)/M_n(x) \leqslant p$ for all $x \in (0, 1)$ and $n > n_0$.

Claim. $\{M_n\}$ is uniformly equicontinuous on $[0, 1]$. For let $x, y \in [0, 1]$ Then there exists ω_n between x and y such that

$$|M_n(x) - M_n(y)| \leqslant M'_n(\omega_n)\,|x-y| \leqslant p\,\frac{M_n(\omega_n)}{\omega_n}\,|x-y| < p\,|x-y|.$$

So $\{M_n\}$ is uniformly equicontinuous on $[0, 1]$.

$M_n(1) = 1$ for all n implies $\{M_n\}$ is uniformly bounded on $[0, 1]$. So by compactness in $C[0, 1]$, there exists a subsequence $\{M_{n_i}\}$ of $\{M_n\}$ converging to a convex function M on $[0, 1]$. We can select the subsequence such that

$$\sum_{i=1}^{\infty} \sup\{|M_{n_i}(x) - M(x)|\colon x \in [0, 1]\} < \infty.$$

It is easy to see that M is an Orlicz function satisfying the Δ_2 condition, and the subspace generated by $\{e_{n_i}\}$ is isomorphic to l_M. Because $\{e_n\}$ is unconditional, this subspace is complemented in $l\{M_n\}$.

COROLLARY 3.7. *Every modular sequence space contains a subspace isomorphic to l_p for some $p \geqslant 1$.*

Proof. If $\{M_n\}$ is not equivalent to any $\{N_n\}$ that satisfies the uniform Δ_2 condition, $l\{M_n\}$ contains l_∞ by Theorem 3.5 and so contains every l_p, $p \in [1, \infty)$.

Otherwise, by Theorem 3.6, $l\{M_n\}$ contains l_M for some M, which by [4] contains l_p for some $p \in [1, \infty)$.

DEFINITION. A basis is called a *modular basis* if it is equivalent to the unit vector basis of some modular sequence space.

It is easy to see that every normalized block basis of a modular basis is a modular basis.

THEOREM 3.8. *Suppose $\{M_n\}$ satisfies the uniform Δ_2 condition. Then every subspace of $l\{M_n\}$ contains l_p for some $p \in [1, \infty)$.*

Proof. Let X be a subspace of $l\{M_n\}$. By C.2 of [1], X contains a basic sequence equivalent to a normalized block basis of the unit vector basis of $l\{M_n\}$. By our remarks above, X contains a modular sequence space. So by Corollary 3.7, X contains l_p for some $p \in [1, \infty)$.

Finally, Corollaries 3.9 and 3.10 of [3] can be reformulated as the following propositions:

PROPOSITION 3.9. *$l\{M_n\}$ is isomorphic to a separable Orlicz sequence space iff it has a symmetric basis.*

PROPOSITION 3.10. *Let X be a subspace of a separable modular sequence space. Suppose X has an unconditional basis. Then X contains a complemented subspace isomorphic to some Orlicz sequence space.*

4. Duals and reflexivity. The complementary M-function of an M-function has been introduced in Section 2. By Proposition 2.5, every sequence $\{M_n\}$ of Orlicz functions is equivalent to a sequence of M-functions. So we can restrict ourselves to M-functions throughout this section. We are now going to prove $l\{M_n^*\} \cong l\{M_n\}^*$ if $\{M_n\}$ satisfies the uniform Δ_2 condition.

Note that in general $M_n^*(1) \neq 1$, which is rather inconvenient.

We first renorm $l\{M_n\}$ as follows. Let $l^{\#}\{M_n\}$ be $\{\{x_n\}: \sum |x_n y_n| < \infty$ for all $\{y_n\}$ such that $\sum M_n^*(|y_n|) \leqslant 1\}$. Define $|||\{x_n\}|||$ to be $\sup\{\sum |x_n y_n|: \sum M_n^*(|y_n|) \leqslant 1\}$. It is not hard to show that $(l^{\#}\{M_n\}, |||\cdot|||)$ is a Banach space. The following proposition is very easy to prove, using Young's inequality.

