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A note on topological model theory

by

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Abstract. A *topological structure* is endowed with a topology, with first order functions which are continuous in that topology, and with first order relations which are open or closed in it (like order and equality, respectively). In the present paper we investigate questions of continuity concerning the predicates tnd Skolem functions which are definable in a topological structure. An application ao positive definite polynomials is included.

1. In the present paper, we offer several observations on the emerging subject of topological model theory (see problem No. 4 in [4]). This theory is, or will be, concerned with the general model theoretic aspects of algebraic structures endowed with a topology to which the algebraic entities of the structure relate in a natural way. Topological groups or fields are typical of the kind of structure that we have in mind.

We begin with a rather natural definition although we shall see in due course that it is not sufficiently wide to cover several cases that should be taken into account.

A *topological structure* M is (i) a structure in the standard sense of model theory, with respect to a first order language L in which equality (if it occurs) is on a par with other relations, and (ii) a topological space such that the following conditions are satisfied.

1.1. *All basic functions in M (i.e., functions which have a name in L), $n \geqslant 1$, are continuous in the given topology.*

1.2. *If R is an n-place relation in M (which has a name in L), $n \geqslant 1$, then it is either open or closed for the product topology in M^n.*

For a topological group, 1.1 is satisfied by the operations of multiplication and inversion (reciprocation). For a topological field, it is satisfied by addition, subtraction and multiplication, but not by inversion (which is discontinuous at zero, however we may define it there). Thus, in our present framework, the language L for a topological field may include symbols for addition, subtraction and multiplication but not for inversion or division.

As for Condition 1.2, consider first equality, $x_1 = x_2$, and suppose that it coincides with the identity in M (i.e., with the diagonal in M^2).

It is then easy to see that $x_1 = x_2$ is closed if and only if M is a Hausdorff space in the given topology. In an ordered field, the order relation is open.

Let $Q(x_1, \dots, x_n)$ be any *predicate* (well formed formula) in the language L, with free variables $x_1, \dots, x_n$. We shall say that Q is open (closed) in M according as the set determined by Q in M^n is open (closed). We shall say that the *term* $t(x_1, \dots, x_n)$ is continuous in M if the corresponding function $M^n \to M$ is continuous. Thus, for a topological structure M as defined above all terms formulated in the vocabulary of M are continuous in M. As an immediate consequence, we have the following lemma, which will be used in the sequel.

1.3. *Let $Q(y_1, \dots, y_m)$ be open in the topological structure M and let $t_j(x_1, \dots, x_n)$, $j = 1, \dots, m$ be terms in the vocabulary of M. Then the predicate*

$$R(x_1, \dots, x_n) = Q\big(t_1(x_1, \dots, x_n), \dots, t_m(x_1, \dots, x_n)\big)$$

is open in M. Similarly, if $Q(y_1, \dots, y_m)$ is closed in M so is $R(x_1, \dots, x_n)$.

We shall also make use of the following remark. Let $A_1, \dots, A_n$, $n \geqslant 1$, be open sets in a topological space T and, for any subset β of the set of natural numbers $\alpha = \{1, \dots, n\}$, let B_β be the intersection of the sets A_j for $j \in \beta$ and of their complements $\bar{A}_j = T - A_j$ for $j \in \alpha - \beta$. We then have

1.4. *Let S be a nonempty open subset of T. Then there exists a $\beta \subset \alpha$ such that $S \cap B_\beta$ is nonempty and open.*

Proof. We have to show, on the stated assumptions that there is a β such that $S \cap B_\beta$ is nonempty and open, J. Suppose first that $n = 1$ and consider $A_1 \cap S$. If this set is not empty, take $J = A_1 \cap S$ and $\beta = \{1\}$. If $A_1 \cap S = \emptyset$, $\bar{A}_1 \cap S$ equals S so we may take $J = S$. Suppose the assertion has been proved up to some $n \geqslant 1$, and let $A_1, A_2, \dots, A_{n+1}$ and S be given so as to satisfy the hypothesis of 1.4. Then S has an open nonempty intersection D with some B_β where $\beta \subset \alpha = \{1, \dots, n\}$. If $D \cap A_{n+1}$ is not empty, we choose it for J. If $D \cap A_{n+1} = \emptyset$, then $D \subset \bar{A}_{n+1}$, so that $D = J$ is the intersection of S with $B_\beta \cap \bar{A}_{n+1}$. This completes the argument. An immediate consequence is:

1.5. *Let A be a boolean combination of open sets $A_1, \dots, A_n$ in a topological space T, and let S be a nonempty open set in T. Then either $S \cap A$ or $S \cap \bar{A}$ contains interior points.*

For we may write both A and $\bar{A}$ as unions of the sets B_β considered in 1.4.

