the directed system we obtained is nice. But this follows from Lemma 12 and Corollary 11.

We only prove (c) since the proof of (d) is similar. We are given an elementary embedding $j\colon V_{\alpha+1}\to V_{\beta+1}$ where $\beta=j(\alpha)$ and α is the first ordinal moved. It suffices to show that j preserves well-orderings. Suppose, on the contrary, that R' is a well-ordering of V_α, but that $j(R')$ is not a well-ordering of V_β. Then there is a $j(R')$-descending sequence $s\colon \omega\to V_\beta$. Since $\beta>\omega$ is inaccessible, $s\in V_\beta$. Hence, $V_{\beta+1}$ satisfies

$$\exists s\colon\ \omega\to V_{j(\alpha)}\ \forall n\in\omega\ \langle s(n+1), s(n)\rangle\in j(R')\,.$$

Then since $j\colon V_{\alpha+1}\to V_{\beta+1}$ is elementary, $V_{\alpha+1}$ satisfies

$$\exists s\colon\ \omega\to V_\alpha\ \forall n\in\omega\ \langle s(n+1), s(n)\rangle\in R'.$$

But then R' is not a well-ordering.

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Two conjectures regarding the stability of ω-categorical theories

by

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Abstract. It has been conjectured (1) that any ω-categorical first order theory has finite Morley rank, and (2) that any stable ω-categorical theory is totally transcendental. In this paper it is shown that any structure, whose theory is a counterexample to either of these two conjectures, contains a pseudoplane. Here a pseudoplane consists of a universe of "points" and "lines" together with an incidence relation; the axioms are that each line contains infinitely many points, and that two distinct lines meet in at most a finite number of points, together with the duals of these. Thus both conjectures would follow if it could be shown that ω-categorical pseudoplanes do not exist.

The greater part of this paper is motivated by the conjecture:

C1. *If T is stable and ω-categorical then T is totally transcendental.*

In § 1 we prove a conjecture weaker than C1 namely:

C1′. *If T is superstable and ω-categorical then T is totally transcendental.*

The truth of C1′ was first known by Shelah. Allthrough it has not appeared explicity it follows immediately from two lemmas of [6], Lemma 38, p. 106 and Lemma 40, p. 108. The main tool we use namely that of normalizing ranked was also invented by Shelah.

In § 2 we show that if M is a structure which refutes C1 then M contains a pseudoplane. Let "$\bigvee_\omega x$" be read "for at most a finite number of x". A *pseudoplane* is a model for the axioms

$$\bigvee x\big(I(x,y)\vee I(y,x)\big)\,,$$

$$\bigvee x I(x,y)\wedge\bigvee y I(x,y)\,.\to x\neq y\,,$$

$$x_0\neq x_1\to\bigvee_\omega y\big(I(x_0,y)\wedge I(x_1,y)\big)\,,$$

$$y_0\neq y_1\to\bigvee_\omega x\big(I(x,y_0)\vee I(x,y_1)\big)\,,$$

$$\bigvee x I(x,y)\to\neg\bigvee_\omega x I(x,y)\,.\wedge.\bigvee y I(x,y)\to\neg\bigvee_\omega y I(x,y)\,.$$

The "lines are elements of the universe satisfying $\bigvee x I(x, y)$ and the "points" are those satisfying $\bigvee y I(x, y)$.

In [3] Morley associated with each countable first order theory T an ordinal α_T which we shall refer to as the "rank" of T. He also asked what model-theoretic conditions on T imply $\alpha_T < \omega$ and suggested that T being categorical in some power might suffice. In [1] Baldwin showed that indeed α_T is finite when T is ω_1-categorical, but whether the statement:

C2. *If T is ω-categorical then $\alpha_T < \omega$.*

is true or false is not known. In § 3 we shall show that if M is a structure which refutes C2 then M contains a pseudoplane. Thus we have a link between C1 and C2. When we say M "contains" a pseudoplane we mean that there are binary relations $U_0(\bar{x}, \bar{x}')$, $U_1(\bar{y}, \bar{y}')$, and $U(\bar{x}, \bar{y})$ definable in M without any use of parameters such that U_0 and U_1 are equivalence relations and if we take the U_0-equivalence classes as points and the U_1-equivalence classes as lines then U induces an incidence relation I. To prove the results already mentioned we shall use normalization of ranked formulas where the rank is the same kind as was introduced by Morley [3]. Suppose $\varphi(x; \bar{y})$ is a formula such that $\varphi(x; \bar{b})$ has rank α and degree 1 for all $\bar{b}$ satisfying $\models \bigvee x\varphi(x; \bar{b})$ in a given structure M. Suppose further that $\varphi(x; \bar{b}^0) \wedge \neg\varphi(x; \bar{b}^1)$ has rank $< \alpha$ for all $\bar{b}^0$ and $\bar{b}^1$ in M such that $\models \bigvee x\varphi(x; \bar{b}^0) \wedge \bigvee x\varphi(x; \bar{b}^1)$, and that M is ω-saturated. Then there is a formula $\varphi'(x)$ with rank α and degree 1 and no parameters such that $\varphi'(x) \wedge \varphi(x; \bar{b})$ has rank α for each $\bar{b} \in M$ for which $\models \bigvee x\varphi(x; \bar{b})$. The theorem we have just stated is an example of what we mean by normalization in that φ' is said to be obtained by *normalizing* $\varphi(x; \bar{y})$. The first result involving normalization was obtained by Shelah; we shall say more about this in § 1 when we state the first of two normalization lemmas. The other normalization lemma is stated in § 3. In § 4 we prove the second normalization lemma; the proof of the first one is omitted because it can be proved by exactly the same method. In § 5 we discuss the conjectures C1 and C2 in the light of our results.

Most of our notation is copied from Shelah [5]. Throughout the paper, except where the contrary is explicitly stated, we shall be dealing with a countable, complete, ω-categorical theory T. Further M will denote a countable model of T. If φ is a sentence in the language of T augmented by names for elements of $|M|$ then $\models \varphi$ means φ is true in M. Let φ have at most $x_1, \ldots, x_n$ free then φ is called *null* if $\models \bigwedge x_1 \ldots \bigwedge x_n \neg\varphi$. Let $\varphi(\bar{x}; \bar{y})$ be a formula of T such that if $\models \bigvee \bar{x}\varphi(\bar{x}; \bar{b}^i)$ for $i = 0, 1$ then $\bar{b}^0, \bar{b}^1$ realize the same type over $\emptyset$; we say that $\varphi(\bar{x}; \bar{y})$ *fixes the type* of $\bar{y}$. Except where the context requires otherwise all types are complete and over $\emptyset$.

