$\lambda e - a$ does not belong to I. Since, by Lemma 3 of [3] any $\mathfrak{l}$-ideal is contained in an $\mathfrak{l}$-maximal ideal, it is sufficient to prove that there is an $\mathfrak{l}$-ideal J which properly contains I.

Let F_1, F_2 be finite subsets of I and $F_1 \supset F_2$. It is easy to see that

$$Z(F_1, a) \subset Z(F_2, a),$$

where the set $Z(F, a)$ is given by formula (19). Hence for any finite family $\{F_1, \ldots, F_n\}$ of finite subsets of I

$$Z(F_1, a) \cap \ldots \cap Z(F_n, a) \supset Z(F_1 \cup \ldots \cup F_n, a).$$

This means that the family of all subsets $Z(F, a)$, where F is any finite subset of I has the finite intersection property. Any F in I consists of joint topological divisors of zero. Hence by Lemma 4 any $Z(F, a)$ is a non-void compact set. Thus the family has a non-void intersection. Let λ_0 belong to the intersection. We notice, that the set $I \cup \{\lambda_0 e - a\}$ consists of joint topological divisors of zero. By Lemmas 1 and 2 in [3], which jointly state that any subset of A consisting of joint topological divisors of zero is contained in an $\mathfrak{l}$-ideal, there is an $\mathfrak{l}$-ideal J which contains I and $\lambda_0 e - a$. The inclusion is proper, because $\lambda_0 e - a$ does not belong to I. So we have obtained a contradiction of the assumption that I is a $\mathfrak{l}$-maximal ideal.

Remark. Since every maximal ideal is a prime ideal, Proposition 2 in [3], which states that every $\mathfrak{l}$-maximal ideal is a prime ideal, follows immediately from the Theorem.

COROLLARY. *If f is a functional in $\mathfrak{L}(A)$ and B is an extension of A, then f extends to a member F of $\mathfrak{L}(B)$.*

Proof. Since any $\mathfrak{l}$-ideal in A is contained in an $\mathfrak{l}$-ideal of B (Proposition 1 of [3]), the kernel of f is contained in an $\mathfrak{l}$-ideal of B. By the Theorem, the ideal is contained in an ideal in $\mathfrak{L}(B)$. The multiplicative-linear functional F in $\mathfrak{M}(B)$ coresponding to this ideal extends f and belongs to $\mathfrak{L}(B)$.

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Separability of orbits of functions on locally compact groups*

by

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Abstract. Let G be a locally compact group. First, if $f \in L^\infty(G)$ has a separable orbit under left translation by elements of G, then f is locally a.e. equal to a bounded and uniformly continuous function on G. Secondly, if $f \in L^\infty_{\mathrm{loc}}(G)$ has a separable orbit, then f is locally a.e. equal to a continuous function on G.

1. Notation. Let G be a locally compact group, dx a left invariant Haar measure ($\int_G f(sx)\,dx = \int_G f(x)\,dx$). We do not assume that G is compact or Abelian or separable. For $s \in G$ and f a function on G, the left translate of f by s is the function $(\gamma(s)f)(x) = f(s^{-1}x)$. We also write $f_s = \gamma(s)f$. For μ a measure on G, define $(\mu * f)(x) = \int_G f(s^{-1}x)\,d\mu(s)$ whenever this makes sense. In particular, for two complex-valued functions f and g on G we have $(g*f)(x) = \int_G f(s^{-1}x)g(s)\,ds$, which exists when $f(s^{-1}x)g(s) \in L^1$. Also, if ε_u denotes the unit mass at $u \in G$, then $(\varepsilon_u * f)(x) = (\gamma(u)f)(x)$; i.e. $\varepsilon_u * f = \gamma(u)f$. Clearly, then, $\gamma(u)(f*g) = (\gamma(u)f)*g$ whenever both members make sense.