2. Although the basic relations of a topological structure M are supposed to be either open or closed, a general predicate in the vocabulary of M may determine a set that is neither. Let $Q(x_1, \dots, x_n)$ be such a predicate. We shall say that a point $P = (a_1, \dots, a_n) \in M^n$ is *stable for Q* if there exists an open neighborhood U of P in M^n (in the product topology) such that for all $P' = (\xi_1, \dots, \xi_n) \in U$, either $M \models Q(\xi_1, \dots, \xi_n)$ simultaneously or $M \models \neg Q(\xi_1, \dots, \xi_n)$ simultaneously. A point $P \in M^n$ will be called *stable* if it is stable for all predicates $Q(x_1, \dots, x_n)$ in the given language. A point is *unstable* (or is *unstable for a predicate Q*) if it is not stable (not stable for Q). It is obvious that the set of points which are stable for a predicate Q is open. We have

2.1. *The set of points which are unstable for a quantifier free predicate $Q(x_1, \dots, x_n)$ is nowhere dense in M^n.*

We recall that a set A is nowhere dense in a topological space T if the closure of A has no interior points.

For any (atomic) relation symbol R_ν which is contained in Q, put $Q_\nu = R_\nu$ if R_ν is open in M and $Q_\nu = \neg R_\nu$ if R_ν is closed in M. Then $Q(x_1, \dots, x_n)$ is logically equivalent to a boolean polynomial Q' of instances of the Q_ν, and these are all open in M by 1.3. We may identify the set determined by such an instance of a Q_ν in $M^n = T$ with an open set A_j as in 1.5. The assertion of 2.1 now follows immediately from the conclusion of 1.5.

We say that a structure M has *elimination of quantifiers* (in a language L which may contain individual constants for some of the elements of M) if for every predicate $Q(x_1, \dots, x_n)$ in L there is a quantifier-free predicate $Q'(x_1, \dots, x_n)$ in L such that

$$M \models (\forall x_1) \dots (\forall x_n)[Q(x_1, \dots, x_n) \equiv Q'(x_1, \dots, x_n)]\,.$$

Evidently, from 2.1,

2.2. *Suppose that M is a topological structure which has elimination of quantifiers. Then the set of points in M^n, which are unstable for a given predicate $Q(x_1, \dots, x_n)$, is nowhere dense in M^n.*

Using the Baire category argument (compare [2], p. 200) we now obtain immediately

2.3. *Suppose that the topological structure M has elimination of quantifiers in a countable language L such that the topology of M is* (i) *regular and locally compact or* (ii) *that of a complete metric space. Then, for any $n \geqslant 1$, the set of stable points of M^n is dense in M^n.*

3. While the argument leading up to 2.3 is exceedingly simple, the result can be illustrated by some interesting concrete examples. First, let $M = C$ be the field of complex numbers with the usual metric and topology. Let L be formulated in terms of the relation of equality and in terms of the function symbols of addition and multiplication and the individual constants 0 and 1. Then L is countable and C has elimination of quantifiers and so 2.3 applies. We have

3.1. *A point $P = (a_1, \dots, a_n) \in C^n$ is stable if and only if the coordinates of P are algebraically independent over the rational numbers.*

Proof. The condition is necessary. For suppose it is not satisfied. Then there exists a nonzero polynomial $p(x_1, \ldots, x_n)$ with rational coefficients such that $p(a_1, \ldots, a_n) = 0$. But the condition $p(x_1, \ldots, x_n) = 0$ can be expressed by a predicate $Q_p(x_1, \ldots, x_n)$ in L. Since $C \models Q_p(a_1, \ldots, a_n)$ there exists an open neighborhood of P, U, such that $C \models Q_p(\xi_1, \ldots, \xi_n)$ for all points $P' = (\xi_1, \ldots, \xi_n) \in U$, i.e., $p(\xi_1, \ldots, \xi_n) = 0$ for such points. But this would imply that the set of zeros of $p(x_1, \ldots, x_n)$ includes an open set in C^n which is impossible.