1. The superstable case. Here we shall prove that if T is superstable and ω-categorical then T is totally transcendental.

For proof by contradiction let T be a complete countable theory which is superstable and ω-categorical, but not totally transcendental. Let M be a countable model of T. For any formula $\varphi(x; \bar{y})$ of T and any $\bar{a} \in M$, let $\varkappa(\varphi(x; \bar{a}))$ be the cardinality of

$$\{p\colon\ p \in S(|M|) \text{ and } \varphi(x; \bar{a}) \in p\}.$$

For any subset A of a model $S(A)$ denotes the set of 1-types over A. We first observe that there is a formula $\varphi(x; \bar{y})$ of T and $\bar{a} \in M$ such that

(i) $\varkappa(\varphi(x; \bar{a})) = 2^\omega$, and

(ii) there do not exist a formula $\psi(x; \bar{y}')$ of T and sequences $\bar{a}_0, \bar{a}_1, \ldots$ in M such that

$$|\varkappa(\varphi(x; \bar{a}) \wedge \psi(x; \bar{a}_i) \wedge \psi(x; \bar{a}_j))| = 2^\omega \Leftrightarrow i = j$$

for all i and $j < \omega$. The argument we use is implicit in the discussion of superstability in [4]. Since we are assuming that T is not totally transcendental we have $\varkappa(x = x) = 2^\omega$ which leads to (i). We obtain (ii) by noting that if the kind of formula $\varphi(x; \bar{y})$ we want doesn't exist then for any infinite cardinal λ we can construct a subset A of a model M' of T such that $|A| = \lambda$ and $|S(A)| \geqslant \lambda^\omega$. (This would contradict the assumption that T is superstable.) The construction consists of choosing a formula $\varphi_i(x; \bar{y}_i)$ of T for each $i < \omega$ and $\bar{a}_{i,\eta} \in M'$ for each $i < \omega$ and each mapping $\eta \in {}^i\lambda$ (where $i = \{0, 1, \ldots, i-1\}$ by convention) such that if $\tau \in {}^\omega\lambda$ then

$$\{\varphi_i(x; \bar{a}_{i,\eta})\colon\ i < \omega,\ \eta \in {}^i\lambda, \text{ and } \eta \subset \tau\} \cup$$
$$\cup \{\neg\varphi_i(x; \bar{a}_{i,\eta\frown\langle j\rangle})\colon\ i < \omega,\ j < \lambda,\ \eta \in {}^i\lambda,\ \eta \subset \tau, \text{ and } \eta \frown \langle j\rangle \not\subset \tau\}$$

is a consistent set of formulas. The choice of $\varphi_i(x; \bar{y}^i)$ is easy because T is ω-categorical. Now we take

$$A = \bigcup \{\operatorname{rng}\bar{a}_{i,\eta}\colon\ i < \omega \text{ and } \eta \in {}^i\lambda\}.$$

For the rest we may as well assume that $\varphi(x; \bar{y})$ is $x = x$, because this will simplify the notation. Choose $a \in M$ and a formula $\varepsilon(x, y)$ of T such that $\varkappa(\varepsilon(x, a)) = 2^\omega$, ε defines an equivalence relation on M, and for any pair $\langle a', \varepsilon'\rangle$ satisfying the same conditions as $\langle a, \varepsilon\rangle$, if $\varepsilon'(x, a')$ implies $\varepsilon(x, a)$, then $\varepsilon'(x, a')$ and $\varepsilon(x, a)$ are equivalent. We must be able to choose a and $\varepsilon(x, y)$, for otherwise T would have an infinite number of 2-types. (This, of course, would be contrary to the theorem of Ryll-Nardzewski.) Since $\varkappa(\varepsilon(x, a)) = 2^\omega$ there is a formula $\theta(x; y)$ of T and $\bar{a} \in M$ such that $\varkappa(\varepsilon(x, a) \wedge \theta(x; \bar{a})) = 2^\omega$ and $\varkappa(\varepsilon(x, a) \wedge \neg\theta(x; \bar{a})) = 2^\omega$. Let $\chi(x; y, \bar{y})$ denote the formula $\varepsilon(x, y) \wedge \theta(x; \bar{y})$. Let Γ be the set of formulas obtained by Boolean operations from formulas of the form

$\chi(x;\, b, \bar{b})$ where b and $\bar{b}$ are in M. Since T is stable we can define the rank and degree, of any formula with just one free variable, relative to any finite set Δ of formulas of T. The notions of rank and degree stem originally from Morley [3]. Considerable refinements of Morley's ideas have been made by Shelah. The particular notions of rank and degree used here are those of [2]. We get χ-rank and χ-degree by letting Δ be $\{\chi(x;\, y, \bar{y}), x = y\}$. Let $\pi(x;\, \bar{z})$ be chosen in Γ and $\bar{c}$ in M such that $\varkappa\big(\chi(x;\, a, \bar{a}) \wedge \pi(x;\, \bar{c})\big) = 2^{\omega}$, the χ-rank of $\chi(x;\, a, \bar{a}) \wedge \pi(x;\, \bar{c})$ is least possible, and the χ-degree of $\chi(x;\, a, \bar{a}) \wedge \pi(x;\, \bar{c})$ is least possible. Since every type over a finite set is realized in any infinite model of an ω-categorical theory, the χ-degree of $\chi(x;\, a, \bar{a}) \wedge \pi(x;\, \bar{c})$ is 1. Let $\chi(x;\, a, \bar{a}) \wedge \wedge \pi(x;\, \bar{c})$ be denoted $\pi_0(x;\, \bar{a}^0)$ where $\bar{a}^0 = a^\frown \bar{a}^\frown \bar{c}$ and $\pi_0(x;\, \bar{y}^0)$ is a suitable formula of T. Let $\pi_0(x;\, \bar{a}^0)$ have χ-rank k_0. Choose a maximal sequence $\bar{a}^0, \bar{a}^1, \ldots$ in M such that whenever $i < j$: $\bar{a}^i$ and $\bar{a}^j$ realize the same type and $\pi_0(x;\, \bar{a}^i) \wedge \pi_0(x;\, \bar{a}^j)$ has χ-rank $< k_0$. The sequence $\bar{a}^0, \bar{a}^1, \ldots$ must be finite. Otherwise (ii) above would be refuted. Suppose the sequence ends with $\bar{a}^{m_0}$.