The space $L^\infty(G)$ is the usual space of classes of bounded measurable functions. The space $L^\infty_{\mathrm{loc}}(G)$ is the space of classes of measurable functions on G that are bounded on compact subsets of G. Two functions belong to the same equivalence class if they agree except at most on a set that intersects every compact set in a set of zero measure. The topology of $L^\infty_{\mathrm{loc}}(G)$ is given by the seminorms

$$\|f\|_K = \operatorname{ess\,sup}\{|f(x)|:\ x \in K\}$$

as K runs over the compact subsets of G. We use the phrase "locally a.e." to mean "almost everywhere on each compact subset of G". By $C_{BU}(G)$ we mean the class of functions that are bounded and uniformly continuous on G.

2. Statements of results. Our first result is related to known results, but does not appear to be explicity stated in the literature. It was established independently for the circle group by T. Kaczyński (unpubli-

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shed). A closely related result was proved by R. E. Edwards in ([3], Corollary 1, p. 405) for the case $\boldsymbol{R}^n$ and it seems likely that his methods would work in the general case. There is other closely related work on pp. 312–313 of the paper [1] by Dunkl and Ramirez.

THEOREM 1. *Suppose that G is a locally compact group and that $f \in L^\infty(G)$ is such that the set of left translates of f is separable as a subset of $L^\infty(G)$. Then there is a bounded and uniformly continuous function F on G such that $f = F$ locally a.e.*

COROLLARY 1. *Each separable closed subset of $L^\infty(G)$ that is invariant under left translation is contained in $C_{BU}(G)$.*

This is equivalent to the following statement, which is reminiscent of Theorem 3 of [5].

COROLLARY 2. *If B is a separable subspace of $L^\infty(G)$ that is invariant under left translation and that contains $C_{BU}(G)$, then $B = C_{BU}(G)$.*

COROLLARY 3. *On the circle group, if B is a separable subspace of H^∞ that is rotation invariant and that contains the disc algebra A, then $B = A$.*

Here, A is the uniform closure on the circle of the algebraic polynomials.

THEOREM 2. *Suppose that G is a locally compact group and that $f \in L^\infty_{\mathrm{loc}}(G)$ is such that the set of left translates of f is separable as a subset of $L^\infty_{\mathrm{loc}}(G)$. Then there is a continuous function F on G such that $f = F$ locally a.e.*

Notice that the example $G = \boldsymbol{R}$ and $f(x) = e^x$ shows that there is no hope of concluding from the hypotheses of the theorem either that f is bounded or uniformly continuous.

COROLLARY. *Let E be a separable Banach space on which G acts, and let $T\colon E \to L^\infty_{\mathrm{loc}}(G)$ be a bounded linear transformation that commutes with the action of G. Then the range of T consists (up to locally a.e. equivalence) of continuous functions.*

3. Proof of Theorem 1. Our proof uses a minor modification of an argument due to K. W. Tam [7] followed by the use of a theorem of D. A. Edwards [2].

LEMMA. *If G is a locally compact group and if $g \in L^\infty(G)$ has a countable set of left translates $\{g_c\colon c \in C\}$ that is dense in the orbit of g under G, then $\lim_{s\to e} \|g_s - g\|_\infty = 0$, where e is the identity element of G.*

In case G is compact, this lemma follows from ([6], p. 234).

Proof. For each positive integer m, and each $c \in C$, write

$$S_c = S_{c,m} = \{x \in G\colon \|g_x - g_c\|_\infty \leqslant 1/m\}.$$

Then

$$G = \bigcup_{c\in C} S_{c,m}.$$

We prove that $S_{c,m}$ is closed. Choose $x_a \in S_{c,m}$ with $x_a \to x$ and take $f \in L^1(G)$. Then by, ([4], Theorem 20.4), $\|f(x_a s) - f(xs)\|_1 \to 0$. Hence

$$\lim_a \int f(s) g_{x_a}(s)\, ds = \lim \int f(x_a s) g(s)\, ds = \int f(s) g_x(s)\, ds.$$

Hence

$$\Big|\int f(s)\{g_x(s) - g_c(s)\}\, ds\Big| = \lim \Big|\int f(s)\{g_{x_a}(s) - g_c(s)\}\, ds\Big|$$
$$\leqslant \|f\|_1 \limsup \|g_{x_a} - g_c\|_\infty \leqslant \frac{1}{m}\|f\|_1$$

so that $\|g_x - g_c\|_\infty \leqslant 1/m$, and hence S_c is closed.

