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# Nash regulous functions

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*To Kamil Rusek on his 70th birthday*

**Abstract.** A real-valued function on $\mathbb{R}^n$ is *k-regulous*, where $k$ is a nonnegative integer, if it is of class $\mathcal{C}^k$ and can be represented as a quotient of two polynomial functions on $\mathbb{R}^n$. Several interesting results involving such functions have been obtained recently. Some of them (Nullstellensatz, Cartan's theorems A and B, etc.) can be carried over to a new setting of Nash $k$-regulous functions, introduced in this paper. Here a function on a Nash manifold $X$ is called *Nash k-regulous* if it is of class $\mathcal{C}^k$ and can be represented as a quotient of two Nash functions on $X$.

**1. Introduction.** Throughout this paper by a function on a set $S$ we always mean a real-valued function. Given a collection $F$ of functions on $S$, we set

$$Z(F) := \{x \in S \colon f(x) = 0 \text{ for all } f \in F\}$$

and write $Z(f_1, \dots, f_r)$ for $Z(F)$ if $F = \{f_1, \dots, f_r\}$.

A function $f$ on $\mathbb{R}^n$ is said to be *k-regulous*, where $k$ is a nonnegative integer, if it is of class $\mathcal{C}^k$ and there exist polynomial functions $p$ and $q$ on $\mathbb{R}^n$ such that $Z(q) \neq \mathbb{R}^n$ and $f = p/q$ on $\mathbb{R}^n \setminus Z(q)$; a 0-regulous function is called *regulous*. Such functions, which often appear under different names and in more general contexts, have several remarkable properties and applications [2, 5, 6, 10–24, 27, 29]. In particular, the authors of [5] obtained a variant of the classical Nullstellensatz for the ring $\mathcal{R}^k(\mathbb{R}^n)$ of $k$-regulous functions on $\mathbb{R}^n$, a description of the zero locus $Z(F)$ of an arbitrary collection $F \subseteq \mathcal{R}^k(\mathbb{R}^n)$ in terms of Zariski (algebraically) constructible sets, and counterparts of Cartan's theorems A and B for quasi-coherent $k$-regulous sheaves.

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For the sake of clarity, let us recall that a function $f$ on $\mathbb{R}^n$ which is regulous and of class $\mathcal{C}^\infty$ is actually regular, that is, $f = p/q$ for some polynomial functions $p$ and $q$ on $\mathbb{R}^n$ with $Z(q) = \emptyset$ [12, Proposition 2.1]. Therefore one gains no new insight by considering such functions.

In the present paper we introduce *Nash $k$-regulous functions* and show that most results of [5] can be carried over to this new setting. Along the way we establish connections with arc-analytic functions and constructible categories investigated in [1, 25, 26, 28], and also point out a potential application to the problem of continuous solutions of linear equations [4].

Recall that a *Nash manifold* $X$ is an analytic submanifold of $\mathbb{R}^n$, for some $n$, which is also a semialgebraic set. A *Nash function* on $X$ is an analytic function with semialgebraic graph. The ring $\mathcal{N}(X)$ of Nash functions on $X$ is Noetherian [3, Theorem 8.7.18].

DEFINITION 1.1. A function $f$ on $X$ is said to be *Nash $k$-regulous*, where $k$ is a nonnegative integer, if it is of class $\mathcal{C}^k$ and there exist Nash functions $\varphi$ and $\psi$ on $X$ such that the set $Z(\psi)$ is nowhere dense in $X$ and $f = \varphi/\psi$ on $X \setminus Z(\psi)$; a Nash 0-regulous function is called *Nash regulous*.

The set $\mathcal{N}^k(X)$ of all Nash $k$-regulous functions on $X$ forms a ring. Obviously, $\mathcal{R}^k(\mathbb{R}^n)$ is a subring of $\mathcal{N}^k(\mathbb{R}^n)$.

EXAMPLE 1.2. The function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \frac{x^{3+k}}{x^2+y^2} \quad \text{for } (x,y) \neq (0,0) \quad \text{and} \quad f(0,0) = 0$$

is $k$-regulous but not $(k+1)$-regulous. In turn, the function $g\colon \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) := f(x,y)\sqrt{1+x^2}$, is Nash $k$-regulous but not $k$-regulous.

A more interesting example is the following.

EXAMPLE 1.3. Consider the function $h\colon \mathbb{R}^3 \to \mathbb{R}$ defined by

$$h(x,y,z) = \begin{cases} \dfrac{x^3}{x^2 + (1+z^2)^{1/3}xy + (1+z^2)^{2/3}y^2} & \text{if } x^2+y^2 \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $x^2 + (1+z^2)^{1/3}xy + (1+z^2)^{2/3}y^2 \geq \frac{1}{2}(x^2+y^2)$, we conclude that $h$ is a Nash regulous function.

It readily follows from [3, Proposition 2.2.2] that any Nash regulous function is semialgebraic. Furthermore, by [3, Proposition 8.1.8], any function on $X$ which is Nash regulous and of class $\mathcal{C}^\infty$ is actually a Nash function. Evidently, if $\dim X = 1$, then any Nash regulous function on $X$ is a Nash function.