The remainder of the proof consists in showing that there is a formula $\varepsilon'(x, y)$ of T and $a' \in M$ such that

(i) $\varepsilon'(x, y)$ defines an equivalence relation on M,

(ii) $\varepsilon'(x, a')$ is a Boolean combination of formulas of the form $\pi_0(x;\, \bar{b}^0)$ where $\bar{b}^0 \in M$ has the same type as $\bar{a}^0$,

(iii) $\varepsilon'(x, a')$ implies $\varepsilon(x, a)$ and also implies some finite disjunction of formulas $\pi_0(x;\, \bar{b}^0)$ where $\bar{b}^0$ has the same type as $\bar{a}^0$,

(iv) $\varepsilon'(x, a')$ has χ-rank k_0 and χ-degree 1.

Suppose for the moment that ε' and a' exist satisfying (i)-(iv). For some particular $\bar{b}^0$, $\pi_0(x;\, \bar{b}^0) \wedge \varepsilon'(x;\, a')$ has χ-rank k_0 by (iii) and (iv). Hence $\pi_0(x;\, \bar{b}^0) \wedge \neg\varepsilon'(x;\, a')$ has χ-rank $< k_0$. Therefore $\varkappa\big(\varepsilon'(x;\, a')\big) = 2^{\omega}$; otherwise $\varkappa\big(\pi_0(x;\, \bar{b}^0) \wedge \neg\varepsilon'(x;\, a')\big) = 2^{\omega}$ contrary to the choice of π_0. From the way ε and a were chosen it follows that $\varepsilon(x;\, a)$ and $\varepsilon'(x;\, a')$ are equivalent. Now $\varkappa\big(\varepsilon'(x;\, a') \wedge \neg\pi_0(x;\, \bar{b}^0)\big) = \varkappa\big(\varepsilon(x;\, a) \wedge \neg\theta(x;\, \bar{a})\big) = 2^{\omega}$. From (iii) there exists $\bar{b}^1$ in M realizing the same type as $\bar{a}^0$ such that $\varkappa\big(\pi_0(x;\, \bar{b}^1) \wedge \varepsilon'(x;\, a') \wedge \wedge \neg\pi_0(x;\, \bar{b}^0)\big) = 2^{\omega}$. Since $\varepsilon'(x, a') \wedge \neg\pi_0(x;\, \bar{b}^0)$ has χ-rank $< k_0$ the choice of π_0 is again contradicted. This completes the proof given ε' and a'.

To show that ε' and a' exist we use the idea of normalizing ranked formulas. This idea has its origin in a theorem of Shelah communicated to us by letter early in 1971. In its simplest form Shelah's theorem says: if T_0 is a totally transcendental theory and $x = x$ has rank α and degree k in T_0 then there is a formula of T_0 defining an equivalence relation E with a finite number of classes such that in any model M_0 of T_0, k of the equivalence classes have rank α. (Here we are using rank and degree in the sense of Morley [3]. The rank of a formula φ is defined to be the greatest rank of types containing φ. The rank of a definable subset of a model is defined to be the rank of a formula defining the subset.) Let a formula $\psi(x;\, \bar{y})$ be given. A formula $\theta(x;\, \bar{z})$ is said to be *normal with respect to* $\psi(x;\, \bar{y})$ if the following three conditions are satisfied:

(i) $\theta(x;\, \bar{z})$ fixes the type of $\bar{z}$ and any formula $\theta(x;\, \bar{c})$ is equivalent to a Boolean combination of formulas of the form $\psi(x;\, \bar{b})$.

(ii) There exists $k < \omega$ such that if $\bar{c} \in M$ and $\theta(x;\, \bar{c})$ is non-null then $\theta(x;\, \bar{c})$ has ψ-rank k and ψ-degree 1.

(iii) If $\bar{c}^0$ and $\bar{c}^1 \in M$ are such that $\theta(x;\, \bar{c}^0) \wedge \theta(x;\, \bar{c}^1)$ has ψ-rank k then $\theta(x;\, \bar{c}^0)$ and $\theta(x;\, \bar{c}^1)$ are equivalent.

LEMMA 1. *Let T be ω-categorical and stable. Let $\psi(x;\, \bar{y})$ and $\theta(x;\, \bar{z})$ be formulas of T which satisfy* (i) *and* (ii) *above. There exists a formula $\theta^*(x;\, \bar{z})$ of T normal with respect to $\psi(x;\, \bar{y})$ such that for all $\bar{c} \in M$*

$$\mathrm{Rank}_{\psi}\,\theta^*(x;\, \bar{c}) = \mathrm{Rank}_{\psi}\,\theta^*(x;\, \bar{c}) \wedge \theta(x;\, \bar{c}) = \mathrm{Rank}_{\psi}\,\theta(x;\, \bar{c})\,.$$

Further for each $\bar{c} \in M$, $\theta^*(x;\, \bar{c})$ can be expressed as a positive Boolean combination of formulas of the form $\theta(x;\, \bar{c}')$ where

$$\mathrm{Rank}_{\psi}\,\theta(x;\, \bar{c}') \wedge \theta(x;\, \bar{c}) = \mathrm{Rank}_{\psi}\,\theta(x;\, \bar{c})\,.$$

The proof of the lemma is deferred until § 4. A formula θ^* satisfying the conclusion of Lemma 1 is said to be obtained by *normalizing θ with respect to ψ*. We now proceed with the proof of our theorem. Recall the formula $\pi_0(x;\, \bar{y}^0)$ selected above. Without loss of generality we may suppose that $\pi_0(x;\, \bar{y}^0)$ fixes the type of $\bar{y}^0$. By Lemma 1 there is a formula $\pi^*(x;\, \bar{y}^0)$ of T obtained by normalizing π_0 with respect to χ. For every $\bar{b}^0 \in M$ such that $\pi^*(x;\, \bar{b}^0)$ is non-null there exists $i \leqslant m_0$ such that

$$\mathrm{Rank}_{\chi}\,\pi^*(x;\, \bar{b}^0) \wedge \pi^*(x;\, \bar{a}^i) = \mathrm{Rank}_{\chi}\,\pi^*(x;\, \bar{b}^0) = k_0$$

because $\bar{a}^0, \bar{a}^1, \ldots, \bar{a}^{m_0}$ was chosen to be a maximal sequence. Thus up to equivalence there are exactly m_0+1 formulas of the form $\pi^*(x;\, \bar{b}^0)$. Let $\varepsilon'(x, y)$ be the formula $\bigwedge \bar{y}^0\big(\pi^*(x;\, \bar{y}^0) \leftrightarrow \pi^*(y;\, \bar{y}^0)\big)$. Then ε' defines an equivalence relation on $|M|$ one of whose equivalence classes is defined by the formula

$$\pi^*(x;\, \bar{a}^0) \wedge \bigwedge \{\neg\pi^*(x;\, \bar{a}^i)\colon\ 1 \leqslant i \leqslant m\}$$

which is non-null because it has χ-rank k_0. Let $a' \in M$ be chosen satisfying this formula. From the lemma for each $i \leqslant m$ $\pi^*(x;\, \bar{a}^i)$ is a positive Boolean combination of formulas of the form $\pi_0(x;\, \bar{b}^0)$ where $\bar{b}^0$ realizes the same type as $\bar{a}^0$. Since $\pi_0(x;\, \bar{a}^0)$ implies $\varepsilon(x, a)$, so does $\pi^*(x;\, \bar{a}^0)$, and hence so does $\varepsilon'(x, a')$. It is now immediate that ε' and a' satisfy all the required conditions. This completes the proof.

