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On Siegel's theorem

by

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1. Linnik [1] proved in an elementary way the famous theorem of Siegel, according to which the class number $h(-k)$ of the imaginary quadratic field belonging to the fundamental discriminant $-k<0$ satisfies $h(-k)>k^{1/2-\varepsilon}$ if $\varepsilon>0$ and $k>K_0(\varepsilon)$, where $K_0(\varepsilon)$ denotes an ineffective constant depending on ε.

The proof of Linnik is composed of the following 4 parts:

Definition. A real primitive character χ_k has the *property* $A(\beta)$ if there is such a constant $C_0(\chi_k,\beta)$, that for all $N>C_0(\chi_k,\beta)$ there exists an $N_1\in[\sqrt{N},N]$ with the property

$$\Big|\sum_{n\leqslant N_1}\chi_k(n)\mu(n)\Big|>N_1^{\beta}.$$

1. If $h(-k)<k^{1/2-\varepsilon_1}$ with $0<\varepsilon_1<0.01$ for a sufficiently large k depending on ε_1, then $L(s,\chi_k)$ vanishes somewhere in the interval $[1-0.001\varepsilon_1, 1+0.001\varepsilon_1]$.

2. If $L(s,\chi_k)$ vanishes somewhere in the interval $[1-0.001\varepsilon_1,1+{}+0.001\varepsilon_1]$ then χ_k possesses the property $A(\beta)$ for $\beta=1-0.08\varepsilon_1$.

3. If χ_k possesses the property $A(\beta)$ with a $\beta>1/2$ and ε is an arbitrary number with $0<\varepsilon<\beta-1/2$, then there is such a constant $C_1(\beta,\varepsilon,\chi_k)$, that for all $N_1>C_1(\beta,\varepsilon,\chi_k)$ there exists an $N_2\in[N^{2\beta-1-2\varepsilon},N_1]$, such that $\big|\sum_{n\leqslant N_2}\chi_k(n)\lambda(n)\big|>N_2^{\beta-\varepsilon}$.

4. If χ_k possesses the property expressed in the assertion of the 3rd statement for $\beta\geqslant 3/4$, then there is such a constant $C_2(\chi_k,\eta_0)$ that in case of $k>C_2(\chi_k,\eta_0)$ $h(-k)>k^{1/2-\eta(\beta)}$, where $\eta(\beta)=10.5(1-\beta)+\eta_0$ and η_0 is an arbitrary positive number.

In this paper, based on the above sketched order of ideas of Linnik, we give a simpler elementary proof of Siegel's theorem. Our proof will be completely elementary, we shall prove, that for arbitrary positive

ε and for $k > K_0(\varepsilon)$ ($K_0(\varepsilon)$ denotes an ineffective constant depending on ε), $L(1, \chi_k) > k^{-\varepsilon}$, where χ_k is a real primitive character, which can have the form

$$\chi_k(n) = \left(\frac{-k}{n}\right) \quad \text{or} \quad \chi_k(n) = \left(\frac{k}{n}\right).$$

In our proof we don't use the Polya-Vinogradov inequality, we don't need any knowledge of the algebraic theory of numbers; on the other hand from the elementary theory of numbers we need only the multiplicativity of Liouville's λ function and the trivial identities

$$\lambda(a) = \sum_{d|a} 2^{\nu(d)}\lambda(d), \qquad \sum_{d|m}\lambda(d) = \begin{cases} 1 & \text{if} \quad m = l^2, \\ 0 & \text{if} \quad m \neq l^2 \end{cases}$$

besides the basic theorem of number theory. We also use some fundamental properties of the continuous real functions.

We shall prove the following two theorems:

SIEGEL'S THEOREM I. *Let χ_k denote a real primitive character* mod k *and ε an arbitrary positive number. Then*

$$L(1, \chi_k) > k^{-\varepsilon} \quad \textit{if} \quad k > K_0(\varepsilon).$$

SIEGEL'S THEOREM II. *Let χ_k denote a real primitive character* mod k *and ε an arbitrary positive number. Then $L(s, \chi_k)$ doesn't vanish in the interval $[1-k^{-\varepsilon}, 1]$ if $k > K_1(\varepsilon)$. (Here $K_0(\varepsilon)$ and $K_1(\varepsilon)$ denote ineffective constants depending on ε.)*

We define the property $B(\gamma)$ similarly to the property $A(\beta)$ defined by Linnik as follows.

