On hereditarily α-Lindelöf and α-separable spaces, II

by

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Abstract. The aim of this note is to show that the continuum hypothesis implies the existence of the following two kinds of *regular* spaces:

I. hereditarily Lindelöf not separable,

II. hereditarily separable not Lindelöf.

It is also shown that the general construction supplying the second type of examples always produces spaces which are hereditarily collectionwise normal.

It should be noted that both constructions are closely related to forcing arguments with the help of which the same authors already proved the consistency of spaces of both types: I and II, with strong additional properties.

In our paper [2] we have shown that the two properties mentioned in the title are independent of each other within the class of Hausdorff spaces. We have also raised the problem there, whether the same is true in the class of the regular spaces as well. It was more or less clear already then that this problem is, if not undecidable itself, at least closely connected with such problems.

In [3] and [4], respectively, using Cohen's forcing method we have indeed established the consistency of the existence of hereditarily α-Lindelöf regular spaces which are not α-separable and vice versa. In fact, these examples provide solutions to much stronger problems, e.g. a hereditarily α-separable regular space of cardinality $\exp\exp\alpha$ and a hereditarily α-Lindelöf regular space of weight $\exp\exp\alpha$ are constructed.

The main aim of the present paper is to show that the independence of the properties (within the class of regular spaces) can be established without using the forcing method, by ordinary transfinite induction. However we need to assume here $2^\alpha = \alpha^+$, and the resulting examples will not have those very strong additional properties enjoyed by the ones constructed with forcing.

The constructions of these examples will be done in § 1 and § 2. Finally in § 3 we shall show that the hereditarily α-separable regular spaces constructed by our methods — both by forcing and transfinite induction — can even be chosen normal, and in the case $\alpha = \omega$ are always hereditarily normal too. The hereditary normality (and even paracompactness) of the other type of spaces is obtained much more easily. We remark that, using a Suslin tree, M. E. Rudin has also constructed a hereditarily separable, non-Lindelöf normal space (cf. [6]).

For all the relevant concepts and the notation which we do not define here, we refer the reader to [2] and [5].

§ 1. Hereditarily α-Lindelöf spaces. Throughout this section let α be a fixed regular cardinal and we assume that $2^\alpha = \alpha^+$ and $\alpha^{\underline{\alpha}} = \alpha$ (i.e. $\alpha^\beta = \alpha$ for all $\beta < \alpha$).

We denote by $\mathfrak{X}$ the set of all functions mapping α into α^+ in a one-to-one manner (i.e. each $x \in \mathfrak{X}$ is a sequence of type α of distinct ordinals less than α^+), by $\mathcal{E}$ the set of all functions from α into 2 (i.e. all α-type sequences of 0 and 1), and by $\mathcal{A}$ the set of all pairs $\langle E, \nu\rangle$, where $E \subset \alpha^+ \setminus \{\nu\}$, $\nu < \alpha^+$ and $|E| \leqslant \alpha$. Finally let us put

$$\mathcal{S} = \mathfrak{X} \times \mathcal{E} \times \alpha .$$

LEMMA 1. *There exists a function*

$$F\colon \alpha^+ \times \alpha^+ \to 2$$

satisfying the following two properties:

(1) *For each $s = \langle x, \varepsilon, \varrho\rangle \in \mathcal{S}$ there exists a $\nu_0 < \alpha^+$ such that for all $\nu > \nu_0$ there is a $\sigma < \alpha$ with $F\big(x(\varrho\sigma+\tau), \nu\big) = \varepsilon(\varrho\sigma+\tau)$ for every $\tau < \varrho$.*

(2) *For each $e = \langle E, \nu\rangle \in \mathcal{A}$ there is a $\xi < \alpha^+$ for which $F(\xi, \mu) = 0$ if $\mu \in E$ and $F(\xi, \nu) = 1$.*

Proof. Both $\mathcal{S}$ and $\mathcal{A}$ are of cardinality α^+, therefore they can be written in the form

$$\mathcal{S} = \{s_\xi\colon \xi < \alpha^+\} \quad \text{and} \quad \mathcal{A} = \{e_\xi\colon \xi < \alpha^+\} .$$

The definition of F is divided into two steps, as follows. First, by transfinite induction on $\nu < \alpha^+$, we shall define $F(\xi, \nu)$ for α different values of ξ in such a way as to insure the validity of (1). Secondly, for each $\xi < \alpha^+$ we shall extend the definition of (the so far partially defined) function F by assuring that (2) holds.

