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Application of approximation and interpolation methods to the examination of entire functions of n complex variables

by T. WINIARSKI (Kraków)

Abstract. The object of this paper is to give a characterization of the order and type and of the order and type systems of the entire function $f\colon C^n \to C$ by means of the Čebyšev best approximation to f on compact sets $E \subset C^n$ by polynomials. The methods of proof are based on the properties of the Leja–Siciak extremal function $\Phi(z, E)$.

1. Introduction. Let E be a bounded closed set in the space C^n of n complex variables $z = (z_1, \ldots, z_n)$.

We put

$$\|f\|_E = \sup\{|f(z)|\colon z \in E\}$$

for a function f defined and bounded on E.

Let $\mathscr{P}_\nu$ denote the set of polynomials in z of degree $\leqslant \nu$. Write

$$\mathscr{E}_\nu(f, E) = \inf\{\|f-p\|_E\colon p \in \mathscr{P}_\nu\}.$$

In the case of one complex variable the following theorems are known (see [2], [9] and [11]):

THEOREM 1. *A function f, defined and bounded on a closed set E with a positive transfinite diameter d, can be continued to an entire function $\tilde{f}$ of order ϱ $(0 < \varrho < \infty)$ and of type σ $(0 < \sigma < \infty)$, if and only if*

$$\limsup_{\nu\to\infty} \nu^{1/\varrho}\big(\mathscr{E}_\nu(f, E)\big)^{1/\nu} = d(e\sigma\varrho)^{1/\varrho}. \tag{1}$$

THEOREM 2. *If $d > 0$, then the order ϱ of $\tilde{f}$ is given by*

$$\varrho = \limsup_{\nu\to\infty} \frac{\nu \ln \nu}{-\ln \mathscr{E}_\nu(f, E)}.$$

The object of this paper is to extend these results to the case $n \geqslant 2$.

Let B be a complex Banach space with a norm $\|\cdot\|$. Let $f\colon C^n \to B$ be an entire function. Write

$$S_f(r) = \sup\{\|f(z)\|:\ \|z\| = r\},$$

$$\varrho = \limsup_{r\to\infty} \frac{\ln\ln S_f(r)}{\ln r},$$

$$\sigma = \limsup_{r\to\infty} \frac{\ln S_f(r)}{r^{\varrho}}, \quad \text{when } 0 < \varrho < \infty.$$

In connectiom with Theorem 1 the following question arises: does there exist a number $d = d(E)$ independent of the entire function f (of order ϱ and type σ) such that

$$d(f, E) = \limsup_{\nu\to\infty} \frac{\nu^{1/\varrho}\sqrt[\nu]{\mathscr{E}_\nu(f, E)}}{(e\sigma\varrho)^{1/\varrho}} = d(E).$$

The answer is negative. Indeed, if we take $E = E_1 \times E_2$, where E_j $(j = 1, 2)$ is a bounded closed set in the complex z_j-plane with a positive transfinite diameter $d_j = d(E_j)$ $(d_1 \neq d_2)$ and if

$$f_1(z_1, z_2) = g(z_1), \quad f_2(z_1, z_2) = g(z_2),$$

where g is an entire function of the order ϱ $(0 < \varrho < \infty)$ and type σ $(0 < \sigma < \infty)$, we have

$$d(f_1, E) = d_1, \quad d(f_2, E) = d_2.$$

Therefore $d(f, E)$ depends on f.

In the case $n > 1$, we replace the assumption $d(E) > 0$ by the assumption of local boundedness in C^n of the extremal function $\Phi(z, E)$. For such a set E Theorem 2 is true, but in Theorem 1 type σ cannot be determined by (1). It appears that the order ϱ of $\tilde{f}$ is given by

$$\varrho = \inf\{\mu > 0:\ \psi(\mu) < \infty\},$$

where $\psi(\mu) = \limsup_{\nu\to\infty} \nu^{1/\mu}\sqrt[\nu]{\mathscr{E}_\nu(f, E)}$.

Moreover, if $0 < \varrho < \infty$, then the type of $\tilde{f}$ is:

(a) minimal, when $\psi(\varrho) = 0$;

(b) normal, when $0 < \psi(\varrho) < \infty$;

(c) maximal, when $\psi(\varrho) = \infty$.

Furthermore, the function $\tilde{f}$ is given by

$$\tilde{f}(z) = \lim_{\nu\to\infty} L_\nu(z), \quad z \in C^n,$$

where L_ν is the ν-th Lagrange interpolation polynomial with nodes at extremal points of E (def. see [10]).

If $n > 1$, the type σ of $\tilde{f}$ cannot be characterized by means of the measure of the Čebyšev best approximation to f on E by polynomials of degree $\leqslant \nu$ with respect to all variables. So we have to consider the measures $\mathscr{E}_k^*(f, E)$, $k = (k_1, \ldots, k_n)$, of the Čebyšev best approximation to f on $E = E_1 \times \ldots \times E_n$ by polynomials of the degree $\leqslant k_j$ with respect to the j-th variable, $j = 1, \ldots, n$, where E_j is a bounded closed set with a positive transfinite diameter $d_j = d(E_j)$ in the complex z_j-plane.

In the case of $n \geqslant 1$ the following theorem will be proved:

THEOREM. *Two systems $\varrho = (\varrho_1, \ldots, \varrho_n)$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\varrho_j, \sigma_j > 0$, $j = 1, \ldots, n$ are order and type systems of an entire function f, respectively, if and only if*

$$\limsup_{\min\{k_j\} \to \infty} \sqrt[|k|]{\frac{\mathscr{E}_k^*(f, E)}{d^k} \left(\frac{k}{e\sigma\varrho}\right)^{k/\varrho}} = 1,$$

where $|k| = k_1 + \ldots + k_n$, $d^k = d_1^{k_1} \ldots d_n^{k_n}$ *and*

$$\left(\frac{k}{e\sigma\varrho}\right)^{k/\varrho} = \left(\frac{k_1}{e\sigma_1\varrho_1}\right)^{k_1/\varrho_1} \cdots \left(\frac{k_n}{e\sigma_n\varrho_n}\right)^{k_n/\varrho_n}.$$

I should like to thank Prof. J. Siciak for his valuable suggestions and remarks concerning the results presented in this paper.

2. Extremal function. As in [10], let us denote by $A_\nu(E)$ the set of all polynomials p of degree $\leqslant \nu$ such that

$$\|p\|_E \leqslant 1.$$

We define the extremal function [10]

$$\Phi(z) = \Phi(z, E) = \lim_{\nu \to \infty}\{\sup\{|p(z)|^{1/\nu} \colon p \in A_\nu(E)\}\}, \qquad z \in C^n.$$

It follows from the definition that $\Phi(z) \geqslant 1$ for $z \in C^n$ and $\Phi(z) = 1$ for $z \in E$. The following property is known [10].

PROPERTY 1. *If $E = E_1 \times \ldots \times E_n$, where E_j is a compact set in the complex z_j-plane, $j = 1, \ldots, n$, then*

$$\Phi(z, E) = \max\{\Phi(z_1, E_1), \ldots, \Phi(z_n, E_n)\}.$$

PROPERTY 2. *If $E = \{z \in C^n \colon \|z\| = r\}$, then $\Phi(z, E) = \max(1, \|z\|/r)$.*

PROPERTY 3. *If E, F $(E \subset F)$ are compact sets in C^n, then*

$$\Phi(z, F) \leqslant \Phi(z, E), \qquad z \in C^n.$$

PROPERTY 4. *If $p \colon C^n \to B$ (B is a complex Banach space with a norm $\|\cdot\|$) is a polynomial of degree $\leqslant \nu$, then*

$$\|p(z)\| \leqslant \|p\|_E \Phi^\nu(z), \qquad z \in C^n. \tag{2.1}$$

Property 4 follows from the definition of the extremal function and from the Hahn–Banach theorem [1].

THEOREM 2.1. *If the extremal function $\Phi(z, E)$ is locally bounded in C^n and if $R = \sup\{\|z\|: z \in E\}$, then there exists an $S \geqslant 1$ such that*

$$\max\left(1, \frac{\|z\|}{R}\right) \leqslant \Phi(z, E) \leqslant S \max\left(1, \frac{\|z\|}{R}\right), \quad z \in C^n. \tag{2.2}$$

Proof. The first inequality follows from Property 3.

Let $p_\nu \in A_\nu(E)$. Put

$$S = \sup\{\Phi(z, E): z \in B(R)\}.$$

By Property 4 of the extremal function it follows that

$$|p_\nu(z)|^{1/\nu} \leqslant S\Phi\big(z, B(R)\big), \quad z \in C^n.$$

Hence and from the definition of Φ we get

$$\Phi(z, E) \leqslant S\Phi\big(z, B(R)\big).$$

3. Order and type of an entire function. Let E be a bounded closed set in C^n such that $\Phi(z, E)$ is locally bounded.

Let

$$E_r = \{z \in C^n: \Phi(z, E) = r\}, \quad r > 1.$$

Write

$$M_E(r, f) = \sup\{\|f(z)\|: z \in E_r\}, \qquad r > 1,$$
$$S_f(r) = \sup\{\|f(z)\|: \|z\| = r\}, \quad r > 0$$

for an entire function $f: C^n \to B$, where B is a complex Banach space. If $E = B(1) = \{z \in C^n: \|z\| = 1\}$, then

$$M_E(r, f) = S_f(r).$$

The order and type of an entire function f will be determined in the same way as in the case of $n = 1$.