2. The stable but not superstable case. Here all we have been able to establish is that if T is ω-categorical, stable, and not superstable then any model of T contains an infinite pseudoplane. The argument uses a series of propositions which we label P1-P7.

P1. *There exists a formula $\varphi(x;\bar{y})$ of T and $\bar{a}\in M$ such that $\varkappa(\varphi(x,\bar{a}))=2^{\omega}$ and for any formula $\psi(x,y)$ of T and $b\in M$ either*

$$\varkappa\big(\varphi(x;\bar{a})\wedge\psi(x,b)\big)<2^{\omega}\quad\text{or}\quad\varkappa\big(\varphi(x;\bar{a})\wedge\neg\psi(x,b)\big)<2^{\omega}.$$

Proof. Since T is ω-categorical there are only a finite number of formulas $\psi(x,y)$ up to equivalence. Thus there is a single formula $\psi'(x;\bar{y})$ of T such that for any $\psi(x,y)$ and $b\in M$, $\psi(x,b)$ is equivalent to $\psi'(x;\bar{b})$ for some $\bar{b}\in M$. If the proposition fails there are clearly 2^{ω} ψ'-types over $|M|$ whence T is unstable, contradiction.

For the rest of the section suppose that $\varphi(x;\bar{y})$ is a particular formula satisfying P1, and that $\bar{a}$ is a corresponding finite sequence in M.

P2. *There exist a formula $\psi(x;\bar{y}')$ of T and $\bar{a}^0,\bar{a}^1,\ldots$ in M such that $\bar{a}^0,\bar{a}^1,\ldots$ all realize the same type over* $\mathrm{Rng}\,\bar{a}$, *and*

$$\varkappa\big(\varphi(x;\bar{a})\wedge\psi(x;\bar{a}^i)\wedge\psi(x;\bar{a}^j)\big)=2^{\omega}\Leftrightarrow i=j\,.$$

This proposition is immediate because otherwise T would be totally transcendental by exactly the same argument which we used in the superstable case. In the conclusion of P2 let q be the type over $\emptyset$ realized by $\bar{a}^0,\bar{a}^1,\ldots$

P3. *In P2 ψ and $\bar{a}^0,\bar{a}^1,\ldots$ may be chosen such that for any $\bar{b}\in M$ realizing q $\psi(x;\bar{b})$ has ψ-degree 1 and*

$$\begin{aligned}\varkappa\big(\varphi(x;\bar{a})\wedge\psi(x;\bar{a}^0)\wedge\psi(x;\bar{b})\big)=2^{\omega}&\\ \Leftrightarrow\varkappa\big(\varphi(x;\bar{a})\wedge\psi(x;\bar{a}^0)\wedge\neg\psi(x;\bar{b})\big)\neq 2^{\omega}&\\ \Leftrightarrow\mathrm{Rank}_{\psi}\psi(x;\bar{a}^0)\wedge\psi(x;\bar{b})=\mathrm{Rank}_{\psi}\psi(x;\bar{b})\,.&\end{aligned}$$

(*Notice that the conclusion is true if and only if it is true for $\bar{a}^0$ replaced by $\bar{a}^i$ and the same ψ.*)

Proof. Let $\psi^0(x;\bar{y}^0)$ be a particular formula which will serve as $\psi(x;\bar{y}')$ in the conclusion of P2 and let $\bar{b}^0,\bar{b}^1,\ldots$ be corresponding values of $\bar{a}^0,\bar{a}^1,\ldots$ Choose $\psi^1(x;\bar{y}^1)$ and $\bar{c}^0\in M$ such that $\varkappa\big(\varphi(x,\bar{a})\wedge\psi^0(x;\bar{b}^0)\wedge\psi^1(x;\bar{c}^0)\big)=2^{\omega}$, $\psi^1(x;\bar{c}^0)$ is a Boolean combination of formulas of the form $\psi^0(x;\bar{b})$, and $\psi^0(x;\bar{b}^0)\wedge\psi^1(x;\bar{c}^0)$ has least possible ψ^0-rank and ψ^0-degree. Let $\psi(x;\bar{y}')$ be $\psi^0(x;\bar{y}^0)\wedge\psi^1(x;\bar{y}^1)$ where $\bar{y}'$ is $\bar{y}^0{}^\frown\bar{y}^1$. Without loss of generality we may suppose that every instance of ψ is a Boolean combination of instances of ψ^0 and vice-versa. This makes ψ-rank and ψ-degree the same as ψ^0-rank and ψ^0-degree. Let $\bar{a}^0$ be $\bar{b}^0{}^\frown\bar{c}^0$ and for $i>0$ let $\bar{a}^i$ be an extension of $\bar{b}^i$ which realizes the same type over $\mathrm{Rng}\,\bar{a}$ as $\bar{a}^0$. It is now easy to see that ψ and $\bar{a}^0,\bar{a}^1,\ldots$ satisfy both P2 and P3.

For the rest of this section let ψ be chosen satisfying P2 and P3 and let k be the ψ-rank of $\psi(x;\bar{b})$ when $\bar{b}$ realizes q.