The condition is also sufficient. Suppose that the coordinates $a_1, \ldots, a_n$ of the point P are algebraically independent over the rational numbers. Let $Q(x_1, \ldots, x_n)$ be a predicate in the language of L and suppose that $Q(a_1, \ldots, a_n)$ holds in C. Since C has elimination of quantifiers, the set of points of C^n, which satisfy Q, to be denoted by A_Q is a finite union of finite intersections of sets B_ν given either (i) by an equation $p(x_1, \ldots, x_n) = 0$, or (ii) by an inequation $p(x_1, \ldots, x_n) \neq 0$ where p has rational coefficients in both cases. Thus, P belongs to one of these intersections $B_1 \cap B_2 \cap \ldots \cap B_n$, say. But if B_ν is given by an equation $p(x_1, \ldots, x_n) = 0$, then the polynomial on the left-hand side must vanish identically since the numbers $a_1, \ldots, a_n$ are algebraically independent. It follows that $B_1 \cap B_2 \cap \ldots \cap B_n$ is an intersection of open sets, and so P is stable for Q.

Suppose next that $M = R$ is the ordered field of real numbers with the usual topology where the language L has been augmented by the inclusion of the order relation ($<$). Then we still have

3.2. *A point $P = (a_1, \ldots, a_n) \in R^n$ is stable if and only if the coordinates of P are algebraically independent over the rational numbers.*

The proof of necessity is similar to that given for 3.1. For sufficiency we recall that R also has elimination of quantifiers. Thus, the above proof is still applicable except that the inequations $p(x_1, \ldots, x_n) \neq 0$ have to be replaced by inequalities $p(x_1, \ldots, x_n) > 0$.

4. The following relativization of the results of section 2 is of interest. Let V be a nonempty subset of a topological space T. By the topology of V we mean the topology induced by T in V. Then the lemmas 1.4 and 1.5 can be relativized from T to V, thus

4.1. *Suppose the sets A_j, B_β are as defined in section 1 and let S be a nonempty open subset of V. Then there exists a $\beta \subset \alpha$ such that $S \cap B_\beta$ is nonempty and open in the topology of V.*

4.2. *Let A be a boolean combination of open sets $A_1, \ldots, A_n$ in T and let S be a nonempty open subset of V. Then either $S \cap A$ or $S \cap \bar{A}$ contains points which are interior in S relative to V.*

Now let M be a topological structure in a language L. Let V be a nonempty subset of the space M^n, $n \geqslant 1$, with the topology induced in it by M^n. A point $P \in V$ is *stable* for a predicate $Q(x_1, \ldots, x_n)$ *in* V if there exists an open neighborhood U of P in V such that either $M \models Q(\xi_1, \ldots, \xi_n)$ for all $P' = (\xi_1, \ldots, \xi_n) \in U$ or $M \models \neg Q(\xi_1, \ldots, \xi_n)$ for all such P'. A point is *stable in* V if it is stable in V for all predicates $Q(x_1, \ldots, x_n)$ in L. Then

4.3. *The set of points which are unstable in V for a given quantifier free predicate $Q(x_1, \ldots, x_n)$ is nowhere dense in V. If M has elimination of quantifiers then this applies to all predicates.*

4.4. *With one of the assumptions of 2.3 on M and L, let V be a closed nonempty subset of M^n, $n \geqslant 1$. Then the set of points of V which are stable in V is dense in V.*

For example, let V be an algebraic variety in C^n which is defined and irreducible over the field of rational numbers. Then the points which are stable in V are just the points which are generic over the rationals. Also, the generic points of a real algebraic variety $V \subset R^n$ which is defined and irreducible over the rationals is stable in V. However, the converse is no longer true in full generality. Thus, let V be the real elliptic curve which is given by

$$y^2 = (x-1)(x+1)^2.$$

Here the point $P = (-1, 0)$ is isolated and so the set $\{P\}$ is open in V. It follows that P is stable in V. However, P is not a generic point of this curve.

5. Let X be a sentence which holds in a topological structure M (for a given language L). Suppose that X is given in prenex normal form. Then every existential quantifier in X gives rise to a Skolem function (symbol) whose arguments are the universally quantified variables to the left of it. It is natural to ask whether these Skolem functions can be realized by *continuous* functions.