Our main results on geometric and algebraic properties of Nash $k$-regulous functions are Theorem 3.5 (characterization of $Z(F)$ for $F \subseteq \mathcal{N}^k(X)$)

and Theorem 3.6 (Nullstellensatz for $\mathcal{N}^k(X)$). They depend on some constructions that involve Nash sets.

Recall that a subset $V \subseteq X$ is called a *Nash subset* if it can be written in the form $V = Z(F)$ for some collection $F$ of Nash functions on $X$. Since the ring $\mathcal{N}(X)$ is Noetherian, one can find functions $f_1, \ldots, f_r$ in $F$ such that $V = Z(f)$ with $f = f_1^2 + \cdots + f_r^2$. In particular, each Nash subset of $X$ is semialgebraic.

Definition 1.4. A subset $E$ of $X$ is said to be *Nash constructible* if it belongs to the Boolean algebra generated by the Nash subsets of $X$, or equivalently if $E$ is a finite union of sets of the form $V \setminus W$, where $V$ and $W$ are Nash subsets of $X$.

Basic properties of Nash constructible sets are established in Section 2 and used in the remainder of the paper.

Unless explicitly stated otherwise, we consider $X$ endowed with the *Euclidean topology* induced by the standard metric on $\mathbb{R}$. However, the *Nash topology* and the *Nash constructible topology*, both defined in Section 2, are also useful. In Section 4 we introduce a locally ringed space $(X, \mathcal{N}_X^k)$, with underlying Nash constructible topology, and study sheaves of $\mathcal{N}_X^k$-modules on $X$. The main results are Theorem 4.4 (Cartan's theorem A) and Theorem 4.5 (Cartan's theorem B). Essentially, the ringed space $(X, \mathcal{N}_X^k)$ is equivalent to the affine scheme $\mathrm{Spec}(\mathcal{N}^k(X))$. We will see that this scheme is rather unusual, for every open subset of it is affine. Let us recall that Cartan's theorems A and B fail for coherent Nash sheaves [9].

For background material on real algebraic geometry we refer the reader to [3]. The *dimension* of a semialgebraic set $A$ in $\mathbb{R}^n$, written $\dim A$, is defined as the maximum dimension of a Nash submanifold of $\mathbb{R}^n$ contained in $A$. Recall that $\dim A$ is equal to the dimension of the Zariski closure of $A$ in $\mathbb{R}^n$. If $V$ is a Nash subset of $X$, then $\dim V$ is, by definition, the dimension of $V$ regarded as a semialgebraic set.

**2. Nash constructible sets.** There are elementary concepts of irreducible subsets and decomposition into irreducible components in an arbitrary topological space $S$. They are particularly useful if $S$ is a *Noetherian space*, that is, every descending chain of closed subsets of $S$ is stationary. For these notions, [7, pp. 13–15] can be consulted.

Let $X$ be a Nash manifold. The collection of all Nash subsets of $X$ is the family of closed subsets for some topology on $X$, called the *Nash topology*. Since the ring $\mathcal{N}(X)$ of Nash functions on $X$ is Noetherian, the Nash topology is Noetherian. It readily follows that if $Z' \subseteq Z$ are Nash subsets of $X$ with $Z' \neq Z$ and $Z$ Nash irreducible, then $\dim Z' < \dim Z$ (recall our convention on dimension).

Let $V \subseteq X$ be an irreducible Nash subset of dimension $d$. A point $x \in V$ is said to be *regular in dimension* $d$ if there exists an open (in the Euclidean topology) semialgebraic neighborhood $U \subseteq X$ of $x$ such that $V \cap U$ is a Nash submanifold of $U$ of dimension $d$. The set $\operatorname{Reg}_d(V)$ of all regular points of $V$ in dimension $d$ is a semialgebraic subset for which

$$\dim(V \setminus \operatorname{Reg}_d(V)) < d.$$

Denoting by $V^{\mathrm{sing}}$ the closure of $V \setminus \operatorname{Reg}_d(V)$ in the Nash topology on $X$, we get

$$\dim(V \setminus \operatorname{Reg}_d(V)) = \dim V^{\mathrm{sing}}.$$

Now let $W \subseteq X$ be a Nash subset and let $W_1, \ldots, W_r$ be its Nash irreducible components. We set

$$W^{\mathrm{sing}} := \bigcup_i W_i^{\mathrm{sing}} \cup \bigcup_{j \neq k} (W_j \cap W_k) \quad \text{and} \quad W^{\mathrm{ns}} := W \setminus W^{\mathrm{sing}}.$$

Consequently, $W^{\mathrm{sing}}$ is a Nash subset of $X$ with

$$\dim W^{\mathrm{sing}} < \dim W.$$

PROPOSITION 2.1. *Let $X$ be a Nash manifold, $E \subseteq X$ a closed (in the Euclidean topology) Nash constructible subset, and $W$ the Nash closure of $E$ in $X$. Then $W^{\mathrm{ns}} \subseteq E$.*