PROPOSITION 4.1. (a) *For all $\{x_n\} \in l^{\#}\{M_n\}$, $\sum |x_n y_n| \leqslant |||\{x_n\}|||$ if $\sum M_n^*(|y_n|) \leqslant 1$.*

(b) *For all $\{x_n\} \in l^{\#}\{M_n\}$, $\sum |x_n y_n| \leqslant |||\{x_n\}||| \sum M_n^*(|y_n|)$ if $M_n^*(|y_n|) \geqslant 1$.*

(c) *For all $\{x_n\} \in l^{\#}\{M_n\}$,*

$$\sum_n M_n^*(M_n'(|x_n|/|||\{x_i\}|||)) \leqslant 1 \quad \text{and} \quad \sum_n M_n(|x_n|/|||\{x_i\}|||) \leqslant 1.$$

(d) *$l\{M_n\} = l^{\#}\{M_n\}$ and $\|\{x_n\}\| \leqslant |||\{x_n\}||| \leqslant 2\|\{x_n\}\|$ for all $\{x_n\} \in l\{M_n\}$.*

THEOREM 4.2. *$c\{M_n\}^* \cong l\{M_n^*\}$. Hence if $\{M_n\}$ satisfies the uniform Δ_2 condition, $l\{M_n\}^* \cong l\{M_n^*\}$.*

Proof. Suppose $\{y_n\} \in l\{M_n^*\}$. Define $T\{y_n\}: c\{M_n\} \to \boldsymbol{R}$ by $T\{y_n\}(\{x_n\}) = \sum x_n y_n$. By Proposition 4.1 (a), $\sum |x_n y_n| \leqslant |||\{x_n\}|||\, \|\{y_n\}\|$. Hence $T\{y_n\}$ is well defined, and $\|T\{y_n\}\| \leqslant \|\{y_n\}\|$. So $T\{y_n\} \in c\{M_n\}^*$.

We thus have a linear map $T: l\{M_n^*\} \to c\{M_n\}^*$. It is clear that T is injective and bounded. It remains to show that T is surjective. Let $f \in c\{M_n\}^*$ and let $\{e_n\}$ be the unit vector basis of $c\{M_n\}$. Put $y_n = f(e_n)$.

Claim. $\{y_n\} \in l^{\#}\{M_n^*\}$. For suppose $\sum M_n(|x_n|) \leqslant 1$. Let $\omega_n = \operatorname{sgn}(x_n y_n)$. Then for all k, $\sum_{n=1}^{k} |x_n y_n| = \sum_{n=1}^{k} \omega_n x_n y_n = f\left(\sum_{n=1}^{k} \omega_n x_n e_n\right) \leqslant \|f\| \left\|\sum_{n=1}^{k} \omega_n x_n e_n\right\| = \|f\| \left\|\sum_{n=1}^{k} x_n e_n\right\| \leqslant \|f\| \left\|\sum_{n=1}^{\infty} x_n e_n\right\|$. As k is arbitrary,

$$\sum_{n=1}^{\infty} |x_n y_n| \leqslant \|f\|\, \|\{x_n\}\|.$$

So $\sum |x_n y_n| \leqslant \|f\|$ for all $\{x_n\}$ such that $\sum M_n(|x_n|) \leqslant 1$. So $\{y_n\} \in l^{\#}\{M_n^*\} = l\{M_n^*\}$. As $\{e_n\}$ is a basis for $c\{M_n\}$,

$$\sum x_n y_n = \sum x_n f(e_n) = f\left(\sum x_n e_n\right),$$

i.e. $f = T\{y_n\}$.

EXAMPLE. Consider the space X_p, $p > 2$, defined in [7]. X_p can be normed by

$$\|\{x_n\}\| = \max\left\{\left(\sum |x_n|^p\right)^{1/p}, \left(\sum |x_n|^2 w_n^2\right)^{\frac{1}{2}}\right\},$$

where $w_n \to 0$ and $\sum w_n^{2p/(p-2)} = \infty$. Define $M_n(x) = \max\{x^p, w_n^2 x^2\}$, i.e.

$$M_n(x) = \begin{cases} w_n^2 x^2, & x \in [0, w_n^{2/(p-2)}], \\ x^p, & x \geqslant w_n^{2/(p-2)}. \end{cases}$$

Then M_n is an M-function for every $n \in Z^+$. It is not hard to see that the unit vector bases in X_p and $l\{M_n\}$ are equivalent. We also have $M_n(1) = 1$ and

$$x M_n'(x)/M_n(x) = \begin{cases} 2, & x \in [0, w_n^{2/(p-2)}), \\ p, & x \geqslant w_n^{2/(p-2)}. \end{cases}$$

So $\{M_n\}$ satisfies the uniform Δ_2 condition.