DEFINITION 3.1. We call $\varrho = \varrho(f)$ the *order of f* if

$$\varrho(f) = \limsup_{r\to\infty} \frac{\ln\ln S_f(r)}{\ln r}. \tag{3.1}$$

We have $0 \leqslant \varrho(f) \leqslant \infty$. If $0 < \varrho(f) < \infty$, we say that f is of the *type* $\sigma = \sigma(f)$ if

$$\sigma(f) = \limsup_{r\to\infty} \frac{\ln S_f(r)}{r^\varrho}. \tag{3.2}$$

An entire function f is said to be of

a) *minimal type* when $\sigma = 0$,

b) *normal type* when $0 < \sigma < \infty$,

c) *maximal type* when $\sigma = \infty$.

If in (3.1), (3.2) we replace $S_f(r)$ by $M_E(r, f)$, then we get ϱ_E, σ_E, respectively. We call ϱ_E and σ_E the *E-order* and *E-type*, respectively. It appears that ϱ_E is independent of E. Indeed, it follows from Theorem 2.1 that

$$S_f\left(r\frac{R}{s}\right) \leqslant M_E(r, f) \leqslant S_f(rR). \tag{3.3}$$

Hence $\varrho_E = \varrho$ and also we get

$$(R/s)^{\varrho}\sigma \leqslant \sigma_E \leqslant R^{\varrho}\sigma. \tag{3.4}$$

Therefore an entire function f is of minimal (normal, maximal) type if and only if $\sigma_E = 0$ $(0 < \sigma_E < \infty,\ \sigma_E = \infty)$.

LEMMA 3.1. *Let $\{p_\nu\}$ be a sequence of polynomials of degree $\leqslant \nu$, respectively.*

If there exist $\lambda \geqslant 0$, $\alpha > 0$, $\nu_0 \in N$, and $j \in N$ such that

$$\|p_{\nu+j}\|_E \leqslant \lambda\nu^{-\alpha\nu} \quad \text{for } \nu > \nu_0, \tag{3.5}$$

then $f(z) = \sum_{\nu=0}^{\infty} p_\nu(z)$, $z \in C^n$, is an entire function, and there exist $A \geqslant 0$, $B > 0$, $\beta > 0$ such that

$$\|f(z)\| \leqslant \Phi^{\beta}(z, E)\left(A + B\exp\left(\alpha\Phi^{1/\alpha}(z, E)\right)\right), \quad z \in C^n.$$

Proof. By (3.5) and Property 4 we have

$$\|f(z)\| \leqslant \Phi^{\nu_1}(z, E)\sum_{\nu<\nu_1}\|p_\nu\|_E + \lambda\Phi^{j}(z, E)\sum_{\nu\geqslant\nu_0}\left(\frac{\alpha\Phi^{1/\alpha}(z, E)}{\alpha\nu}\right)^{\alpha\nu}, \quad z \in C^n, \tag{3.6}$$

where $\nu_1 = \nu_0 + j$.

Denote by m_ν the entire part of $\alpha\nu$. The sequence $\{m_\nu\}$ is an increasing sequence of natural numbers. It follows from the definition that each number of this sequence is repeated no more than k_0 $(k_0 \geqslant 1/\alpha)$ times.

If $\alpha \geqslant 1$, then from (3.6) we obtain

$$\begin{aligned}
\|f(z)\| &\leqslant \Phi^{\nu_1}(z)\sum_{\nu<\nu_1}\|p_\nu\|_E + \lambda\Phi^{j}(z)\sum_{\nu=\nu_1}^{\infty}\frac{\left(\alpha\Phi^{1/\alpha}(z)\right)^{m_\nu+1}}{m_\nu^{m_\nu}} \\
&\leqslant \Phi^{\nu_1}(z)\sum_{\nu<\nu_1}\|p_\nu\|_E + \alpha\lambda\left(\Phi(z)\right)^{j+1/\alpha}\sum_{\nu=0}^{\infty}\frac{\left(\alpha\Phi^{1/\alpha}(z)\right)^{\nu}}{\nu!} \\
&\leqslant \Phi^{\beta}(z)\left(A + B\exp\left(\alpha\Phi^{1/\alpha}(z)\right)\right), \quad z \in C^n,
\end{aligned}$$

where $\beta = \max(\nu_1, j+1/\alpha)$, $A = \sum_{\nu<\nu_1}\|p_\nu\|_E$, $B = \alpha\nu k_0$.

If $0 < a < 1$, we consider two cases:
1° $a\Phi^{1/a}(z) \geqslant 1$,
2° $0 < a\Phi^{1/a}(z) < 1$,
and reason as in the case of $a \geqslant 1$.

CONCLUSION. *The function f is an entire function of the order $\varrho \leqslant 1/a$.*

LEMMA 3.2. *If there exist $K \geqslant 0$, $\mu > 0$, $j \in N$, $\lambda \geqslant 0$ and ν_0 such that*

$$\|p_{\nu+j}\|_E \leqslant \lambda \left(\frac{ek\mu}{\nu}\right)^{1/\mu} \quad \textit{for } \nu > \nu_0,$$

then the function

$$f(z) = \sum_{\nu=1}^{\infty} p_\nu(z), \quad z \in C^n$$

is an entire function, and for all $\varepsilon > 0$ there exists an $r_0 = r_0(\varepsilon)$ such that

$$\ln M_E(r, f) \leqslant (K+\varepsilon) r^\mu \quad \textit{for } r > r_0.$$

This Lemma may be proved in the same way as in the case $n = 1$ (see [11], Lemma 3.3).

Put

$$\varrho_E^* = \limsup_{\nu\to\infty} \frac{\nu \ln \nu}{-\ln \mathscr{E}_\nu(f, E)}.$$

We shall prove that $\varrho_E^* = \varrho$. Let $\varrho' > \varrho_E^*$. It follows from the definition of limsup that

$$\frac{\nu \ln \nu}{-\ln \mathscr{E}_\nu(f, E)} < \varrho' \quad \text{and} \quad \mathscr{E}_\nu(f, E) < 1$$

for a sufficiently large ν, say $\nu > \nu_0$. Hence, by Lemma 3.1 the function f is an entire function of order $\varrho \leqslant \varrho_E^*$.

It remains to prove that $\varrho \geqslant \varrho_E^*$. Without loss of generality we can assume that $E \subset \{z\colon |z_j| \leqslant 1,\ j = 1, \ldots, n\}$.

Let $L_k(z)$, $k = (k_1, \ldots, k_n)$ be the Lagrange interpolation polynomial for f with nodes $\eta^{(1)} \times \ldots \times \eta^{(n)}$ of degree $\leqslant k_j$ with respect to the j-th variable, where $\eta^{(j)} = \{\eta_{j0}, \ldots, \eta_{jk_j}\}$ is a system of k_j+1 extremal points of the circle $|z_j| \leqslant 1$.

It can be proved that

$$f(z) - L_k(z) = \frac{1}{(2\pi i)^n} \int\limits_{|\zeta_1|=r} \ldots \int\limits_{|\zeta_n|=r} \frac{w_k(z)}{w_k(\zeta)} \frac{f(\zeta)}{(\zeta_1 - z_1)\ldots(\zeta_n - z_n)}\, d(\zeta), \quad z \in E,$$

where $w_k(z) = \prod_{j=1}^{n} (z_j - \eta_{j0})\ldots(z_j - \eta_{jk_j})$, $d\zeta = d\zeta_1 \ldots d\zeta_n$.

Hence

$$\|f(z)-L_k(z)\|_E \leqslant 2^{3n\nu}\frac{\tilde{M}_f(r)}{r^\nu}, \quad k_1+\ldots+k_n \leqslant \nu,\ r>2,$$

where

$$\tilde{M}_f(r) = \sup\{\|f(z)\|:\ |z_j|\leqslant r,\ j=1,\ldots,n\}.$$

Therefore

$$\mathscr{E}_\nu(f,E) \leqslant 2^{3n\nu}\frac{\tilde{M}_f(r)}{r^\nu}, \quad \nu=1,2,\ldots,\ r>2. \tag{3.7}$$

If we take $\varrho_1<\varrho_E^*$, then for any sequence $\{\nu_j\}_{j\in N}$

$$\varrho_1 < \frac{\nu_j \ln \nu_j}{-\ln \mathscr{E}_{\nu_j}(f,E)} \quad \text{for } j=1,2,\ldots$$

From this and by (3.7) we get

$$\ln \tilde{M}_f(r) > \nu_j\left(\ln\frac{r}{8^n} - \frac{1}{\varrho_1}\ln\nu_j\right), \quad j=1,2,\ldots,\ r>2. \tag{3.8}$$

Let $r_j = 8^n(e\nu_j)^{1/\varrho_1}$. Then

$$\ln \tilde{M}_f(r_j) > \frac{\nu_j}{\varrho_1} = Ar_j^{\varrho_1}, \quad j=1,2,\ldots,$$

where

$$A = \frac{8^{-n\varrho_1}}{e\varrho_1}.$$

Hence

$$\limsup_{r\to\infty}\frac{\ln\ln \tilde{M}_f(r)}{\ln r} \geqslant \varrho_E^* \tag{3.9}$$