By the Baire category theorem for locally compact groups ([4], p. 456), there is a c_m such that

$$S_m = \{x\colon \|g_x - g_{c_m}\|_\infty \leqslant 1/m\}$$

has non-empty interior. So there is a neighborhood U_m of e such that for some $x_m \in S_m$, we have $x_m U_m \subseteq S_m$. Now for $s \in U_m$,

$$\|g_s - g\|_\infty = \|g_{sx_m} - g_{x_m}\| \leqslant \|g_{sx_m} - g_{c_m}\| + \|g_{c_m} - g_{x_m}\| \leqslant 2/m.$$

Hence $\lim_{s\to e} \|g_s - g\|_\infty = 0$ and the result is proved.

The proof of the theorem is concluded by a direct application of the main result of [2], where the author states and proves it for Abelian groups, but says correctly afterwards that it works as well for non-Abelian groups.

4. Proof of Theorem 2. Assume first that G is σ-compact, i.e., $G = \bigcup_{m=1}^{\infty} K_m$ where each K_m is compact. Let $\{\gamma(s_n)f\}$, $n = 1, 2, \ldots$, be dense in $\{\gamma(s)f\}_{s\in G}$ for the L^∞_{loc} topology. Consider, for each neighborhood U of the identity e of G, a function $\delta_U(x)$ with the properties that $\delta_U \geqslant 0$, δ_U is continuous, the support of δ_U is contained in U, $\int_G \delta_U(x)\, dx = 1$, and $\delta_U(x^{-1}) = \delta_U(x)$. Assume now that U is a compact symmetric neighborhood of e and that K is a compact subset of G. Then

$$\begin{aligned}
\int_K |f * \delta_U(x) - f(x)|\, dx &= \int_{x\in K} \Big| \int_{s\in KU} \{f(s) - f(x)\}\, \delta_U(s^{-1}x)\, ds \Big|\, dx \\
&\leqslant \int_{x\in K} \int_{s\in KU} |f(s) - f(x)|\, \delta_U(s^{-1}x)\, ds\, dx \\
&= \int_{s\in KU} \int_{x\in K} |f(s) - f(x)|\, \delta_U(s^{-1}x)\, dx\, ds \\
&\leqslant \int_{s\in KU} \int_{x\in UK^{-1}K} |f(s) - f(sx)|\, \delta_U(x)\, dx\, ds \\
&= \int_{x\in UK^{-1}K} \int_{s\in KU} |f(s) - f(sx)|\, ds\, \delta_U(x)\, dx.
\end{aligned}$$

Let

$$\psi(x) = \int_{s\in KU} |f(s)-f(sx)|\,ds.$$

Then we have

$$\int_K |f*\delta_U(x)-f(x)|\,dx \leqslant \int_{x\in UK^{-1}K} \psi(x)\,\delta_U(x)\,dx$$
$$\leqslant \sup\{\psi(x):\ x\in (UK^{-1}K)\cap U\} = \sup\{\psi(x):\ x\in U\}.$$