For $q \in (1, 2)$, X_q is defined to be X_p^*, where $p^{-1} + q^{-1} = 1$. So by Theorem 4.2, X_q is a modular sequence space. We are now going to compute M_n^*.

$$M_n'(x) = \begin{cases} 2w_n^2 x, & x \in [0, w_n^{2/(p-2)}), \\ p x^{p-1}, & x \geqslant w_n^{2/(p-2)}. \end{cases}$$

Hence

$$M_n^{*\prime}(x) = \begin{cases} x/2w_n^2, & x \in [0, 2w_n^{2/(2-q)}], \\ w_n^{2/(p-2)}, & x \in [2w_n^{2/(2-q)}, p w_n^{2/(2-q)}], \\ (x/p^1)^{/(p-1)}, & x \geqslant p w_n^{2/(2-q)}. \end{cases}$$

Therefore

$$M_n^*(x) = \begin{cases} x^2/4w_n^2, & x \in [0, 2w_n^{2/(2-q)}], \\ w_n^{2/(p-2)} x - w_n^{2q/(2-q)}, & x \in [2w_n^{2/(2-q)}, p w_n^{2/(2-q)}], \\ x^q/(q p^{q-1}), & x \geqslant p w_n^{2/(2-q)}. \end{cases}$$

$M_n^*(1)$ is a constant independent of n. So all the theorems in Section 3 hold for $\{M_n^*\}$.

$$\frac{x M_n^{*\prime}(x)}{M_n^*(x)} = \begin{cases} 2 & x \in [0, 2w_n^{2/(2-q)}) \\ [1 - (w_n^{2/(2-q)}/x)]^{-1} & x \in [2w_n^{2/(2-q)}, p w_n^{2/(2-q)}) \\ q & x \geqslant p w_n^{2/(2-q)}. \end{cases}$$

Therefore $x M_n^{*\prime}(x)/M_n^*(x) \in [q, 2]$ for all $x > 0$ and for all n, and $\{M_n^*\}$ satisfies the uniform Δ_2 condition. By Theorem 3.8, every subspace of X_q contains l_r for some $r \in (1, \infty)$.

Remark. It is well known that every subspace of X_p, $p > 2$, contains l_p or l_2. See, for example, Corollary 2 of [2].

We are now going to tackle the problem of $M_n^*(1) \neq 1$. We are going to show that if $\{M_n\}$ satisfies the uniform Δ_2^* condition, then $\{M_n^*\}$ is equivalent to a sequence $\{M_n^\#\}$ satisfying $M_n^\#(1) = 1$ for all n.

PROPOSITION 4.3. *Let $\{M_n\}$ be a sequence of M-functions satisfying the uniform Δ_2^* condition, and let*

$$M_n^\#(x) = M_n^*(x)/M_n^*(1).$$

Then $\{M_n^\#\}$ and $\{M_n^*\}$ are equivalent, and $M_n^\#(1) = 1$ for all $n \in \mathbf{Z}^+$.

Proof. It is enough to show that $\inf M_n^*(1) > 0$, since $M_n^*(1) \leqslant 1$ for all $n \in \mathbf{Z}^+$. Suppose $q > 1$ and n_0 are such that $xM_n'(x)/M_n(x) \geqslant q$ for all $x \in (0, 1)$ and $n > n_0$.

Claim. $M_n^*(1) \geqslant q^{-1/(q-1)} - q^{-q/(q-1)}$. For suppose $x_n = M_n^{*\prime}(1)$. Then by Young's Inequality,

$$x_n - M_n(x_n) = M_n^*(1).$$

By convexity, $y = M_n(x)$ would lie above the line $y = x - x_n + M_n(x_n) = x - M_n^*(1)$. By Proposition 3.2, $M_n(x) \leqslant x^q$ for all $n > n_0$. So $y = x - M_n^*(1)$ can at best be tangent to $y = x^q$. So $y = x - M_n^*(1)$ lies below $y = x - q^{-1/(q-1)} + q^{-q/(q-1)}$ and we have the necessary inequality.