It is clear that

$$B(r) \subset \{z\in C^n:\ |z_j|\leqslant r,\ j=1,\ldots,n\} \subset B(r\sqrt{n}),$$

where $B(r)=\{z\in C^n:\ \|z\|\leqslant r\}$. Therefore

$$S_f(r) \leqslant \tilde{M}_f(r) \leqslant S_f(r\sqrt{n}). \tag{3.10}$$

Now, it is clear that (3.10) implies

$$\limsup_{r\to\infty}\frac{\ln\ln \tilde{M}_f(r)}{\ln r} = \varrho. \tag{3.11}$$

From (3.11) and (3.9) it follows that $\varrho\geqslant\varrho_E^*$.
So finally we have

$$\varrho = \varrho_E = \varrho_E^*. \tag{3.12}$$

If we put

$$\tilde{\sigma} = \limsup_{r\to\infty} \frac{\ln \tilde{M}(r)}{r^{\varrho}},$$

then from (3.10) we obtain

$$\sigma \leqslant \tilde{\sigma} \leqslant \sqrt{n}\sigma. \tag{3.13}$$

Let us take a function $f\colon E \to B$ and write

$$\Gamma(f, E) = \{\mu > 0\colon \psi(\mu) < \infty\},$$

where $\psi(\mu) = \limsup_{\nu\to\infty} \nu^{1/\mu} \sqrt[\nu]{\mathscr{E}_\nu(f, E)}$. Put

$$\tilde{\varrho}_E = \begin{cases} \inf \Gamma(f, E), & \text{when } \Gamma(f, E) \neq \Phi, \\ \infty, & \text{when } \Gamma(f, E) = \Phi \text{ and } \limsup\limits_{\nu\to\infty} \sqrt[\nu]{\mathscr{E}_\nu(f, E)} = 0. \end{cases}$$

If $0 < \tilde{\varrho}_E < \infty$, then there exists exactly one number σ_E^* such that

$$\limsup_{\nu\to\infty} \nu^{1/\tilde{\varrho}_E} \sqrt[\nu]{\mathscr{E}_\nu(f, E)} = (e\sigma_E^* \tilde{\varrho}_E)^{1/\tilde{\varrho}_E}.$$

Now, we shall define the ν-th extremal system $\eta^{(\nu)}$ of a compact set $E \subset C^n$ and the ν-th Lagrange interpolation polynomial for $f\colon E \to B$ in the same way as in [10].

Let $k_{1l}, \ldots, k_{nl}$, $l = 1, 2, \ldots, \nu_*$, denote the sequence of all solutions in non-negative integers of the inequality $k_1 + \ldots + k_n \leqslant \nu$.

Let $p^{(\nu)} = \{p_1, p_2, \ldots, p_{\nu_*}\}$ be a system of ν_* points

$$p_i = (z_{1i}, \ldots, z_{ni}), \quad i = 1, \ldots, \nu_*$$

such that the determinant

$$V(p^{(\nu)}) = \det[z_{1i}^{k_{1l}}, \ldots, z_{ni}^{k_{nl}}], \quad i, l = 1, \ldots, \nu_*, \tag{3.14}$$

is different from zero.

We shall consider a new determinant (3.14), say $V_i(z, p^{(\nu)})$, which corresponds to the system of points

$$\{p_1, \ldots, p_{i-1}, z, p_{i+1}, \ldots, p_{\nu_*}\},$$

z being an arbitrary point of C^n. Let

$$L^{(i)}(z, p^{(\nu)}) = \frac{V_i(z, p^{(\nu)})}{V(p^{(\nu)})}, \quad i = 1, \ldots, \nu_*. \tag{3.15}$$

Definition. If f is a function defined on a set E and if $p^{(\nu)} \subset E$, then

$$L_\nu(z) := \sum_{i=1}^{\nu_*} f(p_i) L^{(i)}(z, p^{(\nu)}), \quad z \in C^n,$$

is called the ν-th *Lagrange interpolation polynomial* for f with nodes $p^{(\nu)}$.

If E is a compact set in C^n, then there exists a system $\eta^{(\nu)} = \{\eta_{\nu 0}, \ldots$ $\ldots, \eta_{\nu\nu_*}\} \subset E$, $\nu = 1, 2, \ldots$, such that

$$V(\eta^{(\nu)}) = \sup\{V(p^{(\nu)}):\ p^{(\nu)} \subset E\}. \tag{3.16}$$

DEFINITION. We call $\eta^{(\nu)} \subset E$ satisfying (3.16) the ν-th *extremal system* of E.

Let $E \subset C^n$ be a compact set such that $V(\eta^{(\nu)}) \neq 0$ for $\nu = 1, 2, \ldots$

LEMMA 3.3. *If f: $E \to B$ is defined and bounded on E, then*

$$\mathscr{E}_\nu(f, E) \leqslant \|f - L_\nu\|_E \leqslant (1+\nu_*)\,\mathscr{E}_\nu(f, E) \qquad \text{for } \nu = 1, 2, \ldots, \tag{3.17}$$

where L_ν is the ν-th Lagrange interpolation polynomial for f with nodes a extremal points $\eta^{(\nu)}$ of E.

Proof. Let p_ν be a polynomial of degree $\leqslant \nu$ and let P_ν be the ν-th Lagrange interpolation polynomial for the function $g(z) = f(z) - p_\nu(z)$.

Since

$$L_\nu(\eta_{\nu j}) = p_\nu(\eta_{\nu j}) + P_\nu(\eta_{\nu j}) \qquad \text{for } j = 0, \ldots, \nu_*,$$

we have

$$L_\nu(z) = p_\nu(z) + P_\nu(z) \qquad \text{for } z \in C^n.$$

Therefore

$$\mathscr{E}_\nu(f, E) \leqslant \|f - L_\nu\|_E \leqslant \|f - p_\nu\|_E + \|P_\nu\|_E$$

$$\leqslant \|f - p_\nu\|_E \Big(1 + \sum_{j=1}^{\nu_*} \|L^{(j)}\|_E\Big) \leqslant \|f - p_\nu\|_E (1 + \nu_*).$$

This implies the assertion of Lemma 3.3.

THEOREM 3.1. *If $\Phi(z, E)$ is locally bounded in C^n, then f: $E \to B$ is the restriction to E of an entire function $\tilde{f}$ of the order ϱ if and only if*

$$\tilde{\varrho}_E = \varrho.$$

Moreover, if $0 < \varrho < \infty$, then $\tilde{f}$ is of

a) *minimal type when* $\psi(\varrho) = 0$,

b) *normal type when* $0 < \psi(\varrho) < \infty$,

c) *maximal type when* $\psi(\varrho) = \infty$.

Proof. (Divided into 3 parts).

1° We shall prove that if f is an entire function of order $\varrho < \infty$, then $\Gamma(f, E) \neq \emptyset$ and $\varrho = \tilde{\varrho}_E$.

Taking (if necessary) $g(z) = f(az)$ we can assume that $E \subset \{z\colon \|z\| \leqslant 1\}$. It follows immediately from (3.7) and (3.11) that for every $\varrho_1 > \varrho$ there exists an $r_0 > 2$ such that

$$\mathscr{E}_\nu(f, E) \leqslant 8^{n\nu} \frac{e^{r_1^{\varrho}}}{r^\nu} \qquad \text{for } r > r_0. \tag{3.18}$$

If we take a sufficiently large ν_0, we have

$$\left(\frac{\nu}{\varrho_1}\right)^{1/\varrho_1} > r_0 \quad \text{for } \nu > \nu_0.$$

Putting $r = (\nu/\varrho_1)^{1/\varrho_1}$ into (3.18) gives

$$\mathscr{E}_\nu(f, E) \leqslant 8^{n\nu}\left(\frac{e\varrho_1}{\nu}\right)^{\nu/\varrho_1}.$$

From this it follows that $\Gamma(f, E) \neq \emptyset$ and $\tilde{\varrho}_E \leqslant \varrho$.

Suppose that $\tilde{\varrho}_E < \varrho$ and take ϱ_2 such that $\tilde{\varrho}_E < \varrho_2 < \varrho$. For a sufficiently large ν we obtain

$$\nu^{1/\varrho}\sqrt[\nu]{\|f-L_\nu\|_E} < \psi(\varrho_2)+\varepsilon.$$

From Lemma 3.2 and after developing the function f into a series

$$f(z) = L_1(z) + \sum_{\nu=1}^{\infty}\big(L_{\nu+1}(z)-L_\nu(z)\big), \quad z \in C^n,$$

it would follow that $\varrho \leqslant \varrho_2$ and this contradicts the definition of ϱ_2.