Let $g = f|_W$, where $W = KUU$, so that $g\in L^1(G)$. Then

$$\psi(x) = \int_{s\in KU} |g(s)-g(sx)|\,ds \leqslant \int_G |g(s)-g(sx)|\,ds.$$

Now by ([4], Theorem 20.4), ψ is continuous, and of course $\psi(e) = 0$. It follows that

$$\|f*\delta_U-f\|_{L^1(K)} \to 0$$

as U shrinks to $\{e\}$. This is true for each compact K, and therefore there is a sequence $\delta_n = \delta_{U_n}$, with $U_1 \supseteq U_2 \supseteq U_3 \supseteq \ldots$ and all U_n compact, such that $\|f*\delta_n-f\|_{L^1(K_m)} \to 0$ as $n\to\infty$ for each m. Since $G = \bigcup_{m=1}^{\infty} K_m$, for a further subsequence denoted again by $\{\delta_n\}$ we have $f*\delta_n \to f$ a.e. on G.

A second subsequence, again denoted by $\{\delta_n\}$ will satisfy $\gamma(s_m)f*\delta_n \to \gamma(s_m)f$ a.e. independently of m, and in particular there are points $x_0\in G$ such that $\big(\gamma(s_m)f*\delta_n\big)(x_0) \to \big(\gamma(s_m)f\big)(x_0)$ as $n\to\infty$, for each m. Now for an arbitrary $s\in G$,

$$|\gamma(s)f*\delta_n(x_0)-\gamma(s)f*\delta_m(x_0)| \leqslant \big|(\gamma(s)f-\gamma(s_j)f)*\delta_n(x_0)\big| +$$
$$+|\gamma(s_j)f*\delta_n(x_0)-\gamma(s_j)f*\delta_m(x_0)| + \big|(\gamma(s)f-\gamma(s_j)f)*\delta_m(x_0)\big|.$$

Since

$$\big|(\gamma(s)f-\gamma(s_j)f)*\delta_n(x_0)\big| \leqslant \int |[\gamma(s)f-\gamma(s_j)f]\,(s^{-1}x_0)|\,\delta_n(s)\,ds$$
$$\leqslant \|\gamma(s)f-\gamma(s_j)f\|_{L^\infty(U_1)},$$

we have $|\gamma(s)f*\delta_n(x_0)-\gamma(s)f*\delta_m(x_0)| \to 0$ as $m,n\to\infty$. Therefore we can define F pointwise as the limit of $f*\delta_n(s)$, since $f*\delta_n(s) = f*\delta_n(sx_0^{-1}x_0) = \gamma(x_0s^{-1})f*\delta_n(x_0)$ converges for all $s\in G$. Since $f*\delta_n\to f$ a.e. on G, we conclude that $f = F$ a.e. on G.

Now we have $|\gamma(s)F(x)-\gamma(t)F(x)| = \lim_{n\to\infty} |\gamma(s)f*\delta_n(x)-\gamma(t)f*\delta_n(x)|$. But

$$|\gamma(s)f*\delta_n(x)-\gamma(t)f*\delta_n(x)| \leqslant \int_G |[\gamma(s)f-\gamma(t)f]\,(u)|\,\delta_n(u^{-1}x)\,du$$
$$\leqslant \|\gamma(s)f-\gamma(t)f\|_{L^\infty(xU_n)} \leqslant \|\gamma(s)f-\gamma(t)f\|_{L^\infty(xU_1)},$$

whence

$$|\gamma(s)F(x)-\gamma(t)F(x)| \leqslant \|\gamma(s)f-\gamma(t)f\|_{L^\infty(xU_1)}$$

for each $x\in G$, $s\in G$, $t\in G$. It follows that

$$\sup_{x\in K} |\gamma(s)F(x)-\gamma(t)F(x)| \leqslant \|\gamma(s)f-\gamma(t)f\|_{L^\infty(KU_1)}$$

for each compact K in G. If t is replaced by s_{m_k} with $\gamma(s_{m_k})f\to\gamma(s)f$ in $L^\infty_{\mathrm{loc}}(G)$, it follows that $\gamma(s_{m_k})F$ converges uniformly to $\gamma(s)F$ on each compact subset of G. However $\gamma(s_{m_k})F = \lim_{n\to\infty}\gamma(s_{m_k})f*\delta_n$ everywhere, so that because $f*\delta_n$ is continuous, we know that $\gamma(s_{m_k})F$ is continuous off a set of first category. By the Baire category theorem for locally compact groups, there is some $z_0\in G$ at which the $\gamma(s_{m_k}F)$ are all continuous. Therefore $\gamma(s)F$ as the uniform limit of $\gamma(s_{m_k})F$ on each compact neighborhood of z_0 must also be continuous at z_0. Thus F is continuous everywhere.