P4. *If $b\in M$, $\bar{b}\in M$ realizes q, and $\models\psi[b;\bar{b}]$ then there exist $\bar{b}^0,\bar{b}^1,\ldots$ all realizing the same type over $\{b\}$ which $\bar{b}$ realizes such that if $i\neq j$ then the ψ-rank of $\psi(x;\bar{b}^i)\wedge\psi(x;\bar{b}^j)$ is $<k$.*

Proof. For proof by contradiction suppose that the hypothesis is true and the conclusion false. Without loss of generality suppose that $\bar{b}$ is $\bar{a}^0$. Let $T(b)$ be obtained from T by adjoining b as a new constant to the language and adding a suitable axiom fixing the type of b. In the new theory let $\pi(\bar{y}')$ be a formula fixing the type of $\bar{b}$. Let $\psi^*(x;\bar{y}')$ be obtained by normalizing $\psi(x;\bar{y}')\wedge\pi(\bar{y}')$ with respect to ψ in the manner of Lemma 1. Since there exist no sequences $\bar{b}^0,\bar{b}^1,\ldots$ satisfying the conclusion of the proposition, up to equivalence there are only a finite number of formulas of the form $\psi^*(x;\bar{b}')$. Thus $\bigvee\bar{y}'\psi^*(x;\bar{y}')$, which contains one parameter with respect to the original language, can be expressed as a positive Boolean combination of formulas of the form $\psi(x;\bar{b}')$ where $\bar{b}'$ realizes q. From P3 it follows that

$$\varkappa\big(\varphi(x;\bar{a})\vee\psi(x;\bar{a}^i)\wedge\bigvee\bar{y}'\psi^*(x;\bar{y}')\big)=2^{\omega}$$

for only finitely many i. Also since we are supposing that $\bar{b}$ is $\bar{a}^0$, $\psi(x;\bar{a}^0)\wedge\wedge\psi^*(x;\bar{a}^0)$ has ψ-rank k whence $\varkappa\big(\varphi(x;\bar{a})\wedge\psi(x;\bar{a}^0)\wedge\bigvee\bar{y}'\psi^*(x;\bar{y}')\big)=2^{\omega}$. We now have a contradiction of P1 if we take $\bigvee\bar{y}'\psi^*(x;\bar{y}')$ to be $\psi(x,b)$. This completes the proof of P4.

P5. *There exists a formula $\theta(x;\bar{z})$ of T such that:*

5.1. $\models\theta[a;\bar{c}]$ *for some a and $\bar{c}$ in M.*

5.2. *Any formula $\theta(x;\bar{c})$ is equivalent to a Boolean combination of formulas of the form $\psi(x;\bar{b})$.*

5.3. *If $\models\theta[a;\bar{c}]$ then there exist $\bar{c}^0,\bar{c}^1,\ldots$ all realizing the same type over $\{a\}$ as $\bar{c}$ such that the ψ-rank of $\theta(x;\bar{c}^i)\wedge\theta(x;\bar{c}^j)$ is less than the ψ-rank of $\theta(x;\bar{c}^i)$ unless $i=j$.*

5.4. *If $\theta(x;\bar{c})$ is non-null then it has ψ-degree 1 and ψ-rank independent of $\bar{c}$.*

5.5. *If $\theta(x;\bar{c})$ is non-null its ψ-rank is the least such that 5.1-5.4 can be satisfied.*

If we ignore P5.5 then $\psi(x;\bar{y}')\wedge\pi(\bar{y}')$ will serve for $\theta(x;\bar{z})$ where $\pi(\bar{y}')$ generates the type q. From this observation the proposition is obvious. Let $\theta(x;\bar{z})$ be chosen satisfying P5 such that $\theta(x;\bar{z})$ fixes the type of $\bar{z}$. Let this type be p_θ. When $\bar{c}$ realizes p_θ let the ψ-rank of $\theta(x;\bar{c})$ be l. From 5.3 and 5.4 we have $l\geqslant 1$.

P6. *There exists a formula* $\sigma(x; \bar{y}^{\#})$ *of* T *such that:*

6.1. *Any formula* $\sigma(x; \bar{b}^{\#})$ *is equivalent to a Boolean combination of formulas of the form* $\psi(x; \bar{b})$.

6.2. *If* $\bar{c}$ *realizes* p_0 *then there exist* $\bar{b}^0, \bar{b}^1, \ldots$ *in* M *such that for all* i *and* $j < \omega$ $\mathrm{Rank}_\psi \sigma(x; \bar{b}^i) = l-1$, $\mathrm{Deg}_\psi \sigma(x; \bar{b}^i) = 1$,

$$\mathrm{Rank}_\psi \sigma(x; \bar{b}^i) \wedge \sigma(x; \bar{b}^j) = l-1 \Leftrightarrow i = j$$

and $\sigma(x; \bar{b}^i)$ *implies* $\theta(x; \bar{c})$.

Proof. It is clear from the definition of rank and the ω-saturatedness of M that there exist formulas $\sigma^i(x; \bar{y}^i)$ and sequences $\bar{b}^i$ for $i < \omega$ such that $\sigma^i(x; \bar{b}^i)$ is equivalent to a Boolean combination of formulas of the form $\psi(x; \bar{b})$, $\mathrm{Rank}_\psi \sigma^i(x; \bar{b}^i) = l-1$, $\mathrm{Deg}_\psi \sigma^i(x; \bar{b}^i) = 1$,

$$\mathrm{Rank}_\psi \sigma^i(x; \bar{b}^i) \wedge \sigma^j(x; \bar{b}^j) = l-1 \Leftrightarrow i = j$$

and $\sigma^i(x; \bar{b}^i)$ implies $\theta(x; \bar{c})$. If $\sigma^i(x; \bar{y}^i)$ will not serve as $\sigma(x; \bar{y}^{\#})$ in the conclusion of the proposition, by using Lemma 1 we can obtain a formula $\tau^i(x; \bar{z})$ such that $\tau^i(x; \bar{c})$ and $\tau^i(x; \bar{c}) \wedge \sigma^i(x; \bar{b}^i)$ both have ψ-rank $l-1$. Thus if no σ^i will serve as σ there are infinitely many pairwise inequivalent formulas of the form $\tau(x; \bar{c})$. This contradicts the ω-categoricity of T.

Let $\sigma(x; \bar{y}^{\#})$ be chosen satisfying P6 and such that $\sigma(x; \bar{y}^{\#})$ fixes the type of $\bar{y}^{\#}$. Let this type be p_σ. Choose a type p' such that in P6.2 $\bar{b}^0, \bar{b}^1, \ldots$ may all be chosen such that $\bar{c} ^\frown \bar{b}^i$ realizes p' for each $i < \omega$. Let $\sigma^*(x; \bar{y}^{\#})$ be obtained by normalizing $\sigma(x; \bar{y}^{\#})$ with respect to ψ. If $\bar{b}^0$ and $\bar{b}^1$ both realize p_σ call them *equivalent* if $\sigma^*(x; \bar{b}^0)$ and $\sigma^*(x; \bar{b}^1)$ are equivalent.

P7. *Let* $\bar{b}$ *realize* p_σ. *There exist* $\bar{c}^0, \bar{c}^1, \ldots$ *such that for all* i *and* $j < \omega$:

7.1. *There exists* $\bar{b}^i$ *equivalent to* $\bar{b}$ *such that* $\bar{c}^i ^\frown \bar{b}^i$ *realizes* p'.