2° We shall prove that if R and S are defined as in Theorem 1 and $0 < \varrho < \infty$, then

$$\left(\frac{R}{S}\right)^{\varrho}\sigma \leqslant \sigma_E \leqslant \sigma_E^* \leqslant 8^{n\varrho}\sqrt{n}\sigma. \tag{3.19}$$

Let $\sigma < \infty$. Then $\tilde{\sigma} < \infty$. Let us fix $K > \tilde{\sigma}$ and $R > 2$ such that

$$\tilde{M}_f(R) < e^{Kr^\varrho} \quad \text{for } r > R.$$

Taking sufficiently large ν_0 we have

$$\left(\frac{\nu}{K\varrho}\right)^{1/\varrho} > R \quad \text{for } \nu > \nu_0.$$

Putting $r = (\nu/K\varrho)$ in (3.7) gives

$$\mathscr{E}_\nu(f, E) \leqslant 8^{n\nu}\left(\frac{eK\varrho}{\nu}\right)^{\nu/\varrho} \quad \text{for } \nu > \nu_0.$$

Hence and from (3.13) we conclude that

$$\limsup_{\nu\to\infty}\nu^{1/\varrho}\sqrt[\nu]{\mathscr{E}_\nu(f, E)} \leqslant 8^n(e\tilde{\sigma}\varrho)^{1/\varrho} \leqslant 8^n(e\sigma\varrho)^{1/\varrho}.$$

Suppose that $\sigma_E^* < \sigma_E$ and take σ_1 such that $\sigma_E^* < \sigma_1 < \sigma_E$. For a sufficiently large ν we would obtain

$$\|f-L_\nu\|_E \leqslant \left(\frac{e\sigma_1\varrho}{\nu}\right)^{\nu/\varrho}.$$

Hence and from Lemma 3.2 it would follow that $\sigma_E < \sigma_1$ and this contradicts our assumption. Therefore

$$(e\sigma_E\varrho)^{1/\varrho} \leqslant (e\sigma_E^*\varrho)^{1/\varrho} \leqslant 8^n n^{1/2\varrho}(e\sigma\varrho)^{1/\varrho}.$$

From this and (3.4) we get

$$\left(\frac{R}{S}\right)^\varrho \sigma \leqslant \sigma_E \leqslant \sigma_E^* \leqslant 8^{n\varrho}\sqrt{n}\sigma.$$

If $\sigma = \infty$, then the above Theorem results from Lemma 3.2. This completes the proof of the sufficient condition.

3° If $\Gamma(f, E) \neq \emptyset$, then for every $\mu \in \Gamma(f, E)$

$$\limsup_{\nu\to\infty} \nu^{1/\mu}\sqrt[\nu]{\mathscr{E}_\nu(f, E)} = \psi(\mu) < \infty.$$

Hence for every $\varepsilon > 0$ there exists a $\nu_1 = \nu_1(\varepsilon, \mu)$ such that

$$\|f - L_\nu\|_E \leqslant \left(\frac{(\psi(\mu)+\varepsilon)^\mu}{\nu}\right)^{\nu/\mu} \quad \text{for } \nu > \nu_1.$$

By Lemma 3.2 the function

$$\tilde{f}(z) = L_1(z) + \sum_{\nu=1}^{\infty}\left(L_{\nu+1}(z) - L_\nu(z)\right) \quad \text{for } z \in C^n$$

is an entire function of a finite order ϱ and $\tilde{f}(z) = f(z)$ for $z \in E$.

Now, applying parts 1° and 2° of this proof, we can prove the necessary condition.

4. Order and type systems. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ be two systems of n real or complex numbers.

We shall use the following notation:

$$\alpha\beta = (\alpha_1\beta_1, \ldots, \alpha_n\beta_n),$$

$$\frac{\alpha}{\beta} = \left(\frac{\alpha_1}{\beta_1}, \ldots, \frac{\alpha_n}{\beta_n}\right), \quad \text{when } \beta_j \neq 0,\ j = 1, \ldots, n,$$

$$\alpha^\beta = \alpha_1^{\beta_1}\cdot \ldots \cdot \alpha_n^{\beta_n},$$

$$|\alpha| = |\alpha_1| + \ldots + |\alpha_n|,$$

$$\alpha < \beta \Leftrightarrow \alpha_j < \beta_j \quad \text{for } j = 1, \ldots, n,$$

$$\alpha \leqslant \beta \Leftrightarrow \alpha_j \leqslant \beta_j \quad \text{for } j = 1, \ldots, n,$$

$$\alpha^+ = (|\alpha_1|, \ldots, |\alpha_n|).$$

Let B be a complex Banach space and let $f\colon C^n \to B$ be an entire function.

The order and type surfaces of f will be determined in the same way as in [7].

Let

$$M_f^*(r) = \sup\{\|f(z)\|\colon z^+ \leqslant r\},$$

where $r = (r_1, \ldots, r_n) > (0, \ldots, 0)$.

Let $P_f \subset R^n$ be a set of points $\mu \in R^n$ such that for every $\mu \in P_f$ there exists an $r^{(0)} = (r_1^{(0)}, \ldots, r_n^{(0)})$ such that

$$\ln M_f^*(r) \leqslant r_1^{\mu_1} + \ldots + r_n^{\mu_n} \quad \text{for } r > r^{(0)}. \tag{4.1}$$

It follows from the definition that the set P_f satisfies the following condition:

$$\text{(W)} \qquad \begin{cases} 1^\circ & \textit{If } \mu \in P_f, \textit{ then } \{\mu' \in R^n\colon \mu' \geqslant \mu\} \subset P_f. \\ 2^\circ & \textit{If } \mu \in P_f, \textit{ then } \{\mu' \in R^n\colon \mu' < \mu\} \subset (R^n \setminus P_f). \end{cases}$$

DEFINITION 1. The boundary ∂P_f of the set P_f is called the *order surface of the entire function* f. A point $\varrho \in \partial P_f$ is called the *order system of* f.

Let us take $\varrho \in \partial P_f$ and denote by $T_f(\varrho)$ the set of all $\gamma \in R^n$ such that

$$\ln M_f^*(r) \leqslant \gamma_1 r_1^{\varrho_1} + \ldots + \gamma_n r_n^{\varrho_n} \quad \text{for } r > r^{(1)},\ r \in R^n. \tag{4.2}$$

One can easily check that the set $T_f(\varrho)$ satisfies condition (W).

DEFINITION 4.2. The boundary $\partial T_f(\varrho)$ of the set $T_f(\varrho)$ is called the *type surface of f corresponding to ϱ*. A point $\sigma \in \partial T_f(\varrho)$ is called the *type system of f corresponding to ϱ.*

Let $E = E^{(1)} \times \ldots \times E^{(n)}$, where $E^{(j)}$ $(j = 1, \ldots, n)$ is a compact set of a positive transfinite diameter $d_j = d(E^{(j)})$, and let $\Phi_j(z_j) = \Phi(z_j, E^{(j)})$ be the extremal function of the compact set $E^{(j)}$ $(j = 1, \ldots, n)$.

Let us denote

$$\begin{aligned} E_{r_j}^{(j)} &= \{z_j\colon d_j \Phi_j(z_j) = r_j\}, \qquad r_j > d_j,\ j = 1, \ldots, n; \\ E_r &= E_{r_1}^{(1)} \times \ldots \times E_{r_n}^{(n)}; \\ M_f(r) &= \sup\{\|f(z)\|\colon z \in E_r\} \qquad \text{for } r > d. \end{aligned}$$

It can be proved [11] that there exist $r^{(2)} > d = (d_1, \ldots, d_n)$, and numbers $a_j > 0$ such that if $z_j \in E_{r_j}^{(j)}$, then

$$r_j - a_j \leqslant |z_j| \leqslant r_j + a_j \quad \text{for } r > r^{(2)},\ j = 1, \ldots, n. \tag{4.3}$$

Now we shall prove that in the definition of the order and type systems of f, $M_f^*(r)$ may be replaced by $M_f(r)$. In order to do this we assume that $\tilde{P}_f$ and $\tilde{T}_f(\varrho)$ are subsets of R^n defined by $M_f(r)$ as P_f and $T_f(\varrho)$ by $M_f^*(r)$.

Let $\mu \in P_f$. Applying (4.3), we easily find that

$$M_f^*(r-a) \leqslant M_f(r) \leqslant M_f^*(r+a) \quad \text{for } r > r^{(2)}, \tag{4.4}$$

where $a = (a_1, \ldots, a_n)$. Hence

$$\ln M_f^*(r) \leqslant (r_1+a_1)^{\mu_1} + \ldots + (r_n+a_n)^{\mu_n} \quad \text{for } r > r(\mu).$$

Let us divide the system $(\mu_1, \ldots, \mu_n)$ into two parts $(\mu_{j_1}, \ldots, \mu_{j_s})$ and $(\mu_{j_{s+1}}, \ldots, \mu_{j_n})$ such that $\mu_{j_l} \leqslant 0$ (for $l = 1, \ldots, s$) and $\mu_{j_q} > 0$ for $q = s+1, \ldots, n$. Taking any $\varepsilon > (0, \ldots, 0)$ and sufficiently large r, say $r > r^{(3)}$, we have

$$\ln M_f(r) \leqslant r_{j_1}^{\mu_{j_1}} + \ldots + r_{j_s}^{\mu_{j_s}} + r_{j_{s+1}}^{(\mu_{j_{s+1}}+\varepsilon_{j_{s+1}})} + \ldots + r_{j_n}^{(\mu_{j_n}+\varepsilon_{j_n})}.$$

Hence if $\mu \in \tilde{P}_f$, then $\mu + \varepsilon \in P_f$, and so

$$\tilde{P}_f \subset \mathrm{cl}P_f = \text{closure of } P_f.$$

Applying the right-hand side of inequality (4.3) in the same way, we may prove that $P_f \subset \mathrm{cl}\tilde{P}_f$. Now applying property (W) we obtain

$$\partial P_f = \partial \tilde{P}_f. \tag{4.5}$$

Therefore in the definition of the order system of the function f_1 we can take $M_f^*(r)$ instead of $M_f(r)$.