THEOREM 4.4. *$l\{M_n\}$ is reflexive iff $\{M_n\}$ is equivalent to a sequence $\{N_n\}$ that satisfies the uniform Δ_2 and Δ_2^* conditions.*

Proof. $\Leftarrow$. By Proposition 2.5 and 2.6, we can assume $\{N_n\}$ to be a sequence of M-functions, and N_n' is continuous and strictly increasing. As $\{N_n\}$ satisfies the uniform Δ_2 condition, $l\{N_n\}^* \cong l\{N_n^*\}$. By Proposition 4.3, $l\{N_n^*\} \cong l\{N_n^\#\}$. We are going to show that $\{N_n^\#\}$ satisfies the uniform Δ_2 condition. Then $l\{N_n^\#\}^* \cong l\{N_n^{\#*}\} = l\{N_n\}$. By the way the isomorphisms are defined, we have $l\{N_n\} \cong l\{N_n\}^{**}$ under the canonical injection.

Let $q > 1$ and n_0 be such that $xN_n'(x)/N_n(x) \geqslant q$ for all $x \in (0, 1)$ and $n > n_0$. N_n' is strictly increasing and continuous. So for all $y \in (0, 1)$, there exists x_n such that $N_n'(x_n) = y$. Because $N_n'(1) \geqslant 1$, $x_n \in (0, 1)$. Hence for all $n > n_0$,

$$\frac{yN_n^{\#\prime}(y)}{N_n^\#(y)} = \frac{N_n'(x_n)N_n^{*\prime}(N_n'(x_n))}{N_n^*(N_n'(x_n))} = \frac{x_nN_n'(x_n)}{x_nN_n'(x_n) - N_n(x_n)} = \frac{1}{1 - \dfrac{N_n(x_n)}{x_nN_n'(x_n)}} \leqslant \frac{1}{1 - \dfrac{1}{q}} = \frac{q}{q-1}.$$

So $\{N_n^\#\}$ satisfies the uniform Δ_2 condition.

$\Rightarrow$. By Theorem 3.5, $\{M_n\}$ is equivalent to a sequence $\{N_n\}$ that satisfies the uniform Δ_2 condition, otherwise $l\{M_n\} \supset l_\infty$. By Propositions 2.5 and 2.6, we can assume N_n is an M-function, and N_n' continuous and strictly increasing.

Claim: $\inf N_n^*(1) > 0$. For let $N_n'(x_n) = 1$. Then as in Proposition 4.3,

$$x \geqslant N_n(x) \geqslant x - N_n^*(1) \quad \text{for all } x \in [0, 1].$$

Thus if $\inf N_n^*(1) = 0$, $N_n(x)$ converges to x uniformly on $[0, 1]$ as n tends to ∞, and $l\{N_n\}$ contains l_1, which is impossible by reflexivity.

We can now define $N_n^\#(x) = N_n^*(x)/N_n^*(1)$ as in Proposition 4.3. If $\{N_n^\#\}$ satisfies the uniform Δ_2 condition, then computing as in the proof of "$\Leftarrow$", we can easily show that $\{N_n\}$ satisfies the uniform Δ_2^* condition, and we are through.

If $\{N_n^\#\}$ does not satisfy the uniform Δ_2 condition, then by Theorem 3.5 (a), $\{N_n^\#\}$ is almost equal to $\{P_n\}$, which satisfies the following:

There exist $\beta > 0$, n_0, q such that

(i) $\inf P_n(\beta) > 0$, and

(ii) $yP_n'(y)/P_n(y) \leqslant q$ for all $y \in (0, \beta]$ and $n > n_0$.

We now use $\{P_n\}$ to construct $\{R_n\}$ equivalent to $\{N_n\}$ and satisfying the uniform Δ_2 and Δ_2^* conditions.