Suppose to begin with that X is an $\forall\exists$-sentence,

$$X = (\forall x_1) \ldots (\forall x_n)(\exists y_1) \ldots (\exists y_m) Q(x_1, \ldots, x_n, y_1, \ldots, y_m), \quad n \geqslant 1,\ m \geqslant 1 \tag{5.1}$$

where Q is free of quantifiers. Suppose, moreover, that M is a model of a set of universal axioms in L, K, say, and that X is not only true in M but actually deducible from K. The Skolem open form of 5.1 is

$$X' = Q\big(x_1, \ldots, x_n, \varphi_1(x_1, \ldots, x_n), \ldots, \varphi_m(x_1, \ldots, x_n)\big).$$

We have

5.2. *Let S be a nonempty open subset of M^n. Then there exists a nonempty open subset W of S such that the Skolem functions $\varphi_1, \ldots, \varphi_m$ can be realized by functions*

$$f_j(x_1, \ldots, x_n)\colon M^n \to M, \quad j = 1, \ldots, m$$

that are continuous on W.

Proof. According to a result which is due to P. Bernays (compare ref. 3, p. 230), the deducibility of X from K implies that there exist terms

$$t_{j1}(x_1, \ldots, x_n), t_{j2}(x_1, \ldots, x_n), \ldots, t_{jm}(x_1, \ldots, x_n), \quad j = 1, \ldots, l, \; l \geqslant 1$$

such that the sentence

$$(\forall x_1) \ldots (\forall x_n) \bigvee_{j=1}^{l} Q\big(x_1, \ldots, x_n, t_{j1}(x_1, \ldots, x_n), \ldots, t_{jm}(x_1, \ldots, x_n)\big)$$

is deducible from K. Now $Q\big(x_1, \ldots, x_n, t_{11}(x_1, \ldots, x_n), \ldots, t_{1m}(x_1, \ldots, x_n)\big)$ is a predicate which is free of quantifiers. Hence, by 2.1, there exists a nonempty open subset W' of S such that either

$$Q\big(\xi_1, \ldots, \xi_n, t_{11}(\xi_1, \ldots, \xi_n), \ldots, t_{1m}(\xi_1, \ldots, \xi_n)\big) = Q_1$$

holds for all $(\xi_1, \ldots, \xi_n) \in W'$ or else $\neg Q_1$ holds for all points in W'. In the former case we put $W = W'$, in the latter case we notice that

$$(\forall x_1) \ldots (\forall x_n) \bigvee_{j=2}^{l} Q\big(x_1, \ldots, x_n, t_{j1}(x_1, \ldots, x_n), \ldots, t_{jm}(x_1, \ldots, x_n)\big)$$

holds for all $(\xi_1, \ldots, \xi_n) \in W'$. We now repeat the procedure for $Q(x_1, \ldots, x_n, t_{21}, \ldots, t_{2m})$, and so on. In any case, we arrive at a nonempty open subset W of S such that for some j,

$$Q\big(\xi_1, \ldots, \xi_n, t_{j1}(\xi_1, \ldots, \xi_n), \ldots, t_{jm}(\xi_1, \ldots, \xi_n)\big)$$

holds in M for all $(\xi_1, \ldots, \xi_n) \in W$. But the terms $t_{j1}(x_1, \ldots, x_n), \ldots$ $\ldots, t_{jm}(x_1, \ldots, x_n)$ are obtained by the composition of the atomic function symbols of L and since these represent continuous functions in M, the same is true of the t_{jk}. Accordingly, the functions φ_k, which represent the t_{jk}, satisfy the conditions of 5.2.

6. To illustrate section 5, we are going to produce an effective result in the theory of positive definite polynomials. Let $p(x_1, \ldots, x_n, y_1, \ldots, y_m)$, $n \geqslant 1$, $m \geqslant 1$, be a polynomial with integer coefficients, where we regard the $y_1, \ldots, y_m$ as parameters. Suppose that p is a nonzero positive definite polynomial of $x_1, \ldots, x_n$, $p \geqslant 0$, for all values of $y_1, \ldots, y_m$ in a ball S: $|y_1 - b_1|^2 + \ldots + |y_m - b_m|^2 < r^2$, $b_1, \ldots, b_m$ and r real, $r > 0$. Then we are going to show

6.1. *There exist a ball $S' \subset S$ and an identity*

$$p(x_1, \ldots, x_n, y_1, \ldots, y_m) = \sum_{k=1}^{j} p_k(y_1, \ldots, y_m)\big(f_k(x_1, \ldots, x_n, y_1, \ldots, y_m)\big)^2$$

where the p_k are polynomials and the f_k rational functions, all with rational coefficients, such that $p_k(y_1, \ldots, y_m) > 0$, for all $(y_1, \ldots, y_m)$ in S', $k = 1, \ldots, j$.

