PDES, SUBMANIFOLDS AND
AFFINE DIFFERENTIAL GEOMETRY
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# INTRODUCTION

It was the aim of this conference at the Banach Center to have a forum for experts in the topics of the conference and of related fields.

This volume presents some of the contributions and reflects progress in topics of the three fields, in particular relations between them and to other fields; in fact, this latter viewpoint was of some importance for the conference; it was also one of the aims to present methods and results from related field as stimulus for research. Some of the articles are surveys describing interesting geometric developments.

A renaissance of the very classical field of Affine Differential Geometry (ADG) began with the global study of affine spheres in the 70s, and a new approach to the theory in the 80s. In the last decade there appeared several monographs and introductory texts on affine differential geometry and related fields; the recent article [SI] in the Handbook of Differential Geometry gives an introduction, a survey up to recent research, and references.

In a problem session we collected a series of open questions and conjectures.

We would like to point out the following:

1. Two topics of recent interest in ADG are homogeneous submanifolds and submanifolds of higher codimension.

— Classification of homogeneous submanifolds: The classification of affinely homogeneous surfaces in affine 3-space is given in the monograph of Nomizu and Sasaki [N-S]. In the decade since then there have appeared a large number of contributions on hypersurfaces and submanifolds of codimension two and three. R. Walter's contribution gives insight into this very interesting, but difficult field.

— Submanifolds of higher codimension: So far, there are different approaches to develop a theory which is geometrically transparent; classification results are rare. Furuhata studies the realization of given affine structures for codimensions 1 and 2.

2. The following papers are typical contributions which we consider as stimuli.

— The so called viewpoint of relative hypersurface theory (see the introductory lecture notes [S-S-V]) unifies different affine theories as well the Euclidean viewpoint for non-degenerate hypersurfaces; in the latter case the second fundamental form plays the role of a relative metric. The paper of Gálvez–Martínez studies problems using this metric.

— From the point of view of relative geometry, a parallel second fundamental form is a vanishing cubic form. This is an "affine background" for the results of Belkhelfa *et al.*

— Vrancken's contribution concerns centroaffine hypersurfaces, but at the same time it points out relations to hypersurfaces in space forms.

— Complex methods are frequently used for the investigation of extremal (minimal, maximal) surfaces in Euclidean and affine geometry; in both cases Weierstrass representation formulas are well known. Sometimes there are interesting interpretations in complex curve theory (see Gollek's contribution).

— Results around the Osserman conjecture like those of Gilkey–Ivanova are of interest to start analogous investigations in ADG.

3. In affine differential geometry there appear some differential geometric structures in a very natural way: e.g. conformal structures, projective structures, conjugate connections, Weyl geometries. Semi-Riemannian metrics appear on so called indefinite hypersurfaces in affine space. Thus this latter class is a resource for examples and also a natural field for applications of general results in semi-Riemannian geometry. This explains our particular interest in contributions from semi-Riemannian geometry as in the survey by Belkhelfa, Deszcz *et al.* Relations to Kaehler geometry as in Cortès's paper are rare, so far.

4. Considering local graph representations of affine hypersurfaces, all intrinsic and extrinsic curvature invariants can be described in terms of PDEs of at least fourth order. Thus geometric problems about affine curvatures are also of interest from the viewpoint of PDEs. Another type of PDEs which also appear in ADG are so called Codazzi equations. Moreover, PDE methods which were developed in the context of ADG are applied elsewhere; a typical example for this latter statement is the contribution of Simon *et al.* in this volume.

## References

[N-S] K. Nomizu and T. Sasaki, *Affine Differential Geometry*, Cambridge Tracts in Math. 111, Cambridge Univ. Press, 1994.

[SI] U. Simon, *Affine differential geometry*, in: Handbook of Differential Geometry, Vol. I, F. Dillen and L. Verstraelen (eds.), Elsevier, 2000, 905–961.

[S-S-V] A. Schwenk-Schellschmidt, U. Simon, and H. Viesel, *Introduction to the Affine Differential Geometry of Hypersurfaces*, Lecture Notes Science Univ. Tokyo, 1991.

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# PROBLEM LIST

This list continues a related list of problems, see T. Binder and U. Simon, *Progress in affine differential geometry—problem list and continued bibliography*, in: Geometry and Topology of Submanifolds $X$, W. H. Chen *et al.* (eds.), World Scientific, Singapore, 2000, 1–17.

## Euclidean geometry

(1) F. Dillen: Study polynomial surfaces and hypersurfaces in $\mathbb{R}^{n+1}$.

(2) H. Furuhata (see his contribution): Give a necessary and sufficient condition for an equiaffine structure $(\nabla, \theta)$ to be realized as minimal affine immersions.

(3) A. Martínez: Let $c$ be a closed curve in $\mathbb{R}^2$. Study surfaces in $\mathbb{R}^3$ with $c$ as prescribed boundary and Euclidean Gauss curvature $K > 0$.

(4) Simon–Vrancken–Voss–Wiehe (see their contribution): Let $x, x^\# : M \to \mathbb{R}^3$ be two Euclidean ovaloids with the same Weingarten operator at corresponding points. Under which additional conditions are $x, x^\#$ congruent modulo a Euclidean motion?

Conjecture for the additional assumption: The set of umbilics has empty open kernel.

Commentary: The conjecture stimulated further research on the local and global behaviour of the Euclidean Weingarten operator. See the papers of R. Bryant and K. Voss in Results Math. 40 (2001).

(5) A. Schwenk, U. Simon, M. Wiehe: Study closed curves in Euclidean $\mathbb{R}^2$ with curvature $K \neq 0$ which are parametrized by the arc length of the Gauss map. Assume that $K$ satisfies a third order eigenvalue equation $(K-c)'' + 9(K-c) = 0$ for an appropriate $0 < c \in \mathbb{R}$. Classify such curves.

## Affine geometry

(6) B. Opozda: Study affine structures on manifolds, in particular study whether given affine connections can be realized as affine immersions (in the sense of Nomizu and Pinkall).

(7) B. Opozda and K. Nomizu, U. Simon and L. Vrancken: Let $x, x^\# : M \to A^{n+1}$ be hyperovaloids with Blaschke structure in real affine space such that the induced connections $\nabla$, $\nabla^\#$ coincide. Are $x, x^\#$ affinely equivalent?

Commentary: The problem has been solved for $n = 2$. See U. Simon, Tohoku J. Math. 44 (1992), 327–334. It can be treated as a local problem under the additional assumption that $\dim(\operatorname{im} R) \geq 1$ for the curvature operator $R$.

(8) U. Simon and L. Vrancken: Which (local and global) examples of centroaffine surfaces with constant sectional curvature are known?

(9) U. Simon: Let $x, x^{\#} : M \to \mathbb{R}^3$ be two Blaschke ovaloids with the same affine Weingarten operator at corresponding points. Which further assumptions are necessary to prove an affine equivalence theorem?

(10) U. Simon: Let $x, x^{\#} : M \to \mathbb{R}^3$ be two Blaschke ovaloids with the same cubic form. Are $x, x^{\#}$ equiaffinely equivalent?