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Multiple integrals evaluated by functional equations

by SHIGERU HARUKI (Waterloo, Canada)

We consider the multiple integral (see [7])

$$(1)\qquad f(a_1, a_2, \ldots, a_n) = \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty}}_{n} \exp(-x_1^2-x_2^2-\ldots-x_n^2)\times$$
$$\times(a_1x_1+a_2x_2+\ldots+a_nx_n)^{2m}dx_1dx_2\ldots dx_n,$$

where a_i, b_i are real numbers and n, m are natural numbers.

The purpose of this note is to prove

$$(2)\qquad f(a_1, a_2, \ldots, a_n) = (\sqrt{\pi})^n\frac{(2m)!}{4^m m!}(a_1^2+a_2^2+\ldots+a_n^2)^m$$

from the standpoint of functional equation theory (e.g. [1]-[6], [8] and [9]).

Let R^n be the Euclidian vector space, and let U, V, be vectors in R^n with

$$(3)\qquad \|V\| = \|U\|;$$

then there exists an orthogonal matrix

$$T = \begin{pmatrix} t_{11} & \ldots & t_{1n} \\ \vdots & & \vdots \\ t_{n1} & \ldots & t_{nn} \end{pmatrix}$$

such that

$$(4)\qquad U = TV.$$

Consider the functional equation

$$(5)\qquad f(TV) = f(V).$$

If f satisfies (5) for all T, then by (4), (5) we obtain

$$(6)\qquad f(V) = f(U).$$

(3) and (6) yields, since U may be chosen as the vector with components $\|V\|, 0, 0, \ldots, 0$, that

$$(7)\qquad f(V) = \varphi(v_1^2+v_2^2+\ldots+v_n^2),$$

where φ is an arbitrary function.

THEOREM. *The function $f(a_1, a_2, \dots, a_n)$ defined in* (1) *satisfies the functional equation*

$$(5') \qquad f(A) = f(BA),$$

where $A = (a_1, \dots, a_n)$ and B is a square orthogonal matrix. This function is given by (2).

Proof. Let $V = A, T = B, U = BA$, where $B = (b_{ij})$ is a square orthogonal matrix. Then replacing x_1 by

$$b_{11}x_1 + b_{21}x_2 + \dots + b_{n1}x_n, \dots, x_n \quad \text{by} \quad b_{1n}x_2 + b_{2n}x_2 + \dots + b_{nn}x_n$$

in (1), we obtain (the Jacobian $= 1$, since B is orthogonal)

$$(8) \quad f(A) = f(a_1, a_2, \dots, a_n)$$
$$= \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty}}_{n} \exp(-x_1^2 - x_2^2 - \dots - x_n^2)\{(a_1b_{11} + a_2b_{12} + \dots + a_nb_{1n})x_1 +$$
$$+ (a_1b_{21} + a_2b_{22} + \dots + a_nb_{2n})x_2 + \dots + (a_1b_{n1} + a_2b_{n2} + \dots +$$
$$+ a_nb_{nn})x_n\}^{2m} dx_1 dx_2 \dots dx_n = f(BA).$$

Hence we get (5′). By (7)

$$f(a_1, a_2, \dots, a_n) = \varphi(a_1^2 + a_2^2 + \dots + a_n^2),$$

$$(9) \qquad \varphi(a_1^2 + a_2^2 + \dots + a_n^2)$$
$$= \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty}}_{n} \exp(-x_1^2 - x_2^2 - \dots - x_n^2)(a_1x_1 + a_2x_2 + \dots +$$
$$+ a_nx_n)^{2m} dx_1 dx_2 \dots dx_n.$$

Set $a_2 = a_3 = \dots = a_n = 0$ in (9) to obtain

$$(10) \quad \varphi(a_1^2)$$
$$= \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty}}_{n-1} \left\{\int_{-\infty}^{+\infty} (a_1x_1)^{2m} e^{-x_1^2} dx_1\right\} e^{-x_2^2} e^{-x_3^2} \dots e^{-x_n^2} dx_2 dx_3 \dots dx_n.$$

Applying the well-known integrals

$$\int_{-\infty}^{+\infty} \exp(-x^2)dx = \sqrt{\pi}, \qquad \int_{-\infty}^{+\infty} x^{2m}e^{-x^2}dx = \sqrt{\pi}\frac{(2m)!}{4^m m!}$$

in (10) we get

$$\varphi(a_1^2) = (\sqrt{\pi})^n \frac{(2m)!}{4^m m!}(a_1^2)^m,$$

which yields

$$f(a_1, a_2, \ldots, a_n) = \varphi(a_1^2 + a_2^2 + \ldots + a_n^2) = (\sqrt{\pi})^n \frac{(2m)!}{4^m m!} (a_1^2 + a_2^2 + \ldots + a_n^2)^m.$$

Q.E.D.

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STUDENT
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO, CANADA

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