*Proof.* The set $E$ can be written as a finite union

$$E = \bigcup_{i \in I} (Z_i \setminus Z_i'),$$

where $Z_i' \subseteq Z_i$ are Nash subsets of $X$ with $Z_i' \neq Z_i$ and $Z_i$ Nash irreducible for all $i \in I$. Note that $Z_i \subseteq W$ because $Z_i \setminus Z_i' \subseteq W$ and the Nash closure of $Z_i \setminus Z_i'$ in $X$ is equal to $Z_i$. Let $W_1, \ldots, W_r$ be the Nash irreducible components of $W$. Since $Z_i$ is Nash irreducible, we have $Z_i \subseteq W_j$ for some $j$. Setting $I_j := \{i \in I \colon Z_i \subseteq W_j\}$, we get

$$W_j = \bigcup_{i \in I_j} Z_i$$

for all $j = 1, \ldots, r$. Furthermore, if

$$W_j' := \bigcup_{i \in I_j} Z_i',$$

then $\dim W_j' < \dim W_j$ and $W_j \setminus W_j' \subseteq E$. In particular, $W_j^{\mathrm{ns}} \setminus W_j' \subseteq E$ for every $j$. Since $W_j^{\mathrm{ns}} \setminus W_j'$ is Euclidean dense in $W_j^{\mathrm{ns}}$, we get $W_j^{\mathrm{ns}} \subseteq E$ ($E$ is closed in the Euclidean topology). Hence

$$W^{\mathrm{ns}} \subseteq W_1^{\mathrm{ns}} \cup \cdots \cup W_r^{\mathrm{ns}} \subseteq E,$$

as required. ∎

Proposition 2.2. *Let $X$ be a Nash manifold and let*

$$E_1 \supseteq E_2 \supseteq \cdots$$

*be a chain of closed (in the Euclidean topology) Nash constructible subsets of $X$. Then the chain is* stationary*, that is, there exists a positive integer $m$ such that $E_m = E_i$ for all $i \geq m$.*

*Proof.* Let $W_i$ be the Nash closure of $E_i$ in $X$. We use induction on $\dim W_1$. The case $\dim W_1 = 0$ is obvious since then $W_1$ is a finite set.

Suppose that $\dim W_1 > 0$. The chain

$$W_1 \supseteq W_2 \supseteq \cdots$$

of Nash subsets of $X$ is stationary, hence there exists a positive integer $l$ such that $W_l = W_i$ for all $i \geq l$. Set $W := W_l$. By Proposition 2.1, we get a chain

$$E_l \setminus W^{\mathrm{ns}} \supseteq E_{l+1} \setminus W^{\mathrm{ns}} \supseteq \cdots$$

of closed (in the Euclidean topology) Nash constructible subsets of $X$. This chain is stationary by the induction hypothesis since $E_l \setminus W^{\mathrm{ns}} \subseteq W^{\mathrm{sing}}$ and $\dim W^{\mathrm{sing}} < \dim W \leq \dim W_1$. It follows that the chain

$$E_l \supseteq E_{l+1} \supseteq \cdots$$

is stationary. ∎

In view of Proposition 2.2, the collection of all closed (in the Euclidean topology) Nash constructible subsets of $X$ forms the family of closed sets for a Noetherian topology on $X$, called the *Nash constructible topology*. The reader may compare this with the constructions of topologies in [5, 25, 26, 28].

**3. Nash $k$-regulous functions.** Recall that a function $f$ on an analytic manifold $M$ is said to be *arc-analytic* if for every analytic arc $\gamma\colon (-1,1) \to M$ the composite $f \circ \gamma$ is an analytic function [25].

Proposition 3.1. *Let $X$ be a Nash manifold and let $f$ be a Nash regulous function on $X$. Then $f$ is a semialgebraic arc-analytic function, and its zero locus $Z(f)$ is a Nash constructible set.*

*Proof.* As we already observed in Section 1, $f$ is a semialgebraic function.

For the proof of the other assertions we may assume that $X$ is connected (in the Euclidean topology). Let $\varphi$ and $\psi$ be Nash functions on $X$ such that $Z(\psi) \neq X$ and $f = \varphi/\psi$ on $X \setminus Z(\psi)$. By the Artin–Mazur theorem [3, Theorem 8.4.4], there exist a nonsingular irreducible algebraic set $V \subseteq \mathbb{R}^m$, an open semialgebraic subset $X' \subseteq V$, a Nash diffeomorphism $\sigma\colon X \to X'$, and polynomial functions $p$ and $q$ on $\mathbb{R}^m$ such that

$$p(\sigma(x)) = \varphi(x) \quad \text{and} \quad q(\sigma(x)) = \psi(x) \quad \text{for all } x \in X.$$

Evidently, the function $g := f \circ \sigma^{-1}\colon X' \to \mathbb{R}$ is continuous and $g = p/q$ on $X' \cap (V \setminus Z(q))$. In view of [10, Proposition 4.2, Theorem 1.12], $g$ is an arc-analytic function. Furthermore, according to [10, Propositions 4.2 and 3.5], $Z(g)$ is a Nash constructible subset of $X'$. The proof is complete since $\sigma$ is a Nash diffeomorphism. ∎

Corollary 3.2. *Let $X$ be a Nash manifold and let $F$ be a collection of Nash regulous functions on $X$. Then the zero locus $Z(F)$ is a closed subset of $X$ in the Nash constructible topology.*

*Proof.* This holds since

$$Z(F) = \bigcap_{f \in F} Z(f)$$

and, by Proposition 3.1, each $Z(f)$ is a closed set in the Nash constructible topology. ∎

It is convenient to generalize Definition 1.1 as follows.