We turn now to the general case, assuming that G is an arbitrary locally compact group. Consider an element $x\in G$ and an open, relatively compact symmetric neighborhood U of e. Let H be the subgroup generated by U, x, s_1, s_2, ... It is clear that H coincides with the union of all possible finite products of sets of the form U, xU, Ux^{-1}, s_nU, Us_n^{-1}, which are all relatively compact. Thus H is σ-compact.

Since $U\subseteq H$, H is also an open and hence closed subgroup of G, and therefore a σ-compact locally compact group. It is clear that $\{\gamma(s_n)f_H\}$, $n = 1,2,\ldots$, is still dense in $\{\gamma(s)f_H\}_{s\in H}$ (where f_H is $f|_H$) for the topology of $L^\infty_{\mathrm{loc}}(H)$ and therefore the first part applies. This shows that f_H is equivalent to a continuous function F_H on H, and in particular on Ux. This means that for each set Ux there is a continuous function F_x on it such that $F_x = F_y$ almost everywhere (and hence everywhere) on intersections $Ux\cap Uy$. Calling the global function F, it is well defined and continuous on each Ux and therefore continuous on G. Since $f = F$ a.e. on each Ux, it follows that $F = f$ locally a.e. and the proof is complete.

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On a problem of moments of S. Rolewicz

by

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Abstract. We solve a problem of moments raised by S. Rolewicz.

1. S. Rolewicz has proved the following result, with applications to minimum time problems of the theory of control:

THEOREM [5]. *Let E, F be two Banach spaces, u a continuous linear mapping of E into F, and y an element of F such that the equation $u(x) = y$ has a solution. If $u(S_E)$ is closed in F, where $S_E = \{x \in E \mid \|x\| \leqslant 1\}$ (the unit ball of E), then*

$$\inf_{\substack{x\in E\\ u(x)=y}} \|x\| = \sup_{g\in F^*} \inf_{\substack{x\in E\\ g(u(x))=g(y)}} \|x\|. \tag{1}$$

It is known (see [5], remark 1 and the references of [5]) that in the particular case when $\dim F < \infty$, formula (1) holds without any additional assumption; in particular, in this case the assumption that $u(S_E)$ is closed in F, is superfluous. At the Conference on Functional Analysis in October 1970 at Oberwolfach, S. Rolewicz has raised the problem whether (1) always holds without any additional assumption. In the present Note we shall solve this problem by giving a necessary and sufficient condition for the validity of (1) and an example in which this condition is not satisfied. Also, using our criterion, we shall show that the assumption that $u(E)$ is closed in F is sufficient, but not necessary, in order that we have (1).

2. The following theorem gives a necessary and sufficient condition for the validity of (1):

THEOREM 1. *Let E, F be two Banach spaces, u a continuous linear mapping of E into F, and y an element of F such that the equation $u(x) = y$ has a solution, say x_0. We have (1) if and only if*

$$\inf_{\substack{x\in E\\ u(x)=y}} \|x\| = \sup_{\substack{f\in u^*(F^*)\\ \|f\|=1}} |f(x_0)|. \tag{2}$$

Proof. If for each $g \in F^*$ we denote

$$H_g = \{x\in E \mid (u^*(g))(x) = g(y)\} = \{x \in E \mid g(u(x)) = g(y)\}, \tag{3}$$