COLLOQUIUM MATHEMATICUM

VOL. 150 2017 NO. 1

BLASCHKE HYPERSURFACES WITH CONSTANT NON-POSITIVE AFFINE MEAN CURVATURE

BY

UDO SIMON (Berlin)

Dedicated to the memory of Witold Roter

Abstract. We consider locally strongly convex complete Blaschke hypersurfaces with constant non-positive affine mean curvature and assume additional convexity properties to prove that the hypersurfaces are affine maximal.

1. Introduction. The locally strongly convex, complete Blaschke hypersurfaces with *constant non-negative* affine mean curvature H are classified, namely:

- if $H = \text{const} > 0$ then the hypersurface is a hyperellipsoid;
- if $H \equiv 0$ and the dimension is $n = 2$ then the hypersurface is an elliptic paraboloid.

The classification is known for two notions of completeness, *affine completeness* and *Euclidean completeness*. See [5, Chapter 2] and [7, Chapter 5].

While in the foregoing two cases there exists a unique example in each case, the situation is completely different when we consider locally strongly convex, complete Blaschke hypersurfaces with *constant negative* affine mean curvature, i.e. $H = \text{const} < 0$.

We know a large class of examples, namely the class of hyperbolic affine spheres. There appear further examples in the literature (see [11], [2] and [9]); the Calabi type construction in [2] shows that there are many such examples, but one is far from any classification. Moreover, a new example in [9, (1.3)] seems to indicate that there exist, besides the ones mentioned in [2], still other types of Calabi type compositions.

2010 *Mathematics Subject Classification*: Primary 53A15; Secondary 53C21.
Key words and phrases: constant affine mean curvature, maximum principle of Omori–Yau, affine maximal hypersurface.
Received 21 November 2016.
Published online 22 September 2017.

DOI: 10.4064/cm7155s-2-2017

Considering the examples we would like to state the following conjecture:

Conjecture. Any affine-complete Blaschke hypersurface with constant negative affine mean curvature is a hyperbolic affine sphere or a certain composition of hyperbolic and possibly parabolic affine spheres.

Besides the study of examples there is another approach to the problem due to A. M. Li and B. Wang; in [12, p. 384], they stated the following problem and proved an existence result:

Problem. *Do there exist locally strongly convex, affine complete Blaschke hypersurfaces with constant negative affine mean curvature that are not hyperbolic affine spheres? Give explicit examples. Classify all such hypersurfaces.*

Existence Theorem. *Given a real constant $H < 0$ and a bounded convex domain $\Omega \subset \mathbb{R}^n$ with prescribed boundary value data in terms of a smooth strictly convex function ϕ, one can construct a locally strongly convex, Euclidean complete hypersurface with constant affine mean curvature $H < 0$.*

For the study of the foregoing Existence Problem see also [7, Chapter 6]. It is the aim of our paper to proceed with the study of complete Blaschke hypersurfaces with non-positive affine mean curvature H. To understand the role of other affine invariants in the context of this classification problem, in [4] we investigated the role of the affine support function. In the present paper we study additional global convexity conditions, expressed in terms of the Euclidean Gauß–Kronecker curvature. The role of this invariant in the unimodular geometry was first investigated by Grambow, a student of Blaschke [3] (see also [10, Section 6.2]).

Notational convention. Consider $\mathbb{R}^{n+1}$ as standard Euclidean space and at the same time as unimodular-affine space. The Euclidean structure is given by the standard inner product

$$\langle\ ,\ \rangle : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}.$$

Let $x : M = \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a strictly convex hypersurface immersion; it carries both the induced Euclidean structure and the unimodular-affine structure as a so-called Blaschke hypersurface. We mark the invariants that belong to the different structures as follows: the mark “E” stands for *Euclidean*, while “e” stands for *equiaffine = unimodular-affine.*

2. Theorems.

The following is our main result.

Main Theorem 2.1. *Consider a strictly convex hypersurface immersion $x : M = \mathbb{R}^n \to \mathbb{R}^{n+1}$, equipped with its Euclidean and Blaschke structures. Assume that:*

- *the Blaschke metric $h(e)$ is complete;*

- *the affine mean curvature satisfies* $H(e) = \mathrm{const} \leq 0$;
- *the Euclidean Gauß–Kronecker curvature* $K_n(E)$ *is bounded:* $\sup K_n(E) < \infty$, *and* $K_n(E)$ *is bounded positively away from zero, i.e. there exists* $0 < \epsilon \in \mathbb{R}$ *such that* $K_n(E) \geq \epsilon > 0$ *on* M.

Then $x(M)$ *is affine-maximal, i.e.* $H(e) \equiv 0$.

COROLLARY 2.2. *In the preceding theorem, if* $n = 2$ *then the affine-maximal surface is an elliptic paraboloid* (*see* [6, Chapter 5] *or* [7, Chapter 5]).