Definition 3.3. Let $X$ be a Nash manifold and let $U \subseteq X$ be an open subset (in the Euclidean topology). A function $f$ on $U$ is said to be *Nash $(X,k)$-regulous*, where $k$ is a nonnegative integer, if it is of class $\mathcal{C}^k$ and there exist Nash functions $\varphi$ and $\psi$ on $X$ such that the set $Z(\psi)$ is nowhere dense in $X$ and $f = \varphi/\psi$ on $U \setminus Z(\psi)$.

The set $\mathcal{N}^k_X(U)$ of all Nash $(X,k)$-regulous functions on $U$ forms a ring. It depends on the triple $(X,U,k)$ and not on the pair $(U,k)$ alone. If $U' \subseteq U$ is an open subset, then there is a well-defined homomorphism

$$\mathcal{N}^k_X(U) \to \mathcal{N}^k_X(U')$$

determined by the restriction of functions. Obviously, $\mathcal{N}^k_X(X) = \mathcal{N}^k(X)$.

The following is a Nash $k$-regulous version of [5, Lemme 5.1].

Proposition 3.4. *Let $X$ be a Nash manifold, $k$ a nonnegative integer, $f\colon X \to \mathbb{R}$ a Nash $k$-regulous function, and $g\colon X \setminus Z(f) \to \mathbb{R}$ a Nash $(X,k)$-regulous function. Then, for each integer $N$ large enough, the function $h\colon X \to \mathbb{R}$ defined by*

$$h = f^N g \quad \text{on } X \setminus Z(f) \quad \text{and} \quad h = 0 \quad \text{on } Z(f)$$

*is Nash $k$-regulous.*

*Proof.* The function $f$ is semialgebraic. Similarly, in view of [3, Proposition 2.2.2], $g$ is also semialgebraic. Consequently, according to [3, Proposition 2.6.4], $h$ is continuous if $N$ is large enough. Note that then $h$ is actually a Nash regulous function. Thus it suffices to prove that for each integer $s$ large enough the product $f^s g$, defined on $X \setminus Z(f)$, extends by zero through the zero locus $Z(f)$ to a $\mathcal{C}^k$ function on $X$. Instead of computing partial derivatives in some local coordinates, it seems more convenient to make use of globally defined vector fields on $X$.

Let $\eta$ be a Nash vector field on $X$ and let $u$ be a function of class $\mathcal{C}^p$, where $p \geq 1$, defined on an open subset $U \subseteq X$. By applying $\eta$ to $u$, we get a function $\eta u = \eta(u)$ of class $\mathcal{C}^{p-1}$ on $U$. If $u$ is a semialgebraic function (hence $U$ is a semialgebraic set), then so is $\eta u$. Given an integer $q$ with $0 \leq q \leq p$, we define $\eta^q u$ recursively:

$$\eta^q u = u \quad \text{for } q = 0 \quad \text{and} \quad \eta^q u = \eta(\eta^{q-1} u) \quad \text{for } q \geq 1.$$

Now let $\xi = (\xi_1, \ldots, \xi_n)$ be an $n$-tuple of Nash vector fields that generate the tangent bundle to $X$. For any $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers with $|\alpha| := \alpha_1 + \cdots + \alpha_n \leq k$, the function

$$\xi^\alpha(g) := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} g$$

is continuous semialgebraic on $X \setminus Z(f)$. By [3, Proposition 2.6.4], we can choose a positive integer $r$ so that the functions

$$f^r \xi^\alpha(g) \quad \text{for all } \alpha \text{ with } |\alpha| \leq k$$

extend by zero to continuous functions on $X$. Hence, by the Leibniz rule, $f^{r+k} g$ extends by zero to a function of class $\mathcal{C}^k$ on $X$ which is $k$-flat on $Z(f)$. ∎

Proposition 3.4 plays an essential role in the proof of the following.

Theorem 3.5. *Let $X$ be a Nash manifold, $E \subseteq X$ some subset, and $k$ a nonnegative integer. Then the following conditions are equivalent:*

(a) *$E$ is closed in the Nash constructible topology.*
(b) *$E = Z(f)$ for some Nash $k$-regulous function $f$ on $X$.*
(c) *$E = Z(F)$ for some collection $F$ of Nash $k$-regulous functions on $X$.*

*Furthermore, if* (c) *holds, then there exist functions $g_1, \ldots, g_r$ in $F$ such that*

$$E = Z(g) \quad \textit{with} \quad g = g_1^2 + \cdots + g_r^2.$$

*Proof.* Suppose that condition (a) is satisfied, and let $W \subseteq X$ be the closure of $E$ in the Nash topology. We use induction on $\dim W$ to prove that (b) holds. The case $\dim W = 0$ is obvious since then $W$ is a finite set.