Step 1. Let $\nu < \alpha^+$ and assume that for each $\mu < \nu$ there is a set $a_\mu \subset \alpha^+$ with $|a_\mu| \leqslant \alpha$ so that if $\xi \in a_\mu$ then $F(\xi, \mu)$ has already been defined.

Since $\nu \leqslant \alpha^+$ we can select an enumeration of the set $\{s_\xi\colon \xi < \nu\}$ of type α, i.e.

$$\mathfrak{X}_\nu = \{s_\xi\colon \xi < \nu\} = \{t_\eta\colon \eta < \alpha\} ,$$

where we put

$$t_\eta = \langle x_\eta, \varepsilon_\eta, \varrho_\eta\rangle .$$

Now we define a sequence of ordinals $\sigma_\eta < \alpha$ by induction on $\eta < \alpha$ as follows. Suppose that for all $\zeta < \eta < \alpha$, σ_ζ has already been defined. Then we choose $\sigma_\eta < \alpha$ so that

$$A^{(x_\eta)}_{\varrho_\eta, \sigma_\eta} \subset \alpha^+ \setminus \bigcup_{\zeta<\eta} A^{(x_\zeta)}_{\varrho_\zeta, \sigma_\zeta} ,$$

where

$$A^{(x)}_{\varrho,\sigma} = \{x(\xi)\colon \varrho\sigma \leqslant \xi < \varrho(\sigma+1)\} .$$

This is possible because

$$\Big|\bigcup_{\zeta<\eta} A^{(x_\zeta)}_{\varrho_\zeta, \varrho_\zeta}\Big| < \alpha ,$$

and the sets

$$A^{(x_\eta)}_{\varrho_\eta, \sigma} \quad \text{for} \quad \sigma < \alpha$$

are pairwise disjoint.

Having defined σ_η for all $\eta < \alpha$, we put

$$a_\nu = \bigcup \{A^{(x_\eta)}_{\varrho_\eta, \sigma_\eta}\colon \eta < \alpha\} \quad \text{and} \quad F\big(x_\eta(\xi), \nu\big) = \varepsilon_\eta(\xi)$$

for $\varrho_\eta \sigma_\eta \leqslant \xi < \varrho_\eta(\sigma_\eta+1)$ and $\eta < \alpha$. By the choice of the σ_η this definition is justified.

It is quite easy to see now that no matter how we extend F to a full function on $\alpha^+ \times \alpha^+$ it will satisfy (1). Indeed, if s_ξ is arbitrary then for $\nu > \xi$ we have $s_\xi \in \mathfrak{X}_\nu$, and therefore if $s_\xi = t_\eta$ in $\mathfrak{X}_\nu$, σ_η will be as required by (1).

Step 2. Observe that in step 1. $F(\xi, \nu)$ has been defined only for α values of ξ for each $\nu < \alpha^+$. Therefore we can find a $\mu_0 < \alpha^+$ so that $F(\xi, \nu)$ is undefined for all $\xi \geqslant \mu_0$ and $\nu \in E_0 \cup \{\nu_0\}$, where $e_0 = \langle E_0, \nu_0\rangle$ (and in general $e_\xi = \langle E_\xi, \nu_\xi\rangle$).

The extension of F will now again be done by induction. Initially we put

$$F(\mu_0, \nu) = \begin{cases} 0, & \text{if} \quad \nu \in E_0 , \\ 1, & \text{if} \quad \nu = \nu_0 . \end{cases}$$

Having defined μ_η for each $\eta < \zeta$ for a fixed $\zeta < \alpha^+$, we choose μ_ζ in such a way that $F(\xi, \nu)$ is still undefined for $\xi \geqslant \mu_\zeta$ and $\nu \in E_\zeta \cup \{\nu_\zeta\}$. Since, for fixed ν, $F(\xi, \nu)$ has only been defined for at most α values of ξ, this is possible. Then again we put

$$F(\mu_\zeta, \nu) = \begin{cases} 0, & \text{if} \quad \nu \in E_\zeta , \\ 1, & \text{if} \quad \nu = \nu_\zeta . \end{cases}$$

Having completed this induction we see immediately that no matter how we extend F to a full function on $\alpha^+ \times \alpha^+$, in addition to (1), it will also satisfy (2). Thus the proof of the Lemma is completed.

Now we come to the main result of this section.