Let $\varrho \in \partial P_f$ and $\gamma \leqslant \tilde{T}_f(\varrho)$. By (4.3) for sufficiently large r we have

$$\ln M_f^*(r) \leqslant \gamma_1 (r_1+a_1)^{\varrho_1} + \ldots + \gamma_n (r_n+a_n)^{\varrho_n}.$$

Because for every $\gamma_j \neq 0$ we have

$$\lim_{r_j \to \infty} \frac{(r_j+a_j)^{\varrho_j}}{r_j^{\varrho_j}} = 1,$$

so for every $\varepsilon > (0, \ldots, 0)$ there exists an $r^{(5)}$ such that

$$\ln M_f(r) \leqslant (\gamma_1+\varepsilon_1) r_1^{\varrho_1} + \ldots + (\gamma_n+\varepsilon_n) r_n^{\varrho_n} \quad \text{for } r > r^{(5)}.$$

Therefore $\tilde{T}_f(\varrho) \subset \mathrm{cl}T_f(\varrho)$ and analogously $T_f(\varrho) \subset \mathrm{cl}\tilde{T}_f(\varrho)$. Thus, the conclusion is analogous as in the case of the adjoint order systems of f.

Write the following two remarks resulting from the definitions of the sets P_f, $T_f(\varrho)$ and from Liouville's Theorem.

1° If $\mu = (\mu_1, \ldots, \mu_n) \in P_f$ (or $T_f(\varrho)$) and for any j we have $\mu_j \leqslant 0$, then f does not depend on the j-th variable.

2° If $\varrho \in \partial P_f$ and for any j we have $\varrho_j \leqslant 0$, then f does not depend on the j-th variable or $T_f(\varrho) = \emptyset$.

With reference to these remarks and considering the order system ϱ and the type system σ corresponding to ϱ, it suffices to confine ourselves to the case $\varrho_j > 0$, $\sigma \geqslant 0$ for $j = 1, \ldots, n$. In the opposite case our considerations will reduce to the entire function in C^m, $m < n$.

Suppose that $\partial P_f \neq \emptyset$ and take a straight line $l \subset R^n$ defined by

$$l(\tau) = (\tau, \ldots, \tau), \qquad \tau \in R.$$

By applying property (W) it can be proved that $\partial P_f \cap l$ is a one-point set.

DEFINITION 3. We call $p = p(f)$ the *adjoint order of f* if

$$\tilde{p} = (p, \ldots, p) \in \partial P_f.$$

We have $0 \leqslant p \leqslant \infty$. If $0 < p < \infty$, we say that f is of the *adjoint type* $q = q(f)$ if

$$\tilde{q} = (q, \ldots, q) \in \partial T_f(\tilde{p}).$$

Observe that the adjoint order of the entire function f can be determined as the infimum p of the set of numbers p' for which there exists an $r^{(0)} = (r_1^{(0)}, \ldots, r_n^{(0)})$ such that

$$\ln M_f(r) \leqslant r_1^{p'} + \ldots + r_n^{p'} \quad \text{ for } r > r^{(0)}$$

and, analogously, the adjoint type of f can be determined as the infimum q of the set of numbers q' for which there exists an $r^{(1)} = (r_1^{(1)}, \ldots, r_n^{(1)})$ such that

$$\ln M_f(r) \leqslant q' r_1^p + \ldots + q' r_n^p \quad \text{ for } r > r^{(1)}.$$

Let us examine the relationship between the order ϱ and the adjoint order p and between the type σ and the adjoint type q. It will follow that

$$\varrho = p \tag{4.6}$$

and, if $0 < \varrho < \infty$, then

$$\frac{\sigma}{n} \leqslant q \leqslant \sqrt{n}\sigma. \tag{4.7}$$

Let $\varrho' > \varrho$. From (3.11) it follows that

$$\ln \tilde{M}_f(s) \leqslant s^{\varrho'} \quad \text{ for a sufficiently large } s.$$

Since $\tilde{M}_f(s) = M_f^*(s, \ldots, s)$, so

$$\ln M_f^*(r) \leqslant \ln \tilde{M}_f(r_1) + \ldots + \ln \tilde{M}_f(r_n). \tag{4.8}$$

Hence

$$\ln M_f^*(r) \leqslant r_1^{\varrho'} + \ldots + r_n^{\varrho'} \quad \text{ for } r_j > s_0,\ j = 1, \ldots, n.$$

Thus, owing to optional $\varrho' > \varrho$, it follows that $\varrho \geqslant p$.

To obtain the opposite inequality we fix $p' > p$ and $r^{(0)} = (r_1^{(0)}, \ldots, r_n^{(0)})$ so that

$$\ln M_f^*(r) \leqslant r_1^{p'} + \ldots + r_n^{p'} \quad \text{for } r > r^{(0)}.$$

In particuliar, if $s > \max\{r_1^{(0)}, \ldots, r_n^{(0)}\}$, then

$$\ln \tilde{M}_f(s) \leqslant n s^{p'}.$$

Hence, by a standard argument, $p \geqslant \varrho$ and so (4.6) is proved.

In order to prove (4.7) let us take $\sigma' > \tilde{\sigma}$, where

$$\tilde{\sigma} = \limsup_{s\to\infty} \frac{\ln \tilde{M}_f(s)}{s^p}.$$

By the definition of $\tilde{\sigma}$ and (4.8) we have

$$\ln M_f^*(r) \leqslant \sigma' r_1^p + \ldots + \sigma' r_n^p \quad \text{for } r > r^{(1)}.$$

Hence, and by (3.13)

$$q \leqslant \tilde{\sigma} \leqslant \sqrt{n}\sigma.$$

On the other hand, if $q' > q$, then for a sufficiently large s

$$\ln \tilde{M}_f(s) \leqslant n q' s^{\varrho},$$

so $\tilde{\sigma} \leqslant nq$. But at the same time by (3.13) we have $\sigma \leqslant \tilde{\sigma}$. Therefore

$$\frac{\sigma}{n} \leqslant q \leqslant \sqrt{n}\sigma.$$

5. Best approximation and interpolation in a set $E = E^{(1)} \times \ldots \times E^{(n)}$. Let $\mathscr{P}_k = \mathscr{P}_k(C^n, B)$, $k = (k_1, \ldots, k_n)$ be the set of all polynomials $p: C^n \to B$ of degree $\leqslant k_j$ with respect to the j-th variable, respectively.

Let E be a compact set in C^n and let $f: E \to B$ be a function defined and bounded on E.

Write

$$\mathscr{E}_k^*(f, E) = \inf\{\|f - p\|_E : p \in \mathscr{P}_k\}.$$

Let $E = E^{(1)} \times \ldots \times E^{(n)}$, where $E^{(j)}$ $(j = 1, \ldots, n)$ is a compact set in C containing infinitely many different points.

Let $\eta_j^{(k_j)} = (\eta_{j0}, \ldots, \eta_{jk_j})$, $j = 1, \ldots, n$, be a system of $k_j + 1$ extremal points of $E^{(j)}$ (see [8]).

Let us write

$$L^{(\mu_j)}(z_j) = L^{(\mu_j)}(z_j, E^{(j)}) = \frac{(z_j - \eta_{j0}) \ldots}{(\eta_{j\mu_j} - \eta_{j0}) \ldots}\Bigg|_{\mu_j} \frac{\ldots (z_j - \eta_{jk_j})}{\ldots (\eta_{j\mu_j} - \eta_{jk_j})},$$

where $|_{\mu_j}$ means that the factor μ_j is omitted.

The polynomial

$$L_k(z) = \sum_{\mu_1,\ldots,\mu_n=0}^{k_1,\ldots,k_n} f(\eta_{1\mu_1}, \ldots, \eta_{n\mu_n}) L^{(\mu_1)}(z_1) \ldots L^{(\mu_n)}(z_n)$$

is the Lagrange interpolation polynomial for f with nodes $\eta_1^{(k_1)} \times \ldots \times \eta_n^{(k_n)}$ of degree $\leqslant k_j$ with respect to the j-th variable.

The inequality

$$(5.1) \qquad \mathscr{E}_k^*(f, E) \leqslant \|f - L_k\|_E \Big(1 + \prod_{j=1}^{n} (k_j+1)\Big) \mathscr{E}_k^*(f, E)$$

can be proved in the same way as (3.17).

Now we shall prove the Lemma, which together with inequality (5.1) and Property 4 of the extremal function (see Section 2) will be of primary importance in successive investigations.