Assume that $\dim W > 0$. By Proposition 2.1, $W^{\mathrm{ns}} \subseteq E$ and hence

$$W = E \cup Z, \quad \text{where} \quad Z = W^{\mathrm{sing}} = W \setminus W^{\mathrm{ns}}.$$

Let $\varphi$ and $\psi$ be Nash functions on $X$ with $Z(\varphi) = W$ and $Z(\psi) = Z$. Consider the functions

$$\frac{1}{\varphi^2} \colon X \setminus W = (X \setminus E) \setminus (Z \setminus (E \cap Z)) \to \mathbb{R}, \qquad \psi|_{X \setminus E} \colon X \setminus E \to \mathbb{R}.$$

Since $Z(\psi|_{X\setminus E}) = Z \setminus (E \cap Z)$, it follows from Proposition 3.4 that for a sufficiently large integer $N$ the function $\alpha\colon X \setminus E \to \mathbb{R}$ defined by

$$\alpha = \frac{\psi^{2N}}{\varphi^2} \quad \text{on } X \setminus W \quad \text{and} \quad \alpha = 0 \quad \text{on } Z \setminus (E \cap Z)$$

is Nash $k$-regulous. Clearly, $\alpha$ is actually Nash $(X, k)$-regulous.

We claim that the function $\beta\colon X \setminus (E \cap Z) \to \mathbb{R}$ defined by

$$\beta = \frac{\varphi^2}{\varphi^2 + \psi^{2N}} \quad \text{on } X \setminus Z \quad \text{and} \quad \beta = 1 \quad \text{on } Z \setminus (E \cap Z)$$

is Nash $(X, k)$-regulous. Indeed, since

$$X \setminus (E \cap Z) = (X \setminus E) \cup (X \setminus Z) \quad \text{and} \quad (X \setminus E) \cap (X \setminus Z) = X \setminus W,$$

it suffices to show that $\beta$ is of class $\mathcal{C}^k$ on $X \setminus E$. However, on $X \setminus E$ we have $\beta = \frac{1}{1+\alpha}$, which proves the claim.

By construction,

$$Z(\beta) = W \setminus Z = E \setminus (E \cap Z).$$

Since $\dim Z < \dim W$, it follows from the induction hypothesis that there exists a Nash $k$-regulous function $\gamma\colon X \to \mathbb{R}$ with $Z(\gamma) = E \cap Z$. In view of Proposition 3.4 once again, if $M$ is a sufficiently large integer, then the function $f\colon X \to \mathbb{R}$ defined by

$$f = \gamma^M \beta \quad \text{on } X \setminus (E \cap Z) \quad \text{and} \quad f = 0 \quad \text{on } E \cap Z$$

is Nash $k$-regulous. Evidently, $Z(f) = E$, hence (b) holds.

It is clear that (b) implies (c), while in turn (c) implies (a) by Corollary 3.2. In conclusion, conditions (a), (b), (c) are equivalent.

For the last assertion, suppose that condition (c) holds. It suffices to prove that $E = Z(g_1, \ldots, g_r)$ for some functions $g_1, \ldots, g_r$ in $F$. If this were not the case, we could choose functions $g_1, g_2, \ldots$ in $F$ for which

$$Z(g_1) \supsetneq Z(g_1, g_2) \supsetneq \cdots .$$

However, such a chain cannot exist, the Nash constructible topology on $X$ being Noetherian. ∎

There are analogous results to Theorem 3.5 for $k$-regulous functions [5, Théorèmes 5.21, 6.4] and for arc-analytic functions [1, Theorem 1.1].

In the remainder of this section we obtain counterparts of suitable results for $k$-regulous functions [5].

We investigate ideals in the ring of Nash $k$-regulous functions on a Nash manifold $X$. Given a subset $E \subseteq X$, denote by $J_{\mathcal{N}^k}(E)$ the ideal of $\mathcal{N}^k(X)$ consisting of all functions vanishing on $E$,

$$J_{\mathcal{N}^k}(E) \coloneqq \{f \in \mathcal{N}^k(X)\colon f(x) = 0 \text{ for all } x \in E\}.$$

As usual, the radical of an ideal $I$ of $\mathcal{N}^k(X)$ will be denoted by $\mathrm{Rad}(I)$.

We have the following variant of the Nullstellensatz for the ring $\mathcal{N}^k(X)$.

THEOREM 3.6. *Let $X$ be a Nash manifold and let $I$ be an ideal of the ring $\mathcal{N}^k(X)$, where $k$ is a nonnegative integer. Then*

$$J_{\mathcal{N}^k}(Z(I)) = \mathrm{Rad}(I).$$

*Proof.* It is clear that $\mathrm{Rad}(I) \subseteq J_{\mathcal{N}^k}(Z(I))$.