DEFINITION. A real character χ_k *possesses the property $B(\gamma)$* if there is such a constant $N_0(\chi_k, \gamma)$ that for all $N \geq N_0(\chi_k, \gamma)$ there exists an $N_1 \in [N^{1-\gamma/5}, N]$ with the property

$$\Big|\sum_{n \leq N_1} \chi_k(n)\lambda(n)\Big| > N^{1-\gamma}.$$

Let ε denote an arbitrary number, for which $0 < \varepsilon < 1$. Siegel's Theorem I will be the consequence of the following three lemmas. (χ_k denotes a real character mod k.)

LEMMA 1. *If there are infinitely many k's with $L(1, \chi_k) \leq k^{-\varepsilon}$, then for all sufficiently large such k's $L(s, \chi_k)$ vanishes somewhere in the interval* $\left[1-\frac{\varepsilon}{120}, 1\right]$.

LEMMA 2. *If $L(s, \chi_k)$ vanishes somewhere in the interval* $\left[1-\frac{\varepsilon}{120}, 1\right]$, *then χ_k possesses the property $B(\varepsilon/12)$.*

LEMMA 3. *If there exists a real primitive character possessing the property $B(\varepsilon/12)$, then Siegel's Theorem I is true for ε.*

We shall prove further a Lemma called Lemma 0 that makes it possible to give a much simpler proof of Lemma 1, than that of Linnik. After this we prove our Lemma 2 similarly to the 2nd assertion of Linnik, but directly applying it for the λ function, so we completely avoid the 3rd assertion of Linnik, proved by the sieve-method. Finally using a consequence of our Lemma 0 — without using any knowledge of the algebraic theory of numbers — we shall prove our Lemma 3 which is analogous to the 4th assertion of Linnik.

So at first we prove the easy

LEMMA 0. *Let χ_k denote a real character* mod k *and* $g(n) = \sum_{d|n}\chi_k(d)$. *Then for an arbitrary τ with $0 < \tau < 1$, there exists a c_τ, $0 < c_\tau < 1$, such that for all $x \geq 3k/\tau$*

$$\sum_{n \leq x}\frac{g(n)}{n^{1-\tau}} = \left(c_\tau - \frac{1}{\tau}\right)L(1-\tau) + \frac{1}{\tau}x^\tau L(1) \pm \frac{2\sqrt{3}\sqrt{k}\,x^\tau}{\sqrt{\tau}\sqrt{x}},$$

where $\pm a$ denotes a real number with an absolute value not exceeding a.

Proof. We use the following lemma of the elementary real analysis: *For an arbitrary τ, for which $0 < \tau < 1$, there exists a c_τ, $0 < c_\tau < 1$, such that for all $u \geq 1$*

$$\sum_{m \leq u}\frac{1}{m^{1-\tau}} = \frac{1}{\tau}(u^\tau - 1) + c_\tau \pm \frac{1}{u^{1-\tau}} < \frac{1}{\tau}u^\tau.$$

Let z denote a number — to be chosen later — for which $1 \leq z \leq x$. Then

$$\sum_{n \leq x}\frac{g(n)}{n^{1-\tau}} = \sum_{n \leq x}\frac{1}{n^{1-\tau}}\sum_{d|n}\chi_k(d) = \sum_{d \leq x}\frac{\chi_k(d)}{d^{1-\tau}}\sum_{m \leq x/d}\frac{1}{m^{1-\tau}} = \sum_{d \leq z}{}_1 + \sum_{z < d \leq x}{}_2,$$

$$\Sigma_1 = \sum_{d \leq z}\frac{\chi_k(d)}{d^{1-\tau}}\sum_{m \leq x/d}\frac{1}{m^{1-\tau}} = \sum_{d \leq z}\frac{\chi_k(d)}{d^{1-\tau}}\left\{\frac{1}{\tau}\left(\frac{x^\tau}{d^\tau} - 1\right) + c_\tau \pm \frac{d^{1-\tau}}{x^{1-\tau}}\right\}$$

$$= \left(c_\tau - \frac{1}{\tau}\right)\sum_{d \leq z}\frac{\chi_k(d)}{d^{1-\tau}} + \frac{x^\tau}{\tau}\sum_{d \leq z}\frac{\chi_k(d)}{d} \pm \sum_{d \leq z}\frac{1}{x^{1-\tau}}$$

$$= \left(c_\tau - \frac{1}{\tau}\right)L(1-\tau) + \frac{x^\tau}{\tau}L(1) \pm \left(\frac{1}{\tau} - c_\tau\right) \times$$

$$\times \sum_{d > z}^{\infty}\frac{\chi_k(d)}{d^{1-\tau}} \pm \frac{x^\tau}{\tau}\sum_{d > z}^{\infty}\frac{\chi_k(d)}{d} \pm \frac{z}{x^{1-\tau}}.$$