THEOREM 1. *There exists a hereditarily α-Lindelöf, regular space X of cardinality α^+ in which every subset of power $\leqslant \alpha$ is closed and discrete.*

Proof. Let us put for each $\xi < \alpha^+$

$$A_\xi = \{\nu < \alpha^+ \colon F(\xi, \nu) = 1\}.$$

We shall simply write $-A_\xi$ for $\alpha^+ \setminus A_\xi = \{\nu \colon F(\xi, \nu) = 0\}$.

Then we take α^+ as the underlying set of our space X and choose the family of all sets A_ξ and $-A_\xi$ as an α-subbase for the topology of X. In other words all intersections of sets A_ξ and $-A_\xi$, less than α in number, will provide a basis for the topology of X. Obviously, this topology is an α-topology, i.e. the intersection of less than α open sets in X is always open.

Denoting by $H_\alpha(\alpha^+)$ the set of all partial functions of cardinality $< \alpha$ from α^+ into two, the above defined basic open sets can be labeled by the $\varepsilon \in H_\alpha(\alpha^+)$ in the following manner:

$$B_\varepsilon = \bigcap \{A_\xi \colon \varepsilon(\xi) = 1\} \cap \bigcap \{-A_\xi \colon \varepsilon(\xi) = 0\}.$$

Obviously, $\nu \in B_\varepsilon$ if and only if

$$F(\eta, \nu) = \varepsilon(\eta) \quad \text{for all } \eta \in D(\varepsilon).$$

Now it is immediate from the definition that the sets B_ε are both open closed, hence X is 0-dimensional. Moreover, it follows from (2) that any $E \subset X$, $|E| \leqslant \alpha$ is closed and discrete, applying (2) to the pair $\langle E, \mu\rangle$ if $\mu \notin E$ and to $\langle E \setminus \{\mu\}, \mu\rangle$ if $\mu \in E$. In particular X is T_1, hence a Tychonov space.

Now we prove that X is hereditarily α-Lindelöf. For this it will suffice to show that every right-separated subspace of X is of cardinality $< \alpha^+$ (cf. [2] or [5]).

Suppose, on the contrary, that $\{\nu_\xi \colon \xi < \alpha^+\}$ is a right-separated subspace of X so that for each $\xi < \alpha^+$ there is an $\varepsilon_\xi \in H_\alpha(\alpha^+)$ with

$$\nu_\eta \notin B_{\varepsilon_\xi} \quad \text{for} \quad \eta > \xi.$$

Let us denote by D_ξ the domain of ε_ξ. Then according to the regularity of α^+, $\alpha^{\underline{\alpha}} = \alpha$, and a well-known result of Erdös and Rado (cf. [1] or [5]) we can assume that every D_ξ is of the form

$$D_\xi = D \cup E_\xi \quad (D \cap E_\xi = \emptyset),$$

where the sets E_ξ are pairwise disjoint and all have the same order type $\varrho < \alpha$. Moreover, we can as well assume that the restrictions of all ε_ξ to D are the same.

Now let us define $\varepsilon \in \mathcal{E}$ and $x \in \mathfrak{X}$ by the following stipulations:

$$\begin{aligned} \varepsilon(\varrho\sigma + \tau) &= \varepsilon_\sigma(\eta_\tau^{(\sigma)}) \\ x(\varrho\sigma + \tau) &= \eta_\tau^{(\sigma)} \end{aligned} \quad \text{for} \quad \sigma < \alpha \text{ and } \tau < \varrho,$$

where $\eta_\tau^{(\sigma)}$ is the τth element of E_σ in its natural order. That x is one-to-one follows immediately from the fact that the E_σ are pairwise disjoint.

Now the triple $\langle x, \varepsilon, \varrho\rangle$ belongs to $\mathcal{S}$, hence by (1) there is a $\nu < \alpha^+$ such that if $\mu > \nu$ then for some $\sigma < \alpha$ we have

$$F(\eta_\tau^{(\sigma)}, \mu) = F\big(x(\varrho\sigma + \tau), \mu\big) = \varepsilon(\varrho\sigma + \tau) = \varepsilon_\sigma(\eta_\tau^{(\sigma)})$$

for all $\tau < \varrho$.