For the converse inclusion, we first pick a function $g \in I$ with $Z(g) = Z(I)$; this is possible by Theorem 3.5. Consider a function $f \in J_{\mathcal{N}^k}(Z(I))$. Note that $X \setminus Z(f) \subseteq X \setminus Z(g)$, and $1/g$ is a Nash $(X, k)$-regulous function on $X \setminus Z(g)$. Hence, by Proposition 3.4, there exists a positive integer $N$ such that the function $h$ on $X$ defined by

$$h = \frac{f^N}{g} \quad \text{on } X \setminus Z(g) \quad \text{and} \quad h = 0 \quad \text{on } Z(g)$$

is Nash $k$-regulous. Since $f^N = gh$, we get $f \in \mathrm{Rad}(I)$, as required. ∎

A straightforward consequence of Theorems 3.5 and 3.6 is the following.

COROLLARY 3.7. *Let $X$ be a Nash manifold and let $k$ be a nonnegative integer. Then the assignment*

$$X \supseteq E \mapsto J_{\mathcal{N}^k}(E) \subseteq \mathcal{N}^k(X)$$

*gives rise to one-to-one correspondences:*

(i) *between the closed subsets of $X$ in the Nash constructible topology and the radical ideals of $\mathcal{N}^k(X)$;*
(ii) *between the irreducible closed subsets of $X$ in the Nash constructible topology and the prime ideals of $\mathcal{N}^k(X)$;*
(iii) *between the points of $X$ and the maximal ideals of $\mathcal{N}^k(X)$.* ∎

For a function $f$ in $\mathcal{N}^k(X)$, we let $\mathrm{Rad}(f)$ denote the radical of the principal ideal generated by $f$.

The following is a direct consequence of Theorem 3.6.

COROLLARY 3.8. *Let $X$ be a Nash manifold and let $f$ and $g$ be functions in the ring $\mathcal{N}^k(X)$, where $k$ is a nonnegative integer. Then $\mathrm{Rad}(f) = \mathrm{Rad}(g)$ if and only if $Z(f) = Z(g)$.* ∎

The radical of an arbitrary ideal can be described as follows.

COROLLARY 3.9. *Let $X$ be a Nash manifold and let $I$ be an ideal of the ring $\mathcal{N}^k(X)$, where $k$ is a nonnegative integer. Then*

$$\mathrm{Rad}(I) = \mathrm{Rad}(f)$$

*for some $f \in I$. Furthermore, the equality of radicals holds if and only if $Z(I) = Z(f)$.*

*Proof.* By Theorem 3.5, $Z(I) = Z(f)$ for some $f \in I$. The proof is complete in view of Theorem 3.6. ■

As usual, for $f$ in $\mathcal{N}^k(X)$, we let $\mathcal{N}^k(X)_f$ denote the localization of the ring $\mathcal{N}^k(X)$ with respect to the multiplicatively closed subset

$$\{1\} \cup \{f^m : m = 1, 2, \ldots\}.$$

In particular, $\mathcal{N}^k(X)_f$ is the zero ring if $f$ is identically equal to 0. By convention, $\mathcal{N}^k_X(\emptyset)$ is also the zero ring. Proposition 3.4 allows us to give a geometric description of $\mathcal{N}^k(X)_f$, which will be useful in Section 4.

PROPOSITION 3.10. *Let $X$ be a Nash manifold, $k$ a nonnegative integer, and $f$ a Nash $k$-regulous function on $X$. Then the restriction homomorphism $\mathcal{N}^k(X) \to \mathcal{N}^k_X(X \setminus Z(f))$ induces an isomorphism between the localization $\mathcal{N}^k(X)_f$ and $\mathcal{N}^k_X(X \setminus Z(f))$.*

*Proof.* By Proposition 3.4, the induced homomorphism

$$\mathcal{N}^k(X)_f \to \mathcal{N}^k_X(X \setminus Z(f))$$

is surjective.

If an element $g/f^m \in \mathcal{N}^k(X)_f$ is sent to $0 \in \mathcal{N}^k_X(X \setminus Z(f))$, then $fg = 0$ in $\mathcal{N}^k(X)$. Hence $g/f^m = 0$, which means that the induced homomorphism is injective. ■

If $\dim X = 1$, then $\mathcal{N}^k(X) = \mathcal{N}(X)$ is a Noetherian ring. The case $\dim X \geq 2$ is entirely different.

REMARK 3.11. The ring $\mathcal{N}^k(\mathbb{R}^n)$ is not Noetherian if $n \geq 2$ and $k \geq 0$. Indeed, by Proposition 3.1, Nash regulous functions are arc-analytic, and hence the argument in [7, Example 6.11] for the ring of arc-analytic functions can be easily adapted to the case under consideration (cf. also [5, Proposition 4.16]).

It would not be much harder to prove that the ring $\mathcal{N}^k(X)$ is not Noetherian for any Nash manifold $X$ of dimension at least 2. We leave the details to the interested reader.

Our interest in Nash regulous functions originated from an attempt, not conclusive yet, to strengthen a result of [4].