7.2. $\mathrm{Rank}_\psi\big(\theta(x; \bar{c}^i) \wedge \theta(x; \bar{c}^j)\big) = l \Leftrightarrow i = j$.

Proof. Given $\bar{b}$ we can find $a \in M$ such that $\models \sigma[a; \bar{b}]$ and such that there is a maximal sequence $\bar{b} = \bar{b}^0, \ldots, \bar{b}^m$ satisfying (i) $\bar{b}^0, \ldots, \bar{b}^m$ all realize the same type over $\{a\}$, and (ii) $\sigma(x; \bar{b}^i) \wedge \sigma(x; \bar{b}^j)$ has ψ-rank $l-1$ if and only if $i = j$. If not, then $\sigma(x; \bar{y}^{\#})$ would contradict P5.5. Let $\bar{c}$ be such that $\bar{c} ^\frown \bar{b}$ realizes p'. We can find $\bar{c}^0, \bar{c}^1, \ldots$ all realizing the same type as $\bar{c}$ over $\{a\}$ such that $\theta(x; \bar{c}^i) \wedge \theta(x; \bar{c}^j)$ has ψ-rank l if and only if $i = j$. For each $j < \omega$ there exists $\bar{b}^{(j)}$ realizing the same type over $\{a\}$ as $\bar{b}$ such that $\bar{c}^j ^\frown \bar{b}^{(j)}$ realizes p'. Since $\langle \bar{b}^i \colon i \leqslant m \rangle$ cannot be extended there exists $k \leqslant m$ such that $\bar{b}^{(j)}$ is equivalent to $\bar{b}^k$ for infinitely many j. Thus the conclusion of the proposition is satisfied for $\bar{b}^k$ in place of $\bar{b}$. This suffices since $\bar{b}^k$ and $\bar{b}$ realize the same type.

Let $\theta^*(x; \bar{z})$ be obtained by normalizing $\theta(x; \bar{z})$ with respect to ψ. If $\bar{c}^0$ and $\bar{c}^1$ both realize p_0 call them *equivalent* if $\theta^*(x; \bar{c}^0)$ and $\theta^*(x; \bar{c}^1)$ are equivalent formulas. Let $\bar{c}$ realize p_0. Choose $\bar{b}^0, \bar{b}^1, \ldots$ such that for each $i < \omega$, $\bar{c}' ^\frown \bar{b}^i$ realizes p' for some $\bar{c}'$ equivalent to $\bar{c}$ and such that $\bar{b}^i$ is equivalent to $\bar{b}^j$ only if $i = j$. This choice is possible from P6.2 and the choice of p'. Notice that if $\bar{b}^0, \bar{b}^1, \ldots$ are chosen so that $\bar{c} ^\frown \bar{b}^i$ realizes p' for each $i < \omega$, then $\mathrm{Rank}_\psi \theta^*(x; \bar{c}) \wedge \sigma(x; \bar{b}^i) = l-1$ for all but a finite number of $i < \omega$ since $\mathrm{Rank}_\psi \theta(x; \bar{c}) \wedge \neg\theta^*(x; \bar{c}) < l$. Therefore, for *any* $\bar{b}$ such that $\bar{c}' ^\frown \bar{b}$ realizes p' for some $\bar{c}'$ equivalent to $\bar{c}$, we have $\mathrm{Rank}_\psi \theta^*(x; \bar{c}) \wedge \sigma(x; \bar{b}) = l-1$. Let m be the greatest number such that $\bar{c}^0, \bar{c}^1, \ldots$ can be found satisfying the following two conditions:

c1. For each $i \leqslant m$ and $j < \omega$ there exist $\bar{b}'$ and $\bar{c}'$ equivalent to $\bar{b}$ and $\bar{c}^j$ respectively such that $\bar{c}' ^\frown \bar{b}'$ realizes p'.

c2. $\bar{c}^i$ and $\bar{c}^j$ are equivalent only if $i = j$.

Finally, choose $\langle \bar{b}^i \colon i < \omega \rangle$ so that m is as large as possible. From P7 we have $m \geqslant 0$. Let p'' be the type of $\bar{b}^0 ^\frown \bar{b}^1 ^\frown \ldots ^\frown \bar{b}^m$.

Now we can extract the desired pseudoplane from M as follows. The *lines* are the equivalence classes of finite sequences $\bar{c}$ realizing p under the notion of equivalence just defined. The *points* are the equivalence classes of finite sequences $\bar{b}^0 ^\frown \bar{b}^1 ^\frown \ldots ^\frown \bar{b}^m$ realizing p'' where $\bar{b}^{0,0} ^\frown \ldots ^\frown \bar{b}^{0,m}$ and $\bar{b}^{1,0} ^\frown \ldots ^\frown \bar{b}^{1,m}$ are equivalent if

$$\models \bigwedge x \Big(\bigvee \{ \sigma^*(x; \bar{b}^{0,i}) \colon i \leqslant m \} \leftrightarrow \bigvee \{ \sigma^*(x; \bar{b}^{1,i}) \colon i \leqslant m \} \Big).$$

The line represented by $\bar{c}$ contains the point represented by $\bar{b}^0 ^\frown \ldots ^\frown \bar{b}^m$ if for each $i \leqslant m$ there exist $\bar{c}'$ and $\bar{b}'$ equivalent to $\bar{c}$ and $\bar{b}^i$ respectively such that $\bar{c}' ^\frown \bar{b}'$ realizes p'. There are infinitely many points on each line from P6.2 and the way p' was chosen. From the existence of $\bar{c}^0, \bar{c}^1, \ldots$ satisfying c1 and c2, there are infinitely many lines through each point. Since m is chosen as large as possible through two distinct points there are at most a finite number of lines. Two distinct lines have at most a finite number of points in common because the rank of the intersection is $< l$.

3. The case in which the Morley rank is infinite. We now consider an ω-categorical theory T whose Morley rank is infinite. In [3] Morley conjectured that no such T exists. Here we show that any model of such a theory contains an infinite pseudoplane. As before let M be a countable model of T. We shall again need to normalize ranked formulas this time within the context of Morley rank. A formula $\theta(x; \bar{z})$ is called *normal* if the following three conditions are satisfied:

(i) $\theta(x; \bar{z})$ fixes the type of $\bar{z}$.

(ii) There exists α such that if $\bar{c} \in M$ and $\theta(x; \bar{c})$ is non-null then $\theta(x; \bar{c})$ has rank α and degree 1.

(iii) If $\bar{c}^0$ and $\bar{c}^1 \in M$ are such that $\theta(x; \bar{c}^0) \wedge \theta(x; \bar{c}^1)$ has rank α then $\theta(x; \bar{c}^0)$ and $\theta(x; \bar{c}^1)$ are equivalent.