REMARK 2.3. (i) The general convexity assumptions for x (x is locally strongly convex) imply that $K_n(E) > 0$ on M. Thus the assumption $K_n(E) \geq \epsilon > 0$ improves the local convexity properties of $x(M)$.

(ii) If we modify the assumptions of the Main Theorem we get the following result; its proof goes along the lines of the proof of the Main Theorem.

THEOREM 2.4. *Consider a strictly convex hypersurface immersion* $x : M = \mathbb{R}^n \to \mathbb{R}^{n+1}$, *equipped with its Euclidean and Blaschke structures. Assume that:*

- *the Blaschke metric* $h(e)$ *is complete;*
- *the affine mean curvature satisfies* $H(e) = \mathrm{const} < 0$.

Then the Euclidean Gauß–Kronecker curvature $K_n(E)$ *satisfies the condition*

$$\inf K_n(E) = 0.$$

3. Blaschke hypersurfaces. We use the above Euclidean inner product to describe duality in the affine case:

$$\langle\ ,\ \rangle : \mathbb{R}^{(n+1)*} \times \mathbb{R}^{n+1} \to \mathbb{R}.$$

Regarding the affine calculus, by the same symbol $\bar{\nabla}$ we denote the canonical flat connections on $\mathbb{R}^{n+1}$ and on $\mathbb{R}^{(n+1)*}$.

Let M be a connected, oriented, differentiable manifold of dimension $n \geq 2$. We use both the invariant Koszul and the standard local calculus. We denote local coordinates by $(x_1, \ldots, x_n)$ and the associated Gauß basis by $\partial_i := \partial/\partial x_i$ for $i = 1, \ldots, n$.

We restrict ourselves to locally strongly convex hypersurface immersions $x : M \to \mathbb{R}^{(n+1)}$ with Blaschke normalization (Y, y), where y is the affine Blaschke normal and Y is a conormal such that $\langle Y, y\rangle = 1$ and $\langle Y, dy(v)\rangle = 0$ for all tangent vectors v on M. The triple (x, Y, y) is called a *Blaschke hypersurface* in affine space $\mathbb{R}^{n+1}$.

The affine normal induces a unimodular volume form

$$\omega(v_1, \ldots, , v_n) := \det(dx(v_1), \ldots, dx(v_n), y);$$

here $(v_1, \dots, v_n)$ is a local frame and det the determinant form fixing the unimodular structure on $\mathbb{R}^{n+1}$; ω is compatible with the Euclidean structure on $x(M)$.

The unimodular geometry of the triple (x, Y, y) can be described in terms of the induced volume form and further affine-geometric invariants defined via the *affine structure equations* of *Gauß* and *Weingarten*:

$$\begin{aligned} \bar{\nabla}_v dx(w) &= dx(\nabla_v w) + h(v, w)y, \\ dy(v) &= dx(-B(v)). \end{aligned}$$

Here and in the following, $u, v, w, \dots$ denote tangent vectors and fields. The *induced connection* ∇ is torsion free and Ricci symmetric (i.e. its Ricci tensor is symmetric), h is bilinear, symmetric and definite; we choose the orientation of y such that h is positive definite; thus (M, h) is a Riemannian manifold in this case; its Riemannian volume form equals the induced volume form.

h is called the *Blaschke metric*; B is the *affine shape* or *affine Weingarten operator*. The shape operator B is h-self adjoint. The *affine mean curvature* is defined by $H := \frac{1}{n} \operatorname{trace} B$.

All coefficients in the structure equations depend on the normalization; they are invariant under the unimodular-affine group of transformations in $\mathbb{R}^{n+1}$ (including parallel translations).

For further details we refer to [10, Chapters 3–4] and the beginning sections in [5] and [6].

Notational convention. In a local notation we use components, adopt the Einstein convention and raise and lower indices with the Blaschke metric h. The notation $\nabla(h)$ in formulas indicates covariant differentiation; for $f \in C^\infty$, in local notation we use a comma, and we denote covariant derivatives in terms of $\nabla(h)$ also by $f_{,i}$, $f_{,ij}$, etc.

3.1. Completeness of Blaschke hypersurfaces. There are different notions of completeness for a Blaschke hypersurface; in our paper we need:

(i) A Blaschke hypersurface is called *affine-complete* if its Blaschke metric h is complete.

(ii) A Blaschke hypersurface is called *Euclidean complete* if the ambient affine space $\mathbb{R}^{n+1}$ carries an additional Euclidean structure, and the induced first fundamental form is complete; one easily verifies that this notion does not depend on the special choice of the ambient Euclidean structure on $\mathbb{R}^{n+1}$; moreover, this notion is affinely invariant.