Now let us choose a $\xi < \alpha^+$ with $\xi \geqslant \alpha$ and $\nu_\xi > \nu$ and pick a $\sigma < \alpha$ satisfying the above equalities with $\mu = \nu_\xi$. Since $D_\sigma = D \cup E_\sigma$, $D_\xi = D \cup E_\xi$, and ε_ξ and ε_σ agree on D, we get from here that for all $\eta \in D_\sigma$

$$F(\eta, \nu_\xi) = \varepsilon_\sigma(\eta),$$

which is equivalent to $\nu_\xi \in B_{\varepsilon_\sigma}$. However, this shows that the sequence $\{\nu_\xi \colon \xi < \alpha^+\}$ cannot be right-separated, and thus completes the proof that X is hereditarily α-Lindelöf.

Let us remark that, as can be easily seen, every α-Lindelöf regular α-topological space is paracompact (this is well-known for $\alpha = \omega$), hence our space X enjoys very strong separation properties.

Finally, since every α-element subspace of X is closed, it is obviously not α-separable.

§ 2. Hereditarily α-separable spaces. Let now α be an arbitrary cardinal number with $2^\alpha = \alpha^+$. We shall again start with a general set-theoretic lemma, from which the construction of our spaces will be easy.

LEMMA 2. *There exists a mapping*

$$F \colon \alpha^+ \times \alpha^+ \to 2$$

such that if $A \subset \alpha^+$, $|A| = \alpha$ then there is a $\nu < \alpha^+$ so that for every $\varepsilon \in H(\alpha^+ \setminus \nu)$ we have a $\varrho \in A$ for which

$$F(\eta, \varrho) = \varepsilon(\eta) \quad \textit{for all } \eta \in D(\varepsilon). \tag{3}$$

(Here, as usual, cf. [4], $H(S)$ denotes the set of all finite partial functions from S into 2.)

Proof. We shall construct the function F by transfinite induction as follows. Let us first take an enumeration $\{A_\mu: \mu < \alpha^+\}$ of all α-element subsets of α^+, which is possible by $2^\alpha = \alpha^+$.

Now assume that $\nu < \alpha^+$ and F has already been defined for each pair belonging to $\alpha^+ \times \nu$. Then for any $\varepsilon \in H(\nu)$ we can define the set

$$F_\varepsilon^{(\nu)} = \{\mu < \nu:\ F(\eta, \mu) = \varepsilon(\eta) \text{ for all } \eta \in D(\varepsilon)\}.$$

Next we define the collection Z_ν as the set of all $Y \subset \alpha^+$, $|Y| = \alpha$, for which there are a $\mu < \nu$ and an $\varepsilon \in H(\nu)$ so that $A_\mu \subset \nu$, and

$$Y = A_\mu \cap F_\varepsilon^{(\nu)}.$$

In particular taking for ε the empty function we see that every A_μ with $\mu < \nu$ and $A_\mu \subset \nu$ belongs to Z_ν. Moreover it is obvious from the definition that $|Z_\nu| \leqslant \alpha$.

Thus Z_ν is a collection of at most α α-element subsets of ν, hence as is well-known we can decompose ν into a disjoint union $\nu = H_0^{(\nu)} \cup H_1^{(\nu)}$ so that for every $Y \in Z_\nu$ we have both

$$|Y \cap H_0^{(\nu)}| = \alpha \quad \text{and} \quad |Y \cap H_1^{(\nu)}| = \alpha.$$

Then we extend the definition of F to pairs of the form $\langle \eta, \nu \rangle$ as follows:

$$F(\eta, \nu) = \begin{cases} 0, & \text{if} \quad \eta \in H_0^{(\nu)}, \\ 1, & \text{if} \quad \eta \in H_1^{(\nu)}, \\ \text{arbitrary} & \text{otherwise}. \end{cases}$$

We claim that the full function F: $\alpha^+ \times \alpha^+ \to 2$ obtained in this manner does indeed have the required properties. To see this let $A = A_\mu$ be an arbitrary α-element subset of α^+, and let $\nu < \alpha^+$ be so that $\mu < \nu$ and $A_\mu \subset \nu$.

Now let $\varepsilon \in H(\alpha^+ \setminus \nu)$, where we put $D(\varepsilon) = \{\eta_1, \ldots, \eta_k\}$ in increasing order, and for the sake of simplicity we denote by $\varepsilon_1, \ldots, \varepsilon_k$ the corresponding values of ε. By the choice of ν we have $A_\mu \in Z_\nu \subset Z_{\eta_1}$, hence

$$|A_\mu \cap H_{\varepsilon_1}^{(\eta_1)}| = \alpha.$$

But $A_\mu \cap H_{\varepsilon_1}^{(\eta_1)} \subset A_\mu \cap F_{\langle \eta_1, \varepsilon_1 \rangle}^{(\eta_2)}$, hence this latter set belongs to Z_{η_2}. Continuing this reasoning in k steps and choosing any $\sigma > \eta_k$ we see that

$$A_\mu \cap F_\varepsilon^{(\sigma)} \in Z_\sigma$$

holds as well, but then $|A_\mu \cap F_\varepsilon^{(\sigma)}| = \alpha$, and for any $\varrho \in A_\mu \cap F_\varepsilon^{(\sigma)}$ (3) is satisfied. This completes the proof of our lemma.