LEMMA 2. *Let T be ω-categorical and $\theta(x; \bar{z})$ be a formula of T satisfying* (i) *and* (ii) *above. There exists a normal formula $\theta^*(x; \bar{z})$ of T such that for all $\bar{c} \in M$*

$$\operatorname{Rank} \theta^*(x; \bar{c}) = \operatorname{Rank} \theta^*(x; \bar{c}) \wedge \theta(x; \bar{c}) = \operatorname{Rank} \theta(x; \bar{c}) .$$

The proof of this lemma is deferred until § 4.

To find a pseudoplane in M let l be the least number such that there exist $\bar{c} \in M$ and a formula $\theta(x; \bar{z})$ of T satisfying: (i) $\theta(x; \bar{c})$ has rank l and degree 1, and (ii) if $a \in M$ and $\models \theta[a; \bar{c}]$ then there exist $\bar{c}^0, \bar{c}^1, \ldots$ all realizing the same type over $\{a\}$ as $\bar{c}$ such that for all i and $j < \omega$, $\theta(x; \bar{c}^i) \wedge \theta(x; \bar{c}^j)$ has rank l only if $i = j$. To see that l exists, observe that since T has infinite rank for arbitrarily large $k < \omega$ we can find $\bar{c}^k \in M$ and $\theta^k(x; \bar{z}^k)$ such that $\theta^k(x; \bar{c}^k)$ has rank k and degree 1. If (ii) fails when we take θ and $\bar{c}$ to be θ^k and $\bar{c}^k$ respectively then there exists $a^k \in M$ such that $\models \theta[a^k; \bar{c}^k]$ and a maximal sequence $\bar{c}^{k,0}, \ldots, \bar{c}^{k,m}$ of sequences all realizing the same type over $\{a^k\}$ such that $\theta^k(x; \bar{c}^{k,i}) \wedge \theta^k(x; \bar{c}^{k,j})$ has rank k only if $i = j$. By normalizing θ^k we can find a formula of rank k with only one parameter. (This is exactly the same argument as was used to prove P4 in § 2.) Since up to equivalence there can exist at most a finite number of formulas $\psi(x; y)$ and since any parameter realizes one of a finite number of types, there cannot be a formula of rank k with only one parameter for every $k < \omega$. This shows that l exists.

Having shown the existence of l we can follow the same line as in § 2. We choose $\theta(x; \bar{z})$ corresponding to l which fixes the type of $\bar{z}$ and we denote this type by p_θ. Next we choose a formula $\sigma(x; \bar{y}^{\#})$ such that if $\bar{c}$ realizes p_θ then there exist $\bar{b}^0, \bar{b}^1, \ldots$ in M such that for all i and $j < \omega$ $\operatorname{Rank} \sigma(x; \bar{b}^i) = l-1$, $\operatorname{Deg} \sigma(x; \bar{b}^i) = 1$,

$$\operatorname{Rank} \sigma(x; \bar{b}^i) \wedge \sigma(x; \bar{b}^j) = l-1 \Leftrightarrow i = j$$

and $\sigma(x; \bar{b}^i)$ implies $\theta(x; \bar{c})$. The whole argument has the same pattern as before from this points onwards.

4. The normalization lemmas.

Before discussing the proofs of Lemmas 1 and 2 we need a simple combinatorial result.

LEMMA 3. *Let $\{X(i): i \in I\}$ be an infinite collection of infinite sets such that for some $n < \omega$, $|X(i_0) - X(i_1)| \leqslant n$ for all i_0 and i_1 in I. Let k be the greatest number such that for any finite subset F of I there exists j such that*

$$|\bigcap \{X(i): i \in F\} - X(j)| \geqslant k .$$

Let

$$\mathfrak{J} = \{F: F \subset I, |F| < \omega, \text{ and } |\bigcap \{X(i): i \in F\} - X(j)| \leqslant k \text{ for all } j \in I\} .$$

Then there is a finite subset $\mathfrak{J}_0$ of $\mathfrak{J}$ such that

$$\bigcup \{\bigcap \{X(i): i \in F\}: F \in \mathfrak{J}\} = \bigcup \{\bigcap \{X(i): i \in F\}: F \in \mathfrak{J}_0\} .$$

Proof. Let $F \in \mathfrak{J}$. By choice of k we can choose $i_0, i_1, \ldots$ in I such that for each $j < \omega$

$$|\bigcap \{X(i): i \in F \cup \{i_0, i_1, \ldots, i_{j-1}\}\} - X(i_j)| \geqslant k .$$

Since $F \in \mathfrak{J}$, for each $j < \omega$

$$|\bigcap \{X(i): i \in F\} - X(i_j)| \leqslant k .$$

It follows that the sets $\bigcap \{X(i): i \in F\} - X(i_j)$ for $j = 0, 1, \ldots$ all have cardinality k and are pairwise disjoint. Let $x \in \bigcap \{X(i): i \in F'\}$ for some $F' \in \mathfrak{J}$ and suppose $x \notin \bigcap \{X(i): i \in F\}$. For all sufficiently large $j < \omega$ we have

$$\bigcap \{X(i): i \in F\} - X(i_j) \subset \bigcap \{X(i): i \in F'\} .$$

Otherwise $X(i) - X(i')$ would be infinite for some i and i' in F and F' respectively. Thus $x \in X(i_j)$ for all sufficiently large $j < \omega$. Otherwise there would exist j such that

$$|\bigcap \{X(i): i \in F'\} - X(i_j)| > k$$

contrary to F' being in $\mathfrak{J}$. If there were an infinite number of possibilities for x then $|X(i_j) - \bigcap \{X(i): i \in F\}|$ would be unbounded as j varied. This is impossible and the desired conclusion follows easily.

Since Lemmas 1 and 2 can be proved by exactly the same method it will be sufficient to treat only one of them.