PROBLEM 3.12. Consider a linear equation

$$f_1 y_1 + \cdots + f_r y_r = g,$$

where $g$ and the $f_i$ are polynomial (or regular) functions on $\mathbb{R}^n$. Assume that it admits a solution where the $y_i$ are continuous functions on $\mathbb{R}^n$. Then, according to [4, Section 2], it also has a continuous semialgebraic solution. One could hope to prove that it has a regulous solution. This is indeed the case for $n = 2$ [22, Corollary 1.7], but fails for any $n \geq 3$ [11, Example 6].

It would be interesting to decide whether or not the equation has a Nash regulous solution.

**4. Nash $k$-regulous sheaves.** In order to avoid awkward repetitions, we begin by fixing some notation.

NOTATION 4.1. Throughout this section, $X$ stands for a Nash manifold and $k$ stands for a nonnegative integer. We will consider $X$ endowed with the *Nash constructible topology* (Section 2). Thus, a subset $E \subseteq X$ is closed if and only if $E = Z(f)$ for some $f \in \mathcal{N}^k(X)$ (Theorem 3.5). We set $X(f) := X \setminus Z(f)$.

For every open subset $U$ of $X$, the ring $\mathcal{N}^k_X(U)$ of Nash $(X, k)$-regulous functions is defined (Definition 3.3). It readily follows that the assignment

$$\mathcal{N}^k_X : U \mapsto \mathcal{N}^k_X(U)$$

is a sheaf of rings on $X$, and $(X, \mathcal{N}^k_X)$ is a locally ringed space.

There is a close connection between the ringed space $(X, \mathcal{N}^k_X)$ and the affine scheme $\operatorname{Spec}(\mathcal{N}^k(X))$. We first describe relationships between the underlying topological spaces.

By definition, a subset $V \subseteq \operatorname{Spec}(\mathcal{N}^k(X))$ is closed if and only if $V = V(F)$ for some collection $F \subseteq \mathcal{N}^k(X)$, where

$$V(F) := \{\mathfrak{p} \in \operatorname{Spec}(\mathcal{N}^k(X)) \colon F \subseteq \mathfrak{p}\}.$$

If $I$ is the ideal generated by $F$, then $V(F) = V(\operatorname{Rad}(I))$. Hence, according to Corollary 3.9, $V(F) = V(f)$ for some $f \in I$; here $V(f) := V(\{f\})$. Consequently, each open subset of $\operatorname{Spec}(\mathcal{N}^k(X))$ is of the form

$$D(f) := \operatorname{Spec}(\mathcal{N}^k(X)) \setminus V(f)$$

for some $f \in \mathcal{N}^k(X)$. In particular, each open subset of $\operatorname{Spec}(\mathcal{N}^k(X))$ is affine.

Define a map

$$\iota \colon X \to \operatorname{Spec}(\mathcal{N}^k(X))$$

by $\iota(x) = \mathfrak{m}_x$ for all $x \in X$, where

$$\mathfrak{m}_x := \{f \in \mathcal{N}^k(X) \colon f(x) = 0\}$$

($\mathfrak{m}_x = J_{\mathcal{N}^k}(\{x\})$ with notation as in Section 3).

PROPOSITION 4.2. *The map $\iota$ is a topological embedding of $X$ onto the subspace* $\operatorname{Max}(\mathcal{N}^k(X))$ *of* $\operatorname{Spec}(\mathcal{N}^k(X))$ *consisting of the maximal ideals of* $\mathcal{N}^k(X)$. *Furthermore:*

(1) $Z(f) = \iota^{-1}(V(f))$ *for every* $f \in \mathcal{N}^k(X)$.
(2) *For each closed subset $Z \subseteq X$ there exists a unique closed subset $\tilde{Z} \subseteq$* $\operatorname{Spec}(\mathcal{N}^k(X))$ *such that* $Z = \iota^{-1}(\tilde{Z})$.

(3) *$X(f) = \iota^{-1}(D(f))$ for every $f \in \mathcal{N}^k(X)$.*
(4) *For each open subset $U \subseteq X$ there exists a unique open subset $\tilde{U} \subseteq \operatorname{Spec}(\mathcal{N}^k(X))$ such that $U = \iota^{-1}(\tilde{U})$.*

*Proof.* By Corollary 3.7, $\iota$ includes a bijection of $X$ onto $\operatorname{Max}(\mathcal{N}^k(X))$. It follows immediately that (1) holds.

Suppose that $Z \subseteq X$ and $V \subseteq \operatorname{Spec}(\mathcal{N}^k(X))$ are closed subsets with $Z = \iota^{-1}(V)$. Then $Z = Z(f)$ and $V = V(g)$ for some $f, g \in \mathcal{N}^k(X)$. Hence $Z(f) = Z(g)$ and, by Corollary 3.8, $\operatorname{Rad}(f) = \operatorname{Rad}(g)$. Consequently, $V(f) = V(g) = V$, which proves (2).

Conditions (3) and (4) follow from (1) and (2), respectively.

It is now clear that $\iota$ induces a homeomorphism between $X$ and $\operatorname{Max}(\mathcal{N}^k(X))$. ■

It is worthwhile to record the following.