Making use of $\left|\sum_{d=a}^{b}\chi_k(d)\right| \leqslant \varphi(k) < k$, applying the inequality of Abel, considering that $\frac{1}{d^{1-\tau}}\sum_{m\leqslant x/d}\frac{1}{m^{1-\tau}}$ is monotonically decreasing in d and that $0 < \frac{1}{\tau} - c_\tau < \frac{1}{\tau}$, we get

$$\left(\frac{1}{\tau}-c_\tau\right)\left|\sum_{d>z}^{\infty}\frac{\chi_k(d)}{d^{1-\tau}}\right| < \frac{1}{\tau}\cdot k\cdot\frac{1}{z^{1-\tau}} \leqslant \frac{kx^\tau}{\tau z};$$

$$\frac{x^\tau}{\tau}\left|\sum_{d>z}^{\infty}\frac{\chi_k(d)}{d}\right| \leqslant \frac{x^\tau k}{\tau z};$$

$$|\Sigma_2| = \left|\sum_{z<d\leqslant x}\frac{\chi_k(d)}{d^{1-\tau}}\sum_{m\leqslant x/d}\frac{1}{m^{1-\tau}}\right| \leqslant \frac{k}{z^{1-\tau}}\cdot\frac{1}{\tau}\cdot\frac{x^\tau}{z^\tau} = \frac{kx^\tau}{\tau z}.$$

Now set $z = \sqrt{\frac{3kx}{\tau}}$ $(\leqslant x)$. Then

$$\begin{aligned}\sum_{n\leqslant x}\frac{g(n)}{n^{1-\tau}} &= \Sigma_1+\Sigma_2\\ &= \left(c_\tau-\frac{1}{\tau}\right)L(1-\tau)+\frac{x^\tau}{\tau}L(1)\pm\frac{kx^\tau}{\tau z}\pm\frac{kx^\tau}{\tau z}\pm\frac{zx^\tau}{x}\pm\frac{kx^\tau}{\tau z}\\ &= \left(c_\tau-\frac{1}{\tau}\right)L(1-\tau)+\frac{x^\tau}{\tau}L(1)\pm\frac{2\sqrt{3}\sqrt{k}x^\tau}{\sqrt{\tau}\sqrt{x}}.\end{aligned}$$

Q.E.D.

This gives the following

COROLLARY. *If $0<\tau<1/2$ and $x>y\geqslant 3k/\tau$, then*

$$\sum_{n=y+1}^{x}\frac{g(n)}{n^{1-\tau}} = \frac{1}{\tau}(x^\tau-y^\tau)L(1)\pm\frac{4\sqrt{3}\sqrt{k}x^\tau}{\sqrt{\tau}\sqrt{y}}.$$

2. Proof of Lemma 1. For an arbitrary natural number n we have

$$g(1)=1;\qquad g(n)=\prod_{p_i^{\alpha_i}\| n}\left(1+\chi_k(p_i)+\chi_k(p_i^2)+\ldots+\chi_k(p_i^{\alpha_i})\right)\geqslant 0,$$

thus

$$\sum_{n\leqslant x}\frac{g(n)}{n^{1-\tau}}\geqslant 1.$$

From this and the Lemma 0 we get the well-known $L(1,\chi_k)\geqslant 0$, because if we suppose that $L(1,\chi_k)<0$, then with a fixed τ, $0<\tau<\frac{1}{2}$, for $x\to\infty$ we get that

$$1\leqslant\sum_{n\leqslant x}\frac{g(n)}{n^{1-\tau}} = \left(c_\tau-\frac{1}{\tau}\right)L(1-\tau)+\frac{1}{\tau}x^\tau L(1)\pm\frac{2\sqrt{3}\sqrt{k}}{\sqrt{\tau}x^{1/2-\tau}}\to-\infty.$$