DEFINITION. Let

$$R = \times\{R(\xi):\ \xi < \alpha^+\}$$

be the product of α^+ spaces. (The elements of R are regarded as functions f defined on α^+ with $f(\xi) \in R(\xi)$.) A subspace $S \subset R$ is called *α-hereditarily finally dense* (or shortly α-HFD) if for any $A \subset S$, $|A| = \alpha$ there is a $\nu < \alpha^+$ such that if U is any elementary open set whose ξth projection is always $R(\xi)$ if $\xi < \nu$, then $U \cap A \neq \emptyset$.

Obviously S is α-HFD in R if and only if for any $A \subset S$ with $|A| = \alpha$ there is a $\nu < \alpha^+$ such that the "tails" of the members of A "cut off" at ν are dense in the partial product

$$\times\{R(\xi):\ \nu \leqslant \xi < \alpha^+\}.$$

This explains the term α-HFD.

THEOREM 2. *Let*

$$R = \times\{R(\xi):\ \xi < \alpha^+\},$$

where $w(R(\xi)) \leqslant \alpha$ for each $\xi < \alpha^+$. If $S \subset R$ is α-HFD, then the subspace S is hereditarily α-separable.

Proof. Suppose, on the contrary, that S is not hereditarily α-separable. Then, as is well-known (cf. [2] or [5]) S contains a left-separated subspace $\{f_\xi:\ \xi < \alpha^+\}$ of type α^+. That is for each $\xi < \alpha^+$ we can choose an elementary open neighbourhood U_ξ of f_ξ which contains no f_η with $\eta < \xi$.

Fixing an open basis B_ξ with $|B_\xi| \leqslant \alpha$ in each $R(\xi)$ we might assume that every elementary open set U_ξ is determined by a finite subset $D_\xi \subset \alpha^+$ and a function V_ξ defined on D_ξ so that for $\sigma \in D_\xi$

$$V_\xi(\sigma) \in B_\xi,$$

where

$$U_\xi = \{f \in R:\ \sigma \in D_\xi \to f(\sigma) \in V_\xi(\sigma)\}.$$

Now we can obviously assume, similarly as in the proof of Theorem 1, that the sets D_ξ can be written as

$$D_\xi = D \cup E_\xi \quad (D \cap E_\xi = \emptyset),$$

where the E_ξ are pairwise disjoint, moreover that the restrictions of all the V_ξ to D are the same.

Now let us put $A = \{f_\eta:\ \eta < \alpha\}$. Then $|A| = \alpha$, hence because S is α-HFD we can choose a $\nu < \alpha^+$ for A as indicated in the definition of α-HFD.

Since the E_ξ are disjoint, there is a ξ with $\alpha \leqslant \xi < \alpha^+$ such that $E_\xi \subset \alpha^+ \setminus \nu$. But then we have an $f_\eta \in A$, for which

$$f_\eta(\sigma) \in V_\xi(\sigma) \quad \text{for all } \sigma \in D_\xi = D \cup E_\xi ,$$

i.e.

$$f_\eta \in U_\xi \quad \text{where} \quad \eta < \alpha \leqslant \xi ,$$

a contradiction. This completes the proof.

THEOREM 3. *Let F be as in Lemma 2, and for each $\sigma < \alpha^+$ put*

$$f_\sigma(\xi) = F(\xi, \sigma) ,$$

and let

$$S = \{f_\sigma \colon \sigma < \alpha^+\} .$$

Then S as a subspace of $D(2)^{\alpha^+}$ is α-HFD.

The immediate proof is left to the reader.

It is interesting to notice the following property of α-HFD sets.

LEMMA 3. *Let $R = \times\{R(\xi)\colon \xi < \alpha^+\}$ as above and $S \subset R$ be α-HFD. For every $f \in S$ let us choose another point $f' \in R$ so that*

$$|\{\sigma\colon f(\sigma) \neq f'(\sigma)\}| \leqslant \alpha .$$

Then if $S' = \{f'\colon f \in S\}$, so S' is also α-HFD.