THEOREM 3.1 (Completeness criterion; see e.g. [7, pp. 51–52]). *Let x be a locally strongly convex, Euclidean complete Blaschke hypersurface in $\mathbb{R}^{n+1}$. Assume that there exists a constant $0 < N \in \mathbb{R}$ such that the h-norm of the*

Weingarten operator B is bounded from above:

$$\|B\|_h \leq N.$$

Then (M, h) is also affine-complete.

4. The maximum principle of Omori–Yau [13]. Let (M, h) be a complete, non-compact Riemannian n-manifold with Ricci curvature bounded from below: $\mathrm{Ric} \geq \delta h$ for some $\delta \in \mathbb{R}$. Let $f \in C^2(M)$ be bounded from below. Then there is a sequence $\{p_i \in M\}_{i \in \mathbb{N}}$ such that the following **O-Y**-relations are satisfied:

(1) $\lim_i f(p_i) = \inf(f)$,
(2) $\lim_i \|\mathrm{grad}_h(f)\|(p_i) = 0$,
(3) $\lim_i (\Delta f)(p_i) \geq 0$.

5. Proof of the Main Theorem. As above we consider $\mathbb{R}^{n+1}$ as standard Euclidean space and at the same time as unimodular-affine space with respect to the given volume form in $\mathbb{R}^{n+1}$. Let $x : M = \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a strictly convex hypersurface immersion; it carries both the induced Euclidean structure and the unimodular-affine structure as a so-called Blaschke hypersurface (see [5], [6], [8], [10]).

We now proceed to the proof of Theorem 2.1.

Step 1. We consider $x(M)$ as the graph of a strictly convex function $g : M = \mathbb{R}^n \to \mathbb{R}$ such that $x(M)$ can be described by

$$p \in \mathbb{R}^n \mapsto (p, g(p)) \in \mathbb{R}^{n+1}.$$

The convexity condition implies that there exists $b \in \mathbb{R}^{n+1}$ with $\|b\|_E = 1$ such that b is nowhere tangential to the hypersurface. We choose the orientation of b in such a way that b and the Euclidean normal μ satisfy

$$0 < \langle \mu, b \rangle = \|\mu\|_E \cdot \|b\|_E \cdot \cos(\mu, b) = \cos(\mu, b) \leq 1$$

at each $p \in M$.

Step 2. It is well known that the Blaschke conormal $Y(e)$ and the Euclidean normal $\mu(E)$ are related by (see [1, p. 165] and [3])

$$K_n(E)^{1/(n+2)} \cdot Y(e) = \mu(E).$$

As above, $K_n(E)$ denotes the Euclidean Gauß–Kronecker curvature.

Step 3. Consider the function $f : M \to \mathbb{R}$ defined by

$$f(p) := \langle Y(e)(p), b \rangle.$$

The function f has the following properties:

- $0 < f(p) = (K_n(E)(p))^{-1/(n+2)} \cdot \cos(\mu(p), b) \leq (K_n(E)(p))^{-1/(n+2)}$.

- In terms of the Blaschke geometry, f satisfies the PDE (see [6, Section 4.1.4])
$$\Delta(e)f + nH(e)f = 0,$$
where $\Delta := \Delta(e)$ denotes the Laplace operator of the Blaschke metric.
- f is bounded above if and only if there exists $0 < \epsilon \in \mathbb{R}$ such that $\epsilon \leq K_n(E)(p)$ for all $p \in M$.

STEP 4. We study consequences of the above PDE for f in terms of the Blaschke geometry; we consider the case $H := H(e) = \text{const} \leq 0$.

- $f \cdot \Delta f = -n \cdot H \cdot f^2 \geq 0$.
- $\Delta\Delta f = (n \cdot H)^2 \cdot f \geq 0$.

STEP 5. Define $G : M \to \mathbb{R}$ by $G := f \cdot \Delta f$. Then:

- $G = -nH \cdot f^2 \geq 0$, thus G is bounded below, and $\sup G < \infty$ if and only if $\sup f < \infty$; as already stated above, we have $\sup f < \infty$ if and only if $K_n(E)$ is positively bounded away from zero.
- $\Delta G = (\Delta f)^2 + f \cdot \Delta\Delta f = 2(nH \cdot f)^2 \geq 0$.

STEP 6. For an appropriate $0 < \delta \in \mathbb{R}$ define $F := +(G+\delta)^{-1/2}$. Then, in terms of the Blaschke geometry, we have:

(6.1) $F > 0$ on M;
(6.2) $\partial_i F = -\frac{1}{2}F^3 \cdot \partial_i G$;
(6.3) $\Delta F = \frac{3}{4} \cdot F^5 \cdot \|\text{grad}\, G\|_e^2 - \frac{1}{2} \cdot F^3 \Delta G$;
(6.4) $F^4 \Delta G = 3\|\text{grad}\, F\|_e^2 - \frac{1}{2} \cdot F^4 \Delta G$.