LEMMA 5.1. *Let $k^{(\nu)} = (k_1^{(\nu)}, \ldots, k_n^{(\nu)})$, $\nu = 1, 2, \ldots$, be an increasing sequence such that* $\min\{k_j^{(\nu)}\colon j = 1, 2, \ldots, n\} \to \infty$, *when $\nu \to \infty$ and $k_j^{(\nu)}$ are natural numbers.*

Let $E = E^{(1)} \times \ldots \times E^{(n)}$, where $E^{(j)}$ $(j = 1, \ldots, n)$ is a compact set with a positive transfinite diameter $d_j = d(E^{(j)})$ in the complex z_j-plane and let $p_k \in \mathscr{P}_k$, $k = (k_1, \ldots, k_n)$ be polynomials such that

$$p_k(z) \equiv 0, \quad \textit{when } k \notin \{k^{(\nu)}\}.$$

If there exist $K = (K_1, \ldots, K_n) \geqslant 0$, $\mu = (\mu_1, \ldots, \mu_n) > 0$, $\nu_0 \in N$ and $\lambda \geqslant 0$ such that

$$(5.2) \qquad \|p_k\|_E \leqslant \lambda d^{k-\tilde{\gamma}} \left(\frac{eK\mu}{k - \tilde{\gamma}}\right)^{\frac{k-\tilde{\gamma}}{\mu}} \quad \textit{when } |k| > |k^{(\nu_0)}| := \alpha,$$

where $d = (d_1, \ldots, d_n)$, γ is a fixed natural number and $\tilde{\gamma} = (\gamma, \ldots, \gamma) \in R^n$, then

$$f(z) = \sum_k p_k(z), \quad z \in C^n,$$

is an entire function, and for all $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) > 0$ there exists an $r^{(0)} = (r_1^{(0)}, \ldots, r_n^{(0)}) \in R^n$ such that

$$\ln M_f(r) \leqslant \sum_{j=1}^{n} (K_j + \varepsilon_j) r_j^{\mu_j} \quad \textit{for } r > r^{(0)}.$$

Proof. By Property 4 of the extremal function $\Phi(z, E)$ applied to every variable separately we have

$$(5.3) \qquad \|p_k(z)\| \leqslant \|p_k\|_E \Phi^{k_1}(z_1, E^{(1)}) \ldots \Phi^{k_n}(z_n, E^{(n)}) \quad \text{for } z \in C^n.$$

Write $\nu(r) = (2^{\mu_1}eK_1\mu_1 r_1^{\mu_1}, \ldots, 2^{\mu_n}eK_n\mu_n r_n^{\mu_n})$ and take $r^{(1)} = (r_1^{(1)}, \ldots$ $\ldots, r_n^{(1)}) > (1, \ldots, 1)$ in such a way that $\nu(r) > k^{(\nu_0)}$ for $r > r^{(1)}$. Moreover, we assume that $\gamma = 0$ and $\lambda = 1$.

Thus by (5.3) we get

$$(5.4) \qquad M_f(r) \leqslant \sum_{|k|\leqslant a} \frac{\|p_k\|_E}{d^k} r^k + \sum_{a<|k|<|\nu(r)|} \left(\frac{eK\mu}{k}\right)^{k/\mu} r^k + \sum_{|k|>|\nu(r)|} 2^{\frac{1}{|k|}}$$

$$\leqslant \beta r^{\tilde{a}} + \sum_{a<|k|<|\nu(r)|} \left(\frac{eK\mu}{k}\right)^{k/\mu} r^k + 2^n \quad \text{for } r > r^{(1)},$$

where β does not depend on r, $\tilde{a} = (a, \ldots, a) \in R^n$.

Since the maximum value of the expression

$$\left(\frac{eK_j\mu_j}{k_j}\right)^{k_j/\mu_j} r_j^{k_j}$$

for r_j fixed ($j = 1, \ldots, n$) is obtained for $k_j = \mu_j K_j r_j^{\mu_j}$ and is equal to $\exp(K_j r_j^{\mu_j})$, we have

$$M_f(r) \leqslant \beta r^{\tilde{a}} + \left(\binom{\widehat{|\nu(r)|}+n}{n} - \binom{\hat{a}+n}{n}\right) \exp\left(\sum_{\nu=1}^{n} K_j r_j^{\mu_j}\right) + 2^n$$

$$\leqslant \beta r^{\tilde{a}} + \frac{2\,\widehat{|\nu(r)|}}{n!} \exp\left(\sum_{j=1}^{n} K_j r_j^{\mu_j}\right) + 2^n$$

$$\leqslant \left(\frac{\beta r^{\tilde{a}}}{\exp(\sum K_j r_j^{\mu_j})} + \frac{2\,\widehat{|\nu(r)|}}{n!} + \frac{2^n}{\exp(\sum K_j r_j^{\mu_j})}\right) \exp\left(\sum_{j=1}^{n} K_j r_j^{\mu_j}\right)$$

for $r > r^{(1)}$, where $\hat{\delta}$ is the smallest entire number greater than or equal to δ.

Hence, by a standard argument, there exists an $r^{(0)} > r^{(1)}$ such that

$$M_f(r) \leqslant \exp \sum_{j=1}^{n} (K_j + \varepsilon_j) r_j^{\mu_j} \quad \text{for } r > r^{(0)}.$$

Since for any $K' > K$ we have $(K'/k)^{k/\mu} > \left(K/(k-\tilde{\gamma})\right)^{(k-\tilde{\gamma})/\mu}$ when k is sufficiently large, in the case of $\gamma \neq 0$ or $\lambda \neq 1$ the proof is analogous with the only difference that before the second and the third component of the right-hand side of inequality (5.4) there occur (as factors) positive constants which have no decisive influence on the reasoning.

6. Characterization of the order and type systems by $\mathscr{E}_k^*(f, E)$. Let, as before, $E = E^{(1)} \times \ldots \times E^{(n)}$ and $d_j = d(E^{(j)}) > 0$.

Now, we shall present a characterization of the order and type systems of an entire function f by means of the measure $\mathscr{E}_k^*(f, E)$

of the Čebyšev best approximation to f on E by polynomials of degree $\leqslant k_j$ with respect to the j-th variable.

THEOREM 6.1. *If the transfinite diameter $d_j = d(E^{(j)}) > 0$ $(j = 1, \ldots, n)$ and $\varrho = (\varrho_1, \ldots, \varrho_n) > (0, \ldots, 0)$, $\sigma = (\sigma_1, \ldots, \sigma_n) > (0, \ldots, 0)$ are order and type systems of an entire function f, respectively, then*

$$\limsup_{\min(k_j)\to\infty} \sqrt[|k|]{\frac{\mathcal{E}_k^*(f, E)}{d^k}\left(\frac{k}{e\sigma\varrho}\right)^{k/\varrho}} = 1 \tag{6.1}$$

and

$$\limsup_{\min(k_j)\to\infty} \sqrt[|k|]{\frac{\|f - L_k\|_E}{d^k}\left(\frac{k}{e\sigma\varrho}\right)^{k/\varrho}} = 1. \tag{6.2}$$

Proof. Let

$$w_k(z) = \prod_{j=1}^{n} (z_j - \eta_{j0}) \ldots (z_j - \eta_{jk_j}),$$

where $\{\eta_{j0}, \ldots, \eta_{jk_j}\}$ is a system of $k_j + 1$ extremal points of the compact set $E^{(j)}$ $(j = 1, \ldots, n)$.

If r_j is sufficiently large, say $r_j > r_j^{(0)}$, then

$$E_{r_j}^{(j)} = \{z_j\colon\, d_j \Phi(z_j, E^{(j)}) = r\}$$

is a union of a finite number of mutually disjoint analytic Jordan curves in the complex z_j-plane; therefore

$$f(z) - L_k(z) = \frac{1}{(2\pi i)^n} \int\limits_{E_{r_1}^{(1)}} \ldots \int\limits_{E_{r_n}^{(n)}} \frac{w_k(z)}{w_k(\zeta)} \frac{f(\zeta)}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)}\, d\zeta, \tag{6.3}$$

where $d\zeta = d\zeta_1 \ldots d\zeta_n$.

We can prove [11] that for every $\varepsilon_j > 0$ there exist λ_j, $r_j^{(1)}$ and $k_j^{(1)}$ such that

$$\left|\frac{1}{2\pi i} \int\limits_{E_{r_j}^{(j)}} \left|\frac{(z_j - \eta_{j0}) \ldots (z_j - \eta_{jk_j})}{(\zeta_j - \eta_{j0}) \ldots (\zeta_j - \eta_{jk_j})} \frac{d\zeta_j}{\zeta_j - z_j}\right|\right| \leqslant \lambda_j \left(\frac{d_j e^{\varepsilon_j}}{r_j}\right)^{k_j}$$

$$\text{for } r_j > r_j^{(1)},\ k_j > k_j^{(1)}.$$

Hence and from (6.3) we have

$$\|f - L_k\|_E \leqslant \lambda \frac{M_f(r)}{r^k} (de^{\varepsilon})^k \tag{6.4}$$

$$\text{for } r > r^{(1)} = (r_1^{(1)}, \ldots, r_n^{(1)}),\ k > k^{(1)} = (k_1^{(1)}, \ldots, k_n^{(1)}),$$

where $\lambda = \lambda_1 \cdot \ldots \cdot \lambda_n$, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, $e^{\varepsilon} = (e^{\varepsilon_1}, \ldots, e^{\varepsilon_n})$.

Let $\gamma = (\gamma_1, \ldots, \gamma_n) > \sigma$. In consequence of the definition of the type system of f there exists an $r^{(2)} > r^{(1)}$ such that

$$M_f(r) \leqslant \exp(\gamma_1 r_1^{\varrho_1} + \ldots + \gamma_n r_n^{\varrho_n}) \qquad \text{for } r > r^{(2)}.$$

Let $k^{(2)} > k^{(1)}$ such that

$$\left(\frac{k_j}{r_j \varrho_j}\right)^{1/\varrho_j} > r_j \quad \text{for } j = 1, \ldots, n,\ k > k^{(2)}.$$

Putting

$$r = \left(\left(\frac{k_1}{\gamma_1 \varrho_1}\right)^{1/\varrho_1}, \ldots, \left(\frac{k_n}{\gamma_n \varrho_n}\right)^{1/\varrho_n}\right)$$

in (6.4) and presenting γ in the form $\gamma = (\sigma_1 e^{\delta_1}, \ldots, \sigma_n e^{\delta_n})$, we get

$$(6.5) \qquad \|f - L_k\|_E \leqslant \lambda (de^{\varepsilon})^k \left(\frac{e\gamma\varrho}{k}\right)^{k/\varrho} \leqslant \lambda d^k \left(\frac{e\sigma\varrho}{k}\right)^{k/\varrho} \left(e^{\left(\varepsilon + \frac{\delta}{\varrho}\right)}\right)^k \quad \text{for } k > k^{(2)},$$

where $\delta = (\delta_1, \ldots, \delta_n)$.