Proof. Indeed if $A' \subset S'$, $|A'| = \alpha$, where $A \subset S$ and $A' = \{f'\colon f \in A\}$ then we can choose a $\nu' < \alpha^+$ bigger than the ν corresponding to A by the α-HFD property of S, and all the coordinates σ for which $f(\sigma) \neq f'(\sigma)$ for some $f \in A$. Obviously this ν' will be suitable for A'.

COROLLARY. *There is a subspace $T \subset D(2)^{\alpha^+}$ which is hereditarily α-separable but for which $\psi(p, T) = \alpha^+$ for each $p \in T$. (Here $\psi(p, T)$ is the pseudo-character of p in T, cf.* [5].)

Proof. Let S be the α-HFD subspace of $D(2)^{\alpha^+}$ indicated in Theorem 3. Moreover let $\{g_\sigma\colon \sigma < \alpha^+\}$ be an enumeration of all functions $g \in D(2)^{\alpha^+}$ such that

$$|\{\xi\colon g(\xi) \neq 0\}| \leqslant \alpha .$$

Now for each $\sigma < \alpha^+$ let us put

$$p_\sigma(\xi) = \begin{cases} g_\sigma(\xi) , & \text{if} \quad \xi < \sigma , \\ f_\sigma(\xi) , & \text{if} \quad \xi \geqslant \sigma , \end{cases}$$

and

$$T = \{p_\sigma\colon \sigma < \alpha^+\} .$$

Since each p_σ differs from f_σ in at most α coordinates, by Lemma 3 T is α-HFD, hence hereditarily α-separable.

Let $p_\sigma \in T$ be arbitrary and consider any α elementary neighbourhoods of p_σ. Then these specify at most α coordinates of p_σ, hence there is a g_ϱ which coincides with p_σ at these coordinates but differs from p_σ at some others, which are less than ϱ. However then the same is true for p_ϱ, hence p_ϱ belongs to the intersection of the selected α neighbourhoods although $p_\varrho \neq p_\sigma$, which shows indeed that

$$\psi(p_\sigma, T) = \alpha^+ .$$

Since in a hereditarily α-Lindelöf space X we must have $\psi(p, X) \leqslant \alpha$ for all $p \in X$ (cf. [5]), the above space T is what we require: hereditarily α-separable, completely regular, but non-hereditarily α-Lindelöf. Moreover it also presents a very strong counterexample to Problem 2.17 in [5].

Remark. It is fairly easy to see that if α, in addition to $2^\alpha = \alpha^+$, has the property $\alpha^{\underline{\alpha}} = \alpha$ (i.e. $\alpha^\beta = \alpha$ for every $\beta < \alpha$), then the same theorems will be valid for α-product spaces instead of ordinary topological (i.e. ω-) products. E.g. in Lemma 2 an obvious change would be to put $\varepsilon \in H_\alpha(\alpha^+ \setminus \nu)$ (i.e. ε is a partial function from $\alpha^+ \setminus \nu$ into 2 with $|\varepsilon| < \alpha$) instead of a finite $\varepsilon \in H(\alpha^+ \setminus \nu)$.

§ 3. Normality of ω-HFD subspaces.

Throughout this section let

$$R = \times\{R(\xi)\colon \xi < \omega_1\}$$

be the topological product of regular spaces $R(\xi)$ of weight $\leqslant \omega$; for each $\xi < \omega_1$ we fix an open basis B_ξ for $R(\xi)$ with $|B_\xi| \leqslant \omega$.

For any $\nu < \omega_1$ we shall denote by R_ν and R^ν, respectively the partial products

$$R_\nu = \times\{R(\xi)\colon \xi < \nu\}$$

and

$$R^\nu = \times\{R(\xi)\colon \nu \leqslant \xi < \omega_1\} .$$

The natural projections of R onto R_ν and R^ν will be denoted by π_ν and π^ν, respectively. For any $f \in R$ or $A \subset R$ we shall simply write f_ν and f^ν instead of $\pi_\nu(f)$ and $\pi^\nu(f)$, or A_ν and A^ν instead of $\pi_\nu(A)$ and $\pi^\nu(A)$.

Our main aim is to prove the following result.

THEOREM 4. *Let R be as above and $S \subset R$ be ω-HFD. Then S is hereditarily (collectionwise) normal.*

The proof of the theorem is based on the following two lemmas.