Let us suppose that for infinitely many k $L(1,\chi_k)\leqslant k^{-\varepsilon}$ $(0<\varepsilon<1)$. Then we can choose among them k so large that on putting

$$\tau=\frac{\varepsilon}{120}\ \left(<\frac{1}{120}\right)\quad\text{and}\quad x=k^{12}$$

the inequalities

$$x>\frac{3k}{\tau},\qquad\frac{2\sqrt{3}\sqrt{k}x^\tau}{\sqrt{\tau}\sqrt{x}}<\frac{1}{2}$$

and

$$\frac{x^\tau}{\tau}L(1)\leqslant\frac{120}{\varepsilon}k^{\varepsilon/10}\cdot k^{-\varepsilon}<\frac{1}{2}$$

hold. Hence

$$\begin{aligned}1\leqslant\sum_{n\leqslant x}\frac{g(n)}{n^{1-\tau}} &\leqslant \left(c_\tau-\frac{1}{\tau}\right)L(1-\tau)+\frac{1}{\tau}x^\tau L(1)+\frac{2\sqrt{3}\sqrt{k}x^\tau}{\sqrt{\tau}\sqrt{x}}\\ &< \left(c_\tau-\frac{1}{\tau}\right)L(1-\tau)+\frac{1}{2}+\frac{1}{2}.\end{aligned}$$

Since

$$c_\tau-\frac{1}{\tau}<0,\qquad L(1-\tau)=L\left(1-\frac{\varepsilon}{120}\right)<0.$$

Making use of the continuity of $L(s)$ and of $L(1)\geqslant 0$, we infer that indeed $L(s,\chi_k)$ vanishes somewhere in the interval $\left[1-\frac{\varepsilon}{120},1\right]$ indeed.

3. Proof of Lemma 2. Set $\gamma=\frac{\varepsilon}{12}$. Let us suppose that $L(s,\chi_k)$ has a β zero in the interval $\left[1-\frac{\varepsilon}{120},1\right]=\left[1-\frac{\gamma}{10},1\right]$, and contrary to the assertion of Lemma 2 χ_k doesn't possess the property $B(\gamma)$, i.e. there are infinitely many N, such that for all $x\in[N^{1-\gamma/5},N]$

$$\left|\sum_{n\leqslant x}\chi_k(n)\lambda(n)\right|\leqslant N^{1-\gamma}.$$

Then using Abel's inequality and the identity

$$\sum_{d|m}\lambda(d)=\begin{cases}1 & \text{if} \quad m=l^2,\\ 0 & \text{if} \quad m\neq l^2,\end{cases}$$

we get

$$1\leqslant\sum_{\substack{m\leqslant N\\ m=l^2}}\frac{\chi_k(m)}{m^\beta}=\sum_{m\leqslant N}\frac{\chi_k(m)}{m^\beta}\sum_{d|m}\lambda(d)=\sum_{d\leqslant N}\frac{\chi_k(d)\lambda(d)}{d^\beta}\sum_{m\leqslant N/d}\frac{\chi_k(m)}{m^\beta}$$

$$=\sum_{d\leqslant N^{1-\gamma/5}}\frac{\chi_k(d)\lambda(d)}{d^\beta}\sum_{m\leqslant N/d}\frac{\chi_k(m)}{m^\beta}+\sum_{m\leqslant N^{\gamma/5}}\frac{\chi_k(m)}{m^\beta}\sum_{N^{1-\gamma/5}<d\leqslant N/m}\frac{\chi_k(d)\lambda(d)}{d^\beta}$$

$$<\Big(\sum_{d\leqslant N^{1-\gamma/5}}\frac{\chi_k(d)\lambda(d)}{d^\beta}\Big)L(\beta)+\sum_{d\leqslant N^{1-\gamma/5}}\frac{1}{d^\beta}\cdot\frac{k}{(N/d)^\beta}+\sum_{m\leqslant N^{\gamma/5}}\frac{1}{m^\beta}\cdot\frac{2N^{1-\gamma}}{N^{(1-\gamma/5)\beta}}$$

$$\leqslant\frac{kN^{1-\gamma/5}}{N^{1-\gamma/10}}+\frac{N^{\gamma/5}\cdot 2N^{1-\gamma}}{N^{(1-\gamma/5)(1-\gamma/10)}}\leqslant\frac{k}{N^{\gamma/10}}+\frac{2}{N^{\gamma/2}}$$

and this inequality cannot be valid for infinitely many N.

4. Proof of Lemma 3. Let us suppose that χ_k possesses the property $B(\gamma)$ for $\gamma=\varepsilon/12$ $(0<\varepsilon<1)$, then we shall prove that if χ_D is a real primitive character $\bmod D$, where D is sufficiently large depending on ε, further $N=D^6\geqslant\max(N_0(\chi_k,\gamma),k^{10})$, then $L(1,\chi_D)>D^{-\varepsilon}$. (Here χ_k denotes a real primitive character $\bmod k$.) Since $N\geqslant N_0(\chi_k,\gamma)$ there exists an $N_1\in[N^{1-\gamma/5},N]$ that

$$\Big|\sum_{n\leqslant N_1}\chi_k(n)\lambda(n)\Big|\geqslant N^{1-\gamma}.$$

Let $A_l=\{u\leqslant N_1;\,p\ \text{prime},\ p|u\to\chi_D(p)=l\}$ for $l=-1,0,1$. Let $C=\{u\leqslant N_1;\,u=v_1v_0,\ v_1\in A_1,\ v_0\in A_0\}$. Then an arbitrary $n\leqslant N_1$ can be written as $n=abm$, where $a\in A_1$, $b\in A_0$, $m\in A_{-1}$. Let us suppose that $a_l|a$, $b_j|b$, then

$$\lambda(m)=\chi_D(m)=\chi_D\Big(m\cdot\frac{a}{a_l}\Big)$$

and

$$\chi_D\Big(m\cdot\frac{a}{a_l}\cdot\frac{b}{b_j}\Big)=\begin{cases}0 & \text{if} \quad b_j\neq b,\\ \chi_D\Big(m\cdot\dfrac{a}{a_l}\Big) & \text{if} \quad b_j=b.\end{cases}$$

Then using $\lambda(a)=\sum_{d|a}2^{\nu(d)}\lambda(d)$ we get

$$\lambda(n)=\lambda(b)\lambda(a)\chi_D(m)=\lambda(b)\sum_{a_l|a}2^{\nu(a_l)}\lambda(a_l)\chi_D\Big(m\cdot\frac{a}{a_l}\Big)$$

$$=\sum_{b_j|b}\sum_{a_l|a}2^{\nu(a_l)}\lambda(a_l)\chi_D\Big(m\cdot\frac{a}{a_l}\cdot\frac{b}{b_j}\Big)\lambda(b_j)=\sum_{\substack{c_l\in C,\,c_l|n\\ c_l=a_lb_l}}2^{\nu(a_l)}\lambda(c_l)\chi_D\Big(\frac{n}{c_l}\Big).$$

Let $g(n)=\sum_{d|n}\chi_D(d)$ where $n=abm$, then $2^{\nu(a)}\leqslant g(n)\leqslant n$. Taking in account that

$$\chi_k(n)\lambda(n)=\sum_{\substack{c_l\in C,\,c_l|n\\ c_l=a_lb_l}}2^{\nu(a_l)}\lambda(c_l)\chi_k(c_l)\chi_k\Big(\frac{n}{c_l}\Big)\chi_D\Big(\frac{n}{c_l}\Big)$$

we get

$$N^{1-\gamma}\leqslant\Big|\sum_{n\leqslant N_1}\chi_k(n)\lambda(n)\Big|=\Big|\sum_{\substack{c_l\in C\\ c_l=a_lb_l}}2^{\nu(a_l)}\lambda(c_l)\chi_k(c_l)\sum_{d\leqslant N_1/c_l}\chi_k(d)\chi_D(d)\Big|$$

$$\leqslant\sum_{n\leqslant N_1}g(n)\Big|\sum_{d\leqslant N_1/n}\chi_k(d)\chi_D(d)\Big|\leqslant\sum_{n\leqslant D^2}g(n)\cdot kD+\sum_{D^2<n\leqslant N_1}g(n)\cdot\frac{N_1}{n}$$

$$\leqslant D^2\cdot D^2\cdot kD+N_1\sum_{D^2<n\leqslant N_1}\frac{g(n)}{n}\leqslant\frac{1}{3}N^{1-\gamma}+N\sum_{D^2<n\leqslant N}\frac{g(n)}{n}.$$