Proof of Lemma 2. Let T be ω-categorical and $\theta(x; \bar{z})$ be a formula of T fixing the type of $\bar{z}$ to be q say. Let α be an ordinal such that $\theta(x; \bar{c})$ has rank α and degree 1 whenever $\bar{c} \in M$ realizes q. We first of all make the assumption that $\theta(x; \bar{c}^0) \wedge \theta(x; \bar{c}^1)$ has rank α whenever $\bar{c}^0$ and $\bar{c}^1$ both realize q. Later we shall show how this assumption may be removed. Let β be the greatest rank $< \alpha$ such that $\theta(x; \bar{c}^0) \wedge \neg\theta(x; \bar{c}^1)$ can have rank β when $\bar{c}^0$ and $\bar{c}^1$ both realize q. If $\beta = -1$ there is nothing to prove, then suppose $\beta \geqslant 0$. Let Δ consist of all formulas of rank β and degree 1 of the form $\tau(x; \bar{b})$ where $\tau(x; \bar{y})$ is a formula of T and $\bar{b} \in M$. Call two members of Δ *equivalent* if their conjunction is also in Δ; for $\tau \in \Delta$ let $[\tau]$ denote the corresponding equivalence class. For each formula σ having at most x free define

$$X(\sigma) = \{[\tau]: \tau \in \Delta \text{ and } \sigma \wedge \tau \in \Delta\} .$$

For an application of Lemma 3 let $I = \{\theta(x; \bar{c}): \bar{c} \in M \text{ and } \bar{c} \text{ realizes } q\}$.

Let $\mathfrak{J}$ be defined as in the statement of Lemma 3. From the conclusion of Lemma 3 there exists a formula $\theta'(x; \bar{z}')$ of T and $\bar{c}' \in M$ such that

$$X(\theta'(x; \bar{c}')) = \bigcup \{ \bigcap \{X(\sigma): \sigma \in F\}: F \in \mathfrak{J}\}.$$

Notice that the right hand side of this equation is invariant under an automorphism of M. Further, since $\mathfrak{J}$ can be replaced by a finite subset $\theta'(x; \bar{c}') \wedge \neg\theta(x; \bar{c})$ and $\theta(x; \bar{c}) \wedge \neg\theta'(x; \bar{c}')$ both have rank $\leqslant \beta$ for every $\bar{c}$ realizing q. Thus $\theta'(x; \bar{c}')$ has rank α and degree 1. Let q' be the type realized by $\bar{c}$. If $\bar{c}^0$ and $\bar{c}^1$ both realize q' then $X(\theta'(x; \bar{c}^0)) = X(\theta'(x; \bar{c}^1))$ which means that $\theta'(x; \bar{c}^0) \wedge \neg\theta'(x; \bar{c}^1)$ has rank $\leqslant \beta$. Repeating the process by which θ' and q' were generated from θ and q, we eventually get $\theta^{\#}(x; \bar{y}^{\#})$ and $q^{\#}$ such that if $\bar{c}^0$ and $\bar{c}^1$ both realize $q^{\#}$ then $\theta^{\#}(x; \bar{c}^0)$ and $\theta^{\#}(x; \bar{c}^1)$ are equivalent. Further for each $\bar{c}^{\#}$ realizing $q^{\#}$ $\theta^{\#}(x; \bar{c}^{\#})$ has rank α and degree 1 and $\theta^{\#}(x; \bar{c}^{\#}) \wedge \theta(x; \bar{c})$ has rank α if $\bar{c}$ realizes q. Clearly we can suppose that $\theta^{\#}$ has only x free which completes the proof. In the general case where $\bar{c}^0$ and $\bar{c}^1$ exist realizing q such that $\theta(x; \bar{c}^0) \wedge \theta(x; \bar{c}^1)$ has rank $< \alpha$, let $\varphi(\bar{z}^0, \bar{z}^1)$ be a formula of T such that for $\bar{c}^0$ and $\bar{c}^1$ realizing q we have $\models \varphi(\bar{c}^0, \bar{c}^1)$ if and only if $\theta(x; \bar{c}^0) \wedge \theta(x; \bar{c}^1)$ has rank α. Let $\pi(\bar{z})$ generate q. Form T' from T by adjoining a new predicate symbol U and the axiom

$$\bigvee \bar{z}\, U\bar{z} \wedge \bigwedge \bar{z}(U\bar{z} \to \pi(\bar{z})) \wedge \bigwedge \bar{z}^0 \bigwedge \bar{z}^1 (U\bar{z}^0 \to U\bar{z}^1 \Leftrightarrow \varphi(\bar{z}^0, \bar{z}^1)).$$

T' is ω-categorical and we may apply the result already obtained to $\theta(x; \bar{z}) \wedge U(\bar{z})$ in place of $\theta(x; \bar{z})$. As above we get $\theta^{\#}(x)$ which will now contain some occurrences of U. By replacing each atomic part $U(\cdot)$ of $\theta^{\#}$ by $\varphi(\bar{z}; \cdot) \wedge \pi(\bar{z})$ we obtain the desired formula $\theta^*(x; \bar{z})$.

It is worth pointing out that normalization works when T is not ω-categorical. Let T be stable but not necessarily ω-categorical. Let $\psi(x; \bar{y})$ and $\theta(x; \bar{z})$ be formulas of T, $k \in \omega$, and p_0 be a type such that, if $\bar{c} \in M$ realizes p_0, then $\theta(x; \bar{c})$ has ψ-rank k and ψ-degree 1, and is equivalent to a Boolean combination of formulas $\psi(x; \bar{b})$ where $\bar{b} \in M$. Call such θ *normal* if $\theta(x; \bar{c}^0)$ and $\theta(x; \bar{c}^1)$ are equivalent whenever $\bar{c}^0$ and $\bar{c}^1 \in M$ both realize p_0 and $\theta(x; \bar{c}^0) \wedge \theta(x; \bar{c}^1)$ has ψ-rank k. Using the same method as above one can deduce that given any such θ there exists a normal formula $\theta^*(x; \bar{z})$ with $p_{\theta^*} = p_0$ such that for any $\bar{c} \in M$ realizing p, $\theta^*(x; \bar{z}) \wedge \theta(x; \bar{c})$ has ψ-rank k. This theorem can be used to obtain the theorem of Shelah stated in § 1. (This is perhaps a little surprising because Shelah's theorem speaks of Morley rank while our theorem is about ψ-rank.) The method used by Shelah to prove his theorem although more elegant than ours does not seem adequate for Lemma 2 because T is not necessarily stable.

5. Conclusion. Call a structure M *ω-categorical* if the theory of M is ω-categorical, and let other adjectives applicable to theories be transferred to structures similarly. Both C1 and C2 would follow if we could prove:

C3. *There exists no ω-categorical pseudoplane.*

Using the coordinatization theorem it is easy to show that no infinite Desargueian projective plane is ω-categorical. Beyond this we have no information. If C3 is true it will probably be very hard to prove because of the richness of the class of pseudoplanes. Of course, C1 would follow from the non-existence of stable ω-categorical pseudoplanes. Also, it is easy to show that if no totally transcendental pseudoplane is ω-categorical then any ω-categorical theory has rank $\leqslant \omega$. Progress in this area seems most likely to come from constructing pseudoplanes which are either totally transcendental or at least stable.

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