LEMMA 4. *For every $A \subset S$ with $|A| = \omega$ and every $\sigma < \omega_1$ there is a $\nu < \omega_1$ such that $\sigma \leqslant \nu$ and*

(∗) *if U is an elementary open set in R with $U^\nu = R^\nu$ and $|U \cap A| = \omega$, then $(U \cap A)^\nu$ is dense in R^ν.*

Proof. Obviously it will suffice to obtain a ν in such a way that (*) be valid for those U whose projection to $R(\xi)$ is always a member of B_ξ, if not equal to $R(\xi)$ itself. Let $\mathfrak{U}$ denote the family of all such U and for any fixed $\tau < \omega_1$ let

$$\mathfrak{U}^{(\tau)} = \{U \in \mathfrak{U}\colon\ U^\tau = R^\tau\}\,.$$

Obviously if $\tau < \omega_1$ then

$$|\mathfrak{U}^{(\tau)}| \leqslant \omega.$$

Thus in particular there are only countably many $U \in \mathfrak{U}^{(\sigma)}$ such that

$$|A \cap U| = \omega\,.$$

Therefore, since S is ω-HFD, we can find a ϱ_0 with $\sigma < \varrho_0 < \omega_1$ such that $(A \cap U)^{\varrho_0}$ is dense in R^{ϱ_0} for every such U. (Here we use the obvious fact that if X^μ is dense in R^μ then X^ν is dense in R^ν for all $\nu > \mu$.)

Reasoning in the same manner but exchanging σ by ϱ_0 we can obtain a ϱ_1 with $\varrho_0 < \varrho_1 < \omega_1$ so that if

$$U \in \mathfrak{U}^{(\varrho_0)} \quad \text{and} \quad |U \cap A| = \omega$$

then $(U \cap A)^{\varrho_1}$ is dense in R^{ϱ_1}. Continuing this by induction we can define a strictly increasing sequence of ordinals $\varrho_0 < \varrho_1 < \ldots < \varrho_n < \ldots$ for $n < \omega$ so that for each $n < \omega$ if

$$U \in \mathfrak{U}^{(\varrho_n)} \quad \text{and} \quad |U \cap A| = \omega\,,$$

then $(U \cap A)^{\varrho_{n+1}}$ is dense in $R^{\varrho_{n+1}}$. Now let ν be the limit of this sequence. Obviously $\sigma < \nu$ and we claim that (*) is also satisfied.

Indeed, if $U \in \mathfrak{U}^{(\nu)}$ and $|U \cap A| = \omega$, then there is a $n < \omega$ such that $U \in \mathfrak{U}^{(\varrho_n)}$ as well, hence $(A \cap U)^{\varrho_{n+1}}$ is dense in $R^{\varrho_{n+1}}$ and a fortiori it is dense in R^ν.

For any $A \subset S$, $|A| = \omega$ let $J(A)$ denote the set of all $\nu < \omega_1$ for which (*) is satisfied. We have just shown that $J(A)$ is cofinal in ω_1, while a moment's reflection will convince us that $J(A)$ is a closed set of ordinals, i.e. it is both closed and cofinal.

LEMMA 5. *Assume that $A \subset S$, $|A| = \omega$ and $\nu \in J(A)$. Then $(\bar{A})_\nu$ is a closed set in S_ν, where $\bar{A}$ denotes the closure of A in S.*

Proof. It suffices to prove that if $f \in S$ and f_ν is an accumulation point of $(\bar{A})_\nu$ then $f \in \bar{A}$. To see this let W be an arbitrary elementary neighbourhood of f in R, whose projections to the factors $R(\xi)$, when not equal to $R(\xi)$, belong to B_ξ.

Then W can be written in the form $W = U \cap V$, where $U \in \mathfrak{U}^{(\nu)}$ and $V_\nu = R_\nu$. Since f_ν is an accumulation point of $(\bar{A})_\nu$ and thus of A_ν as well, we have $|A_\nu \cap U_\nu| = \omega$, hence $|A \cap U| = \omega$, too. But then, since $\nu \in J(A)$ and $U \in \mathfrak{U}^{(\nu)}$, the set $(A \cap U)^\nu$ is dense in R^ν, hence there is a $g \in A \cap U$ such that $g^\nu \in V^\nu$. However then $V_\nu = R_\nu$ implies $g \in V$ as well, and thus $g \in A \cap U \cap V = A \cap W$, which indeed shows $f \in \bar{A}$.