Hence because of ε and γ being arbitrary we get

$$\limsup_{\min(k_j)\to\infty} \sqrt[|k|]{\frac{\|f - L_k\|_E}{d^k} \left(\frac{k}{e\sigma\varrho}\right)^{k/\varrho}} \leqslant 1.$$

Suppose that

$$\limsup_{\min(k_j)\to\infty} \sqrt[|k|]{\frac{\|f - L_k\|_E}{d^k} \left(\frac{k}{e\sigma\varrho}\right)^{k/\varrho}} = \eta_1 < 1$$

and take η such that $\eta < \eta_1 < 1$. From the definition of the lim sup there would exist a $k^{(3)}$ such that

$$\|f - L_k\|_E \leqslant \eta^{|k|} d^k \left(\frac{e\sigma\varrho}{k}\right)^{k/\varrho} \qquad \text{for } k > k^{(3)}.$$

Taking $\sigma_j' = \sigma\eta^{\varrho_j}$ we would obtain $\sigma' = (\sigma_1', \ldots, \sigma_n') < \sigma$ and

$$(6.6) \qquad \|f - L_k\|_E < d^k \left(\frac{e\sigma'\varrho}{k}\right)^{k/\varrho} \qquad \text{for } k > k^{(3)}.$$

Let $\tilde{\nu} = (\nu, \ldots, \nu) \in R^n$, $\nu = 0, 1, \ldots$ From (6.6), from the triangle inequality and after expanding the function f in the series

$$f(z) = L_{\tilde{0}}(z) + \sum_{\nu=0}^{\infty} \left(L_{\widetilde{\nu+1}}(z) - L_{\tilde{\nu}}(z)\right), \qquad z \in C^n$$

we would obtain

$$\|L_{\widetilde{\nu+1}} - L_{\tilde{\nu}}\|_E \leqslant \|f - L_{\widetilde{\nu+1}}\|_E + \|f - L_{\tilde{\nu}}\|_E \leqslant 2d^{\tilde{\nu}} \left(\frac{e\sigma'\varrho}{\tilde{\nu}}\right)^{\tilde{\nu}/\varrho}.$$

Hence and from Lemma 5.1 and property (W) of $T_f(\varrho)$ it would follow that $\sigma \in \operatorname{int} T_f(\varrho)$, and this is contradicts the assumption.

The other part of Theorem 6.1 is an immediate consequence of inequality (5.1).

Equality (6.1) or (6.2) is at the same time a necessary and sufficient condition for the systems ϱ, σ of positive numbers to be the order and the type systems of the entire function f, respectively. Before carrying out the proof of this fact we shall prove the necessary and sufficient condition for the system $\varrho = (\varrho_1, \ldots, \varrho_n)$ of positive numbers to be an order system.

Under the same assumption on the compact set $E = E^{(1)} \times \ldots \times E^{(n)}$ as in Theorem 6.1 the following is obtained:

THEOREM 6.2. *A system of n positive numbers $\varrho = (\varrho_1, \ldots, \varrho_n)$ is an order system of the entire function f if and only if*

$$\text{(a)} \qquad \limsup_{\min\{k_j\}\to\infty} \frac{\ln k^{k/\varrho}}{-\ln \|f - L_k\|_E} = 1,$$

or equivalently if

$$\text{(b)} \qquad \limsup_{\min\{k_j\}\to\infty} \frac{\ln k^{k/\varrho}}{-\ln \mathscr{E}_k^*(f, E)} = 1.$$

Proof of the sufficient condition. Let δ be a real number such that

$$0 < 2\delta < \varrho_j, \qquad \delta\varrho_j < 1 \qquad \text{for } j = 1, \ldots, n.$$

It follows from the definition of lim sup that there exists a sequence $\{k^{(\nu)} = (k_1^{(\nu)}, \ldots, k_n^{(\nu)})\}$ $(k^{(\nu+1)} > k^{(\nu)})$ convergent to infinity (i.e. $\lim_{\nu\to\infty} k_j^{(\nu)} = \infty$, $j = 1, \ldots, n$) for which

$$\frac{\ln (k^{(\nu)})^{k^{(\nu)}/\varrho}}{-\ln \|f - L_{k^{(\nu)}}\|_E} > 1 - \delta, \qquad \|f - L_{k^{(\nu)}}\| < 1 \qquad \text{for } \nu = 1, 2, \ldots$$

Hence

$$\text{(6.7)} \quad \ln \|f - L_{k^{(\nu)}}\|_E > -\frac{k_1^{(\nu)}}{\varrho_1 - \varepsilon_1} \ln k_1^{(\nu)} + \ldots + \frac{k_n^{(\nu)}}{\varrho_n - \varepsilon_n} \ln k^{(\nu)}, \qquad \nu = 1, 2, \ldots,$$

where $\varepsilon_j = \delta\varrho_j$, $j = 1, \ldots, n$.

By a standard argument from (6.4) and (6.7) we obtain

$$\text{(6.8)} \qquad \ln M_f(r) > \sum_{j=1}^{n} k_j^{(\nu)} \left(\ln \frac{r_j}{d_j} - \frac{\ln k_j^{(\nu)}}{\varrho_j - \varepsilon_j} \right) - \sum_{j=1}^{n} k_j^{(\nu)} \varepsilon_j - \ln \lambda$$

$$\text{for } \nu > \nu_0,\ r > R^{(1)},$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$.

Let us take $\nu_1 > \nu_0$ so large that

$$d_j(ek_j^{(\nu)})^{\frac{1}{\varrho_j-\varepsilon_j}} > R_j^{(1)} \quad \text{for } j = 1, 2, \ldots, n, \ \nu > \nu_1$$

and let us substitute in place of r_j/d_j (in (6.8))

$$\frac{r_j^{(\nu)}}{d_j} = (ek_j^{(\nu)})^{\frac{1}{\varrho_j-\varepsilon_j}}.$$

After this substitution we have

$$\ln M_f(r^{(\nu)}) > \sum_{j=1}^{n} k_j^{(\nu)} \frac{1}{\varrho_j - \varepsilon_j} - \sum_{j=1}^{n} k_j^{(\nu)} \varepsilon_j - \ln\lambda \quad \text{for } \nu > \nu_1.$$

Since $k_j^{(\nu)} = \dfrac{1}{e}\left(\dfrac{r_j^{(\nu)}}{d_j}\right)^{\varrho_j-\varepsilon_j}$, we have

$$\ln M_f(r^{(\nu)}) > \sum_{j=1}^{n} \left(\frac{1}{e(\varrho_j - \varepsilon_j)\, d_j^{\varrho_j-\varepsilon_j}} - \frac{\varepsilon_j}{e d_j^{\varrho_j-\varepsilon_j}} - \frac{\ln\lambda_j}{(r_j^{(\nu)})^{\varrho_j-\varepsilon_j}}\right)(r_j^{(\nu)})^{\varrho_j-\varepsilon_j}$$

for $\nu > \nu_1$.

Hence by a standard argument we conclude that

$$\ln M_f(r^{(\nu)}) > \sum_{j=1}^{n} (r_j^{(\nu)})^{\varrho_j-2\varepsilon_j},$$

and this means that $\varrho - 2\varepsilon \notin P_f$.

Let ν be a natural number, and $\varepsilon > 0$. We shall put $\tilde{\nu} = (\nu, \ldots, \nu) \in R^n$. By the definition of lim sup it follows that for a sufficiently large ν, say $\nu > \nu_0$, the following is true:

(6.9) $$\|f - L_{\tilde{\nu}}\|_E \leqslant (\tilde{\nu})^{\frac{-\tilde{\nu}}{\varrho+\varepsilon}},$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, $\varepsilon_j = \varrho_j\delta$, $j = 1, \ldots, n$.

Hence

$$\|L_{\widetilde{\nu+1}} - L_{\tilde{\nu}}\|_E \leqslant 2(\tilde{\nu})^{-\frac{\tilde{\nu}}{\varrho+\varepsilon}} \quad \text{for } \nu > \nu_0.$$

Let $K = (K_1, \ldots, K_n)$ be such that

$$2(\tilde{\nu})^{-\frac{\tilde{\nu}}{\varrho+\varepsilon}} \leqslant d^{(\tilde{\nu})}\left(\frac{eK(\varrho+\varepsilon)}{\tilde{\nu}}\right)^{\frac{\tilde{\nu}}{\varrho+\varepsilon}} \quad \text{for } \nu = 1, 2, \ldots$$

Since

$$\|L_{\widetilde{\nu+1}} - L_{\tilde{\nu}}\|_E \leqslant d^{\tilde{\nu}}\left(\frac{eK(\varrho+\varepsilon)}{\tilde{\nu}}\right)^{\frac{\tilde{\nu}}{\varrho+\varepsilon}} \quad \text{for } \nu > \nu_0,$$

by (6.9) and Lemma 5.1 we have

$$f(z) = L_{\tilde{1}}(z) + \sum_{\nu=1}^{\infty} \left(L_{\widetilde{\nu+1}}(z) - L_{\tilde{\nu}}(z)\right) \quad \text{for } z \in C^n.$$

Moreover, there exists an $r^{(0)} = (r_1^{(0)}, \ldots, r_n^{(0)})$ such that

$$\ln M_f(r) \leqslant \sum_{j=1}^{n} (K_j + \varepsilon_j) r_j^{\varrho_j + \varepsilon_j} \quad \text{for } r > r^{(0)}.$$

Hence it follows that $\varrho + 2\varepsilon \in P_f$ for sufficiently small δ. Since we also had $\varrho - 2\varepsilon \notin P_f$, this completes the proof of the sufficient condition.