To prove 6.1, let K be a set of universal axioms for an ordered integral domain formulated in terms of $=, <, +, -, \cdot$ and the constants 0 and 1. Then the model completion of K, K^*, is the theory of real closed ordered fields. We then have the following variant of Artin's solution of Hilbert's seventeenth problem.

6.2. *Let M_0 be a model of K and let M be an extension of M_0 which is a model of K^*. Suppose that the polynomial $q(x_1, \ldots, x_n)$ with coefficients in M_0 is positive definite, i.e., that it satisfies*

$$q(\xi_1, \ldots, \xi_n) \geqslant 0 \quad \textit{for all } \xi_1, \ldots, \xi_n \in M.$$

Then there exist positive elements $c_1, \ldots, c_m$ of M_0 and rational functions $g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)$ with coefficients in M_0 such that

$$q(x_1, \ldots, x_n) = \sum_{i=1}^{l} c_i\big(g_i(x_1, \ldots, x_n)\big)^2. \tag{6.3}$$

For the proof, consult [3], pp. 214–224. The present 6.2 differs from 8.5.20 in that reference only inasmuch as M_0 there is supposed to be an ordered *field* and M its real closure. Both differences are inessential since K^* is the model completion of K as well as of the theory of ordered fields. Also, we still have the corollary (compare 8.5.22 in [3])

6.4. *For a given n and for a given bound on the degree of $g(x_1, \ldots, x_n)$ in 6.2, there are bounds on the number l and on the degrees of the numerators and denominators of the functions $g_i(x_1, \ldots, x_n)$. These bounds are independent not only of the coefficients of $q(x_1, \ldots, x_n)$ but even of the particular choice of M_0 and M.*

The method of elimination of quantifiers for real closed fields shows that the condition of positive definiteness for $q(x_1, \ldots, x_n)$ is equivalent to the satisfaction of a quantifier free predicate P of the coefficients of q in M. Suppose in particular that the coefficients of q are themselves polynomials of the variables $y_1, \ldots, y_m$. Then we may formulate P as a predicate of $y_1, \ldots, y_m$, $P = P(y_1, \ldots, y_m)$. On the other hand, the existence of an identity 6.3 subject to bounds such as exist according to 6.4 can be formulated as an existential statement, in the language of K, in which the existential quantifiers refer to the coefficients c_i and to the coefficients of the g_i in some arbitrary but definite order. Thus, 6.2 can be expressed as a sentence

$$(\forall y_1) \ldots (\forall y_m)[P(y_1, \ldots, y_m) \supset (\exists z_1) \ldots (\exists z_\mu) Q(y_1, \ldots, y_m, z_1, \ldots, z_\mu)]$$

(where some of the z_i have replaced the c_i) or, in prenex normal form

$$(\forall y_1) \ldots (\forall y_m)(\exists z_1) \ldots (\exists z_\mu)[P(y_1, \ldots, y_m) \supset Q(y_1, \ldots, y_m, z_1, \ldots, z_\mu)] \tag{6.5}$$

where Q, like P, is free of quantifiers. Then 6.5 is deducible from K. An application of Bernays' theorem, as in the proof of 5.2, now shows that there exist terms p_k in the language of K, in other words polynomials, with the free variables $y_1, \ldots, y_m$, such that the sentence

$$(6.6)\qquad (\forall y_1)\ldots(\forall y_m)\bigvee_{k=1}^{\nu}[P(y_1, \ldots, y_m) \supset Q(y_1, \ldots, y_m, p_{k1}(y_1, \ldots, y_m), \ldots, p_{k\mu}(y_1, \ldots, y_m))]$$

is deducible from K.

Suppose in particular that the assumptions of 6.1 are satisfied. Since the field of real numbers, R, is a model of both K and K^*, we then have, for any $(\eta_1, \ldots, \eta_m)$ which belongs to the ball S, that $P(\eta_1, \ldots, \eta_m)$ holds in R and so therefore does the sentence

$$\bigvee_{k=1}^{\nu} Q(\eta_1, \ldots, \eta_m, p_{k1}(\eta_1, \ldots, \eta_m), \ldots, p_{k\mu}(\eta_1, \ldots, \eta