Now we turn to the proof of Theorem 4. Since by Theorem 2 S is hereditarily separable, every infinite closed subset of S can be represented in the form $\bar{A}$, where $A \subset S$, $|A| = \omega$.

Now let $\bar{A}$ and $\bar{B}$ be any two disjoint closed infinite sets in S. It is well known that the intersection of countably many closed cofinal subsets of ω_1 is closed cofinal (cf. [5]). Therefore so is

$$J = J(A) \cap J(B) \cap J(A \cup B)\,.$$

Obviously, there exists an ordinal $\sigma_{A,B} < \omega_1$ so that $A_\sigma \cap B_\sigma = \emptyset$ for all $\sigma \geqslant \sigma_{A,B}$, because $A \cap B = \emptyset$. Let $\nu \in J$ be such that $\nu \geqslant \sigma_{A,B}$. Since J is cofinal, such a ν can indeed be chosen.

We claim that we have then

$$(\bar{A})_\nu \cap (\bar{B})_\nu = \emptyset\,.$$

Suppose, on the contrary that there are $f \in \bar{A}$ and $g \in \bar{B}$ such that $f_\nu = g_\nu$. Since $\nu \geqslant \sigma_{A,B}$, we have $A_\nu \cap B_\nu = \emptyset$, hence we can assume e.g. that $g_\nu = f_\nu \notin A_\nu$. But then $g_\nu = f_\nu$ is an accumulation point of A_ν in S_ν, hence as was shown in the proof of Lemma 5, we must have $f, g \in \bar{A}$, which contradicts $\bar{A} \cap \bar{B} = \emptyset$. This, together with Lemma 5 shows that $(\bar{A})_\nu$ and $(\bar{B})_\nu$ are disjoint closed sets in S_ν. But S_ν is of course metrizable, being regular and of countable weight, and then in S_ν we can find open sets G and H so that $(\bar{A})_\nu \subset G$ and $(\bar{B})_\nu \subset H$, moreover $G \cap H = \emptyset$. But then the sets $S \cap \pi_\nu^{-1}(G)$ and $S \cap \pi_\nu^{-1}(H)$ are disjoint open sets in S containing the corresponding sets $\bar{A}$ and $\bar{B}$, which completes the proof that S is normal.

However it is well known that every normal space is countably collectionwise normal, which in the case of S coincides with full collectionwise normality, as being hereditarily separable implies that every discrete collection of sets is countable.

Remarks. It is easily seen that Theorem 4 could be immediately generalized to α-HFD subspaces of α-product spaces, provided that α is a regular cardinal. If we compare this with the remark at the end of § 2, we see that if $\alpha^{\underline{\alpha}} = \alpha$, then all our hereditarily α-separable counterexamples, including even the ones obtained by forcing in [4], can be chosen to be (collectionwise) normal.

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A note on topological model theory

by

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Abstract. A *topological structure* is endowed with a topology, with first order functions which are continuous in that topology, and with first order relations which are open or closed in it (like order and equality, respectively). In the present paper we investigate questions of continuity concerning the predicates tnd Skolem functions which are definable in a topological structure. An application ao positive definite polynomials is included.

1. In the present paper, we offer several observations on the emerging subject of topological model theory (see problem No. 4 in [4]). This theory is, or will be, concerned with the general model theoretic aspects of algebraic structures endowed with a topology to which the algebraic entities of the structure relate in a natural way. Topological groups or fields are typical of the kind of structure that we have in mind.

We begin with a rather natural definition although we shall see in due course that it is not sufficiently wide to cover several cases that should be taken into account.

A *topological structure* M is (i) a structure in the standard sense of model theory, with respect to a first order language L in which equality (if it occurs) is on a par with other relations, and (ii) a topological space such that the following conditions are satisfied.

1.1. *All basic functions in M (i.e., functions which have a name in L), $n \geqslant 1$, are continuous in the given topology.*

1.2. *If R is an n-place relation in M (which has a name in L), $n \geqslant 1$, then it is either open or closed for the product topology in M^n.*

For a topological group, 1.1 is satisfied by the operations of multiplication and inversion (reciprocation). For a topological field, it is satisfied by addition, subtraction and multiplication, but not by inversion (which is discontinuous at zero, however we may define it there). Thus, in our present framework, the language L for a topological field may include symbols for addition, subtraction and multiplication but not for inversion or division.

As for Condition 1.2, consider first equality, $x_1 = x_2$, and suppose that it coincides with the identity in M (i.e., with the diagonal in M^2).