Let $\beta_n \geqslant 0$ be such that $N_n^\#(y) = P_n(y)$ for all $y \geqslant \beta_n$, and such that $\sum N_n^\#(\beta_n) < \infty$. We are going to construct $\alpha_n \geqslant 0$ such that $\sum N_n(\alpha_n) < \infty$, and then we shall define $Q_n(x) = N_n(x)$ for all $x \geqslant \alpha_n$. Because N_n' is strictly increasing and continuous, there exists $\alpha_n \geqslant 0$ such that $N_n'(\alpha_n) = \beta_n$. If n is large enough, $\beta_n \leqslant \beta$. So for n sufficiently large, we have

$$\frac{\beta_n N_n^{\#\prime}(\beta_n)}{N_n^\#(\beta_n)} = \frac{\beta_n P_n'(\beta_n)}{P_n(\beta_n)} \leqslant q.$$

Hence

$$\alpha_n\beta_n = \beta_n N_n^{\#\prime}(\beta_n) N_n^*(1) \leqslant qN_n^\#(\beta_n)N_n^*(1) \leqslant qN_n^\#(\beta_n).$$

Therefore $\sum \alpha_n\beta_n < \infty$. Since $N_n(\alpha_n) \leqslant \alpha_n\beta_n$, $\sum N_n(\alpha_n) < \infty$.

Put $\alpha = \inf N_n^{*\prime}(\beta)$. Claim: $\alpha > 0$. For $N_n^{*\prime}(\beta) = N_n^{\#\prime}(\beta)N_n^*(1) \geqslant N_n^\#(\beta)N_n^*(1)/\beta = P_n(\beta)N_n^*(1)/\beta \geqslant \inf P_n(\beta) \inf N_n^*(1)/\beta > 0$. As $\sum N_n(\alpha_n) < \infty$, there exists $n_1 > n_0$ such that $n > n_1$ implies $\alpha_n \leqslant \alpha$. So for $n > n_1$, $x \in [\alpha_n, \alpha]$ implies $N_n'(x) \in [\beta_n, \beta]$ by continuity of N_n'. This shows that

$$\frac{xN_n'(x)}{N_n(x)} \geqslant \frac{q}{q-1} > 1.$$

Now define $\{Q_n\}$ by

$$Q_n(x) = \begin{cases} N_n(x), & x > \alpha_n, \\ N_n(\alpha_n)\left(\dfrac{x}{\alpha}\right)^{q_n}, & x \in [0, \alpha_n]. \end{cases}$$

where $q_n = a_n N'_n(a_n)/N_n(a_n)$. Clearly $\{Q_n\}$ is a sequence of M-functions almost equal to $\{N_n\}$. Because $\{N_n\}$ satisfies the uniform Δ_2 condition, $\{q_n\}$ is bounded. So $\{Q_n\}$ satisfies the uniform Δ_2 condition. For $n > n_1$ and $x \in (0, a]$,

$$\frac{xQ'_n(x)}{Q_n(x)} \geqslant \frac{q}{q-1} > 1$$

since $q_n \geqslant q/(q-1) > 1$.

Finally, we use the construction in Proposition 3.4 to construct $\{R_n\}$ equivalent to $\{Q_n\}$, and it is easy to see that $\{R_n\}$ satisfies both the uniform Δ_2 and Δ_2^* conditions.

Remark. As we remarked in the proof of Theorem 3.5 (i), the unit vector basis in $l\{M_n\}$ is boundedly complete iff $\{M_n\}$ satisfies the uniform Δ_2 condition. So $l\{M_n\}$ is reflexive iff it does not contain l_1.

We now generalize Lemma 3.1 and Proposition 3.2 of [3] to modular sequence spaces. As the proofs are practically the same, they will not be given.

LEMMA 4.5. *Let $l\{M_n\}$ be a separable modular sequence space. Let $\{B_k\}$ be a seminormalized block basic sequence of the unit vector basis $\{e_n\}$. Also suppose*

$$p \geqslant \frac{xM'_n(x)}{M_n(x)} \geqslant q$$

for all $x \in (0, 1)$ and $n > n_0$. Then $\sum |a_k|^p < \infty$ if $\sum a_k B_k$ converges, and $\sum a_k B_k$ converges if $\sum |a_k|^q < \infty$.

THEOREM 4.6. *Suppose $\{M_n\}$ and $\{N_n\}$ are sequences of Orlicz functions satisfying*

$$p_1 \geqslant \frac{xM'_n(x)}{M_n(x)} \geqslant q_1, \quad p_2 \geqslant \frac{xN'_n(x)}{N_n(x)} \geqslant q_2$$

for all $n > n_0$ and $x \in (0, 1)$. If $q_1 > p_2$, then $l\{M_n\}$ and $l\{N_n\}$ have no common infinite dimensional subspace.

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