Corollary 4.3. *The topological space* $\operatorname{Spec}(\mathcal{N}^k(X))$ *is Noetherian.*

*Proof.* This follows from Proposition 4.2 since $X$ is a Noetherian topological space. ■

It should be mentioned that Proposition 4.2 and Corollary 4.3 are Nash $k$-regulous versions of [5, Théorème 5.29, Corollaires 5.30, 5.31, 5.32].

Next we study sheaves on the spaces under consideration. For any function $f \in \mathcal{N}^k(X)$, we identify the rings $\mathcal{N}^k(X)_f$ and $\mathcal{N}^k_X(X(f))$ via the canonical isomorphism described in Proposition 3.10. Denoting by $\tilde{\mathcal{N}}^k_X$ the structure sheaf on $\operatorname{Spec}(\mathcal{N}^k(X))$, we get

$$\iota_*\mathcal{N}^k_X = \tilde{\mathcal{N}}^k_X \quad \text{and} \quad \mathcal{N}^k_X = \iota^{-1}\tilde{\mathcal{N}}^k_X.$$

In view of Proposition 4.2, the category of sheaves of Abelian groups (resp. sheaves of $\mathcal{N}^k_X$-modules) on $X$ is equivalent to the category of sheaves of Abelian groups (resp. sheaves of $\tilde{\mathcal{N}}^k_X$-modules) on $\operatorname{Spec}(\mathcal{N}^k(X))$. The equivalence is effected by the direct image functor $\mathcal{F} \mapsto \iota_*\mathcal{F}$, whose inverse is the inverse image functor $\mathcal{G} \mapsto \iota^{-1}\mathcal{G}$. Via these equivalences, quasi-coherent sheaves of $\mathcal{N}^k_X$-modules on $X$ correspond to quasi-coherent sheaves of $\tilde{\mathcal{N}}^k_X$-modules on $\operatorname{Spec}(\mathcal{N}^k(X))$.

Henceforth, by a *Nash $k$-regulous sheaf* on $X$ we mean a sheaf of $\mathcal{N}^k_X$-modules. Our goal is to establish basic properties of quasi-coherent Nash $k$-regulous sheaves.

For clarity of exposition, it is convenient to recall Cartan's theorem A for affine schemes [7, Theorem 7.16, Corollary 7.17]. For a commutative ring $R$, consider the affine scheme $Y = \operatorname{Spec}(R)$ with structure sheaf $\mathcal{O}_Y$. Any $R$-module $M$ determines a quasi-coherent sheaf $\tilde{M}$ of $\mathcal{O}_Y$-modules on $Y$. The functor $M \mapsto \tilde{M}$ gives an equivalence of categories between the category of $R$-modules and the category of quasi-coherent sheaves of $\mathcal{O}_Y$-modules on $Y$. Its inverse is the global section functor $\mathcal{G} \mapsto \mathcal{G}(Y)$. In particular, every quasi-coherent sheaf of $\mathcal{O}_Y$-modules on $Y$ is generated by its global sections.

Returning to our main topic, for an $\mathcal{N}^k(X)$-module $M$, we define a presheaf $\tilde{M}_X$ of $\mathcal{N}^k_X$-modules on $X$ by

$$\tilde{M}_X(U) := M \otimes_{\mathcal{N}^k(X)} \mathcal{N}^k_X(U)$$

for every open subset $U \subseteq X$. Since $U = X(f)$ for some $f \in \mathcal{N}^k(X)$, we get

$$\tilde{M}_X(U) = \tilde{M}_X(X(f)) = M \otimes_{\mathcal{N}^k(X)} \mathcal{N}^k(X)_f = \tilde{M}(D(f)),$$

where the last equality is the canonical identification. Hence $\tilde{M}_X$ is actually a sheaf and

$$\tilde{M}_X = \iota^{-1}\tilde{M}.$$

We immediately get the following variant of Cartan's theorem A.

THEOREM 4.4. *The functor $M \mapsto \tilde{M}_X$ gives an equivalence of categories between the category of $\mathcal{N}^k(X)$-modules and the category of quasi-coherent Nash $k$-regulous sheaves on $X$. Its inverse is the global section functor $\mathcal{F} \mapsto \mathcal{F}(X)$. In particular, every quasi-coherent Nash $k$-regulous sheaf on $X$ is generated by its global sections.* ■

According to Cartan's theorem B for affine schemes, if $\mathcal{G}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules on $Y = \operatorname{Spec}(R)$, then $H^i(Y, \mathcal{G}) = 0$ for $i \geq 1$ [8, Théorème 1.3.1].

The equivalence of the categories of sheaves on $X$ and on $\operatorname{Spec}(\mathcal{N}^k(X))$ via the functors $\iota_*$ and $\iota^{-1}$ yields the following variant of Cartan's theorem B.

THEOREM 4.5. *If $\mathcal{F}$ is a quasi-coherent Nash $k$-regulous sheaf on $X$, then*

$$H^i(X, \mathcal{F}) = 0 \quad \textit{for all } i \geq 1. \ ■$$

It should be mentioned that Theorems 4.4 and 4.5 are analogous to the results on quasi-coherent $k$-regulous sheaves obtained in [5].

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