STEP 7. We discuss the consequences of the additonal convexity property, namely that there exists $\epsilon > 0$ such that $K_n(E) \geq \epsilon > 0$ on M. We have the following equivalent conditions:

- there exists $\epsilon > 0$ such that $K_n(E) \geq \epsilon > 0$ on M;
- $\sup f < \infty$;
- $\sup G < \infty$;
- $\inf F > 0$.

STEP 8. We apply the maximum principle of Omori–Yau to the function F; thus assume that there exists a sequence $\{p_k\}_{k \in \mathbb{N}}$ of points such that

(i) $\lim_k \Delta F|p_k \geq 0$;
(ii) $\lim_k \text{grad}\, F|p_k = 0$;
(iii) $\lim_k F|p_k = \inf F$.

The PDE for F gives

$$\begin{aligned}0 \leq \lim_k (F \cdot \Delta F) &= \lim_k \left[3 \cdot \|\operatorname{grad} F\|^2 - \tfrac{1}{2} \cdot F^4 \cdot \Delta G\right] \\ &= -\tfrac{1}{2} \cdot (\inf F)^4 \cdot \lim_k (\Delta G) \\ &\leq -\tfrac{1}{2} \cdot (\inf F)^4 \cdot (2n^2 H)^2 \cdot \lim_k f^2(p_k) \leq 0.\end{aligned}$$

As $\inf F > 0$, the foregoing calculation implies

$$\lim_k (Hf)^2 = 0.$$

According to the assumptions we have

$$\lim_k f^2 = \lim_k (K_n(E)^{-2/(n+2)} \cdot \cos^2(\mu, b)) \neq 0;$$

thus we finally arrive at $H \equiv 0$. This finishes the proof of Theorem 2.1.

The preceding proof can be modified to get a proof of Theorem 2.4.

6. Examples. In [4] our interest was in the study of locally strongly convex Blaschke hypersurfaces that are not hyperbolic affine spheres, but that have constant negative affine mean curvature. Examples are listed in [11], [2] and [9].

L. Vrancken [11] gives a complete classification of all Blaschke surfaces with the property that the equiaffine principal curvatures are constant and the Blaschke metric is regular. Dillen and Vrancken [2] study certain types of Calabi compositions. As already mentioned, Salah and Vrancken [9] present a new type of example.

Acknowledgements. We thank B. Opozda, L. Vrancken and the referee for helpful comments.

REFERENCES

[1] W. Blaschke, *Vorlesungen über Differentialgeometrie: Affine Differentialgeometrie*, Springer, Berlin, 1923.

[2] F. Dillen and L. Vrancken, *Calabi-type composition of affine spheres*, Differential Geom. Appl. 4 (1994), 303–328.

[3] R. Grambow, *Ableitung der Affininvarianten einer krummen Fläche aus den Bewegungsinvarianten*, Dissertation, Univ. Hamburg, 1922.

[4] A.-M. Li, L. Sheng, and U. Simon, *Blaschke hypersurfaces with constant negative affine mean curvature*, Ann. Global Anal. Geom. 47 (2015), 225–238.

[5] A.-M. Li, U. Simon, and G.-S. Zhao, *Global Affine Differential Geometry of Hypersurfaces*, 1st ed., de Gruyter, Berlin, 1993.

[6] A.-M. Li, U. Simon, G.-S. Zhao, and Z. Hu, *Global Affine Differential Geometry of Hypersurfaces*, 2nd ed., de Gruyter, Berlin, 2015.

[7] A.-M. Li, R. Xu, U. Simon, and F. Jia, *Affine Bernstein Problems and Monge–Ampère Equations*, World Sci., Hackensack, NJ, 2010.

[8] K. Nomizu and T. Sasaki, *Affine Differential Geometry*, Cambridge Univ. Press, 1994.

[9] A. C. Salah and L. Vrancken, *Four-dimensional locally strongly convex homogeneous affine hypersurfaces*, J. Geom. 108 (2017), 119–147.

[10] U. Simon, A. Schwenk-Schellschmidt, and H. Viesel, *Introduction to the Affine Differential Geometry of Hypersurfaces*, Lecture Notes, Science Univ. Tokyo, 1991.

[11] L. Vrancken, *Affine surfaces with constant affine curvatures*, Geom. Dedicata 33 (1990), 177–194.

[12] B. Wang and A.-M. Li, *Euclidean complete hypersurfaces with negative constant affine mean curvature*, Results Math. 52 (2008), 383–398.

[13] S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure. Appl. Math. 28 (1975), 201–228.

Udo Simon
Institute Math. MA-8-3
TU Berlin
Straße 17. Juni 136
D-10623 Berlin, Germany
E-mail: simon@math.tu-berlin.de