Proof of the necessary condition. Let $\varrho' = (\varrho_1', \ldots, \varrho_n') > \varrho$. Applying (6.4) and the definition of the order system of f in the same way as in the proof of (6.5), we get

$$\|f - L_k\|_E \leqslant \lambda (de^{\varepsilon})^k \left(\frac{e\varrho'}{k}\right)^{k/\varrho'} \quad \text{for } k > k^{(\varepsilon)}. \tag{6.10}$$

After calculating the logarithm of (6.10) bilaterally and dividing both sides by $\ln\|f - L_k\|_E$ we get

$$1 \geqslant \frac{\ln k^{k/\varrho'}}{-\ln\|f - L_k\|_E} + \frac{\ln \lambda}{\ln\|f - L_k\|_E} + \sum_{j=1}^{n} \frac{\ln d_j + \varepsilon_j + \frac{1}{\varrho_j'} + \frac{\ln \varrho_j'}{\varrho_j'}}{\frac{1}{k_j} \ln\|f - L_k\|_E} \quad \text{for } k > k^{(\varepsilon)}. \tag{6.11}$$

It follows immediately from this that

$$\delta = \limsup_{\min\{k_j\} \to \infty} \frac{\ln k^{k/\varrho'}}{-\ln\|f - L_k\|_E} \leqslant 1.$$

Suppose that $\delta < 1$. Taking ε_0 so small that

$$\delta\varrho + 2\varepsilon < \varrho, \quad \text{where } \varepsilon = (\varepsilon_0 \varrho_1, \ldots, \varepsilon_0 \varrho_n),$$

we would have

$$\|f - L_{k^{(\nu)}}\|_E \leqslant (k^{(\nu)})^{-\frac{k^{(\nu)}}{\delta\varrho + \varepsilon}}$$

for any sequence $\{k^{(\nu)}\}$ convergent to infinity. Now, from Lemma 5.1 it would follow that $(\varrho\sigma + 2\varepsilon) \in P_f$ and consequently $\varrho \in \operatorname{int} P_f$, and this contradicts the assumption.

Condition (b) follows immediately from (a) and (5.1).

Let a compact set $K \subset C^n$ be such that there exists a compact set $E \subset K$ satisfying the assumptions of Theorem 6.2.

Theorem 6.2 implies

THEOREM 6.2a. *A system on f positive real numbers $\varrho = (\varrho_1, \ldots, \varrho_n)$ is an order system of an entire function f if and only if*

$$\limsup_{\min\{k_j\}\to\infty} \frac{\ln k^{k/\varrho}}{-\ln \mathcal{E}_k^*(f, K)} = 1.$$

Keeping the notation of Theorem 6.2 we shall prove the following

THEOREM 6.3. *Let $f\colon E \to B$ be a function defined and bounded on E, where E is as in Theorem 6.2. Let $\varrho = (\varrho_1, \ldots, \varrho_n)$, $\sigma = (\sigma_1, \ldots, \sigma_n)$ be systems of n positive numbers.*

If

$$\text{(i)} \qquad \limsup_{\min\{k_j\}\to\infty} \sqrt[|k|]{\frac{\|f-L_k\|_E}{d^k}\left(\frac{k}{e\sigma\varrho}\right)^{k/\varrho}} = 1,$$

or equivalently if

$$\text{(ii)} \qquad \limsup_{\min\{k_j\}\to\infty} \sqrt[|k|]{\frac{\mathcal{E}_k^*(f, E)}{d^k}\left(\frac{k}{e\sigma\varrho}\right)^{k/\varrho}} = 1,$$

then there exists an entire function $\tilde{f}$ such that

1° $\tilde{f}(z) = f(z)$ *for* $z \in E$;

2° ϱ *is an order system of $\tilde{f}$, and σ is a type system of $\tilde{f}$ corresponding to ϱ.*

Proof. Let us take $\varepsilon > 0$ and $K^{(0)} = (K_1^{(0)}, \ldots, K_n^{(0)})$ such that

$$(6.12) \qquad \|f-L_k\|_E \leqslant d^k\left(\frac{e\sigma\varrho}{k}\right)^{k/\varrho} e^{|k|\varepsilon} \quad \text{for } k > K^{(0)}.$$

From this and by Lemma 5.1 it follows that a function

$$(6.13) \qquad \tilde{f}(z) = L_{\tilde{1}}(z) + \sum_{\nu=1}^{\infty}\left(L_{\widetilde{\nu+1}}(z) - L_{\tilde{\nu}}(z)\right) \quad \text{for } z \in C^n,$$

where $\tilde{\nu} = (\nu, \ldots, \nu) \in R^n$, is an entire function satisfying condition 1°.

Calculating the logarithms of both sides in (6.12) and applying an analogous reasoning to that used in the proof of the necessary condition of Theorem 6.2, we get

$$(6.14) \qquad \limsup_{\min\{k_j\}\to\infty} \frac{\ln k^{k/\varrho}}{-\ln \|f-L_k\|_E} \leqslant 1.$$

From the definition of lim sup it follows that there exists a sequence $\{k^{(\nu)} = (k_1^{(\nu)}, \ldots, k_n^{(\nu)})\}$ such that $\min\{k_j^{(\nu)}\colon j = 1, \ldots, n\} \to \infty$ and

$$\|f-L_{k^{(\nu)}}\|_E > d^{k^{(\nu)}}\left(\frac{e\sigma\varrho}{k^{(\nu)}}\right)^{k^{(\nu)}/\varrho} e^{-|k^{(\nu)}|\varepsilon} \quad \text{for } \nu = 1, 2, \ldots$$

By a similar reasoning we shall obtain the inequality opposite to that in (6.14). Thus from Theorem 6.2 it follows that ϱ is an order system of $\tilde{f}$.

Inequality (6.12) may be written in the form

$$\|f-L_k\|_E \leqslant d^k\left(\frac{e\sigma'\varrho}{k}\right)^{k/\varrho} \quad \text{for } k > K^{(0)}, \tag{6.15}$$

where $\sigma' = (\sigma_1', \ldots, \sigma_n')$, $\sigma_j' = \sigma_j e^{\varepsilon\varrho_j}$ for $j = 1, \ldots, n$.

It follows from Lemma 5.1 that $(\sigma' + \tilde{\varepsilon}) \in T_{\tilde{f}}(\varrho)$, where $\tilde{\varepsilon} = (\varepsilon, \ldots, \varepsilon) \in R^n$. Therefore $\sigma \in T_{\tilde{f}}(\varrho)$.

Suppose that $\sigma \in \operatorname{int} T_{\tilde{f}}(\varrho)$. Since inequality (6.15) holds true for every $\gamma \in \operatorname{int} T_{\tilde{f}}(\varrho)$, taking a sufficiently small $\varepsilon > 0$ such that $\gamma = e^{-\varepsilon}\sigma \in \operatorname{int} T_{\tilde{f}}(\varrho)$ we would have

$$\limsup_{\min\{k_j\}\to\infty} \sqrt[|k|]{\frac{\|f-L_k\|_E}{d^k}\left(\frac{k}{e\gamma\varrho}\right)^{k/\varrho}} \leqslant 1.$$

On the other hand, on the strength of the assumption

$$\limsup_{\min\{k_j\}\to\infty} \sqrt[|k|]{\frac{\|f-L_k\|_E}{d^k}\left(\frac{k}{e\gamma\varrho}\right)^{k/\varrho}} > e^{\varepsilon} > 1,$$

which contradicts the previously obtained inequality. Thus σ is a type system of $\tilde{f}$. From Theorem 6.1-6.3 we get as a conclusion the following generalization of the results obtained in [2], [9] and [11].

THEOREM 6.4. *Function $f\colon E \to B$, where E is as in Theorem 6.2, is prolongable to an entire function $\tilde{f}$ of the order p if and only if*

$$p = \limsup_{\min\{k_j\}\to\infty} \frac{\ln k^k}{-\ln \mathscr{E}_k^*(f, E)},$$

or, equivalently, if and only if

$$p = \limsup_{\min\{k_j\}\to\infty} \frac{\ln k^k}{-\ln \|f-L_k\|_E}.$$

Moreover, if $0 < p < \infty$, then the adjoint type q of $\tilde{f}$ is given by

$$\limsup_{\min\{k_j\}\to\infty} \sqrt[|k|]{\frac{\mathscr{E}_k^*(f, E)}{d^k}\, k^{k/p}} = (epq)^{1/p},$$

or equivalently by

$$\limsup_{\min\{k_j\}\to\infty} \sqrt[|k|]{\frac{\|f-L_k\|_E}{d^k}\, k^{k/p}} = (epq)^{1/p}.$$

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