Thus

$$N\sum_{D^2<n\leqslant N}\frac{g(n)}{n}\geqslant\frac{2}{3}N^{1-\gamma},$$

and as $N=D^6$, $\gamma=\varepsilon/12$ we get

$$\sum_{D^2<n\leqslant D^6}\frac{g(n)}{n}\geqslant\frac{2}{3}\cdot\frac{1}{D^{\varepsilon/2}}.$$

Using the corollary of Lemma 0 with $k=D$, $\tau=\dfrac{1}{\log D}$, $y=D^2$, $x=D^6$ we get using also that $x\geqslant y\geqslant 3D/\tau$ the inequality

$$\frac{2}{3}\cdot\frac{1}{D^{\varepsilon/2}}\leqslant\sum_{n=D^2+1}^{D^6}\frac{g(n)}{n}<\sum_{n=D^2+1}^{D^6}\frac{g(n)}{n^{1-\tau}}$$

$$<\log D\cdot D^{6\frac{1}{\log D}}\cdot L(1)+\frac{4\sqrt{3}\sqrt{D}\cdot D^{2\frac{1}{\log D}}}{\sqrt{D^2}}<\frac{1}{3}D^{\varepsilon/2}\cdot L(1)+\frac{1}{3}\cdot\frac{1}{D^{\varepsilon/2}}.$$

Hence

$$L(1, \chi_D) > D^{-\varepsilon}.$$

So we finished the proof of Lemma 3 and also of Siegel's Theorem I.

5. As it is well-known Siegel's Theorem II follows from Siegel's Theorem I. The usual proof of this fact is that making use of

$$|L'(\sigma, \chi_k)| \leqslant c\log^2 k \quad \text{for} \quad 1-\frac{1}{\log k} \leqslant \sigma \leqslant 1,$$

and of Siegel's Theorem I being valid for $\varepsilon/2$, one gets by the mean value theorem of differential calculus that if

$$L(\beta, \chi_k) = 0$$

then

$$\beta \leqslant 1-\frac{1}{k^{\varepsilon/2}\cdot c\log^2 k} < 1-k^{-\varepsilon} \quad \text{for} \quad k > K(\varepsilon).$$

6. Now we prove in a simple non-elementary way based only on the theory of real valued functions that Siegel's Theorem I follows from Siegel's Theorem II. From this assertion obviously follows the validity of our Lemma 1. Our assertion results from the following lemma which is valid for real characters mod k, where k is sufficiently large.

LEMMA 4. *If*

$$L(1, \chi_k) \leqslant \frac{1}{100\log^3 k}$$

then

$$L\big(1-L(1), \chi_k\big) \leqslant 0.$$

Proof. Set

$$\tau = \frac{1}{10\log^3 k} \quad \left(\leqslant \frac{1}{20}\right)$$

then

$$L(1+\tau) \geqslant \frac{\zeta\big(2(1+\tau)\big)}{\zeta(1+\tau)} \geqslant \frac{3}{2}\tau$$

thus

$$L\left(1+\frac{1}{10\log^3 k}\right) - L(1) \geqslant \frac{3}{2\cdot 10\log^3 k} - \frac{1}{100\log^3 k} = \frac{0.14}{\log^3 k}$$

and so there is a $\xi \in \left[1, 1+\frac{1}{10\log^3 k}\right]$ for which $L'(\xi) \geqslant 1.4$. On the other hand (as one can easily show by partial summation) for sufficiently

large k

$$|L''(\sigma, \chi_k)| \leqslant 2\log^3 k \quad \text{if} \quad 1-\frac{1}{10\log^3 k} \leqslant \sigma \leqslant 1+\frac{1}{10\log^3 k}$$

so if

$$1-\frac{1}{10\log^3 k} \leqslant \eta \leqslant 1$$

then

$$|L'(\xi)-L'(\eta)| \leqslant (\xi-\eta)2\log^3 k \leqslant 0.4.$$

Thus $L'(\eta) \geqslant 1$. Considering that

$$L(1) \leqslant \frac{1}{100\log^3 k} < \frac{1}{10\log^3 k},$$

we get

$$L(1) - L\big(1-L(1)\big) \geqslant L(1)\cdot 1.$$

Hence

$$L\big(1-L(1), \chi_k\big) \leqslant 0.$$

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