

Note on a paper by T. Nagell

by

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T. Nagell [1] proved the following

THEOREM. Let d be a square-free positive integer. The equation

(1)
$$x^2 + y^2 + z^2 = 0$$

has a non-zero solution in the quadratic field $Q(\sqrt{-d})$ if and only if $d \not\equiv -1 \pmod{8}$.He gave two proofs, each of them demanding separate consideration of all possible residues of d modulo 8.

We will show here how to get the result in a few lines. Our proof is effective, we show explicitly a non-trivial solution of (1) if it exists. One of the two basic arguments will be the identity:

(2)
$$(u^2 + v^2)(p^2 + q^2) = (up + vq)^2 + (uq - vp)^2.$$

First show the sufficiency. If $d \not\equiv -1 \pmod{8}$ then, by Gauss's classical theorem, d can be written as a sum of three squares of rational integers, $d = a^2 + b^2 + c^2$, say. Then $(\sqrt{-d})^2 + a^2 + b^2 + c^2 = 0$ and multiplying this by $b^2 + c^2$ and using (2) we get the following non-zero solution of (1) in the field $Q(\sqrt{-d})$:

$$(ab + c\sqrt{-d})^2 + (ac - b\sqrt{-d})^2 + (b^2 + c^2)^2 = 0.$$

Now the proof of necessity. If (1) has a non-trivial solution in $Q(\sqrt{-d})$ then we may assume that

$$w^2 + (u + p\sqrt{-d})^2 + (v + q\sqrt{-d})^2 = 0,$$

where w, u, v, p, q are rational integers. Hence we get $w^2 + u^2 + v^2 = d(p^2 + q^2)$ and $up + vq = 0$. Multiplying the first equality by $p^2 + q^2$ and using (2) we obtain

$$(wp)^2 + (wq)^2 + (uq - vp)^2 = d(p^2 + q^2)^2.$$

Hence by Gauss's theorem $d(p^2+q^2)^2$ is not of the form $4^a(8b+7)$ and consequently $d \not\equiv -1 \pmod{8}$.

We remark here that the above theorem can be used for determining the stufe s of any quadratic field. By definition, the stufe $s = s(k)$ of a field k is the smallest positive integer n such that -1 is a sum of n squares over the field k (or infinity, if such an n does not exist). It was proved first by C. L. Siegel that the stufe of an algebraic number field is always 1, 2, 4 or infinity.

For a non-real quadratic field $\mathbb{Q}(\sqrt{-d})$ we have

- (i) $s = 1$ if and only if $d = 1$,
- (ii) $s = 2$ if and only if $1 \neq d \not\equiv -1 \pmod{8}$,
- (iii) $s = 4$ if and only if $d \equiv -1 \pmod{8}$.

Here (i) is obvious and (ii) follows from the above theorem. If d is any positive integer then by Lagrange's theorem $d = p^2 + q^2 + r^2 + t^2$ for some rational integers p, q, r, t and so

$$((\sqrt{-d})/t)^2 + (p/t)^2 + (q/t)^2 + (r/t)^2 = -1.$$

Hence for any quadratic field $s \leq 4$ and now (i) and (ii) imply (iii).

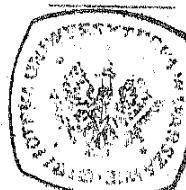
Reference

- [1] T. Nagell, *Sur la résolubilité de l'équation $x^2 + y^2 + z^2 = 0$ dans un corps quadratique*, Acta Arith. 21 (1972), pp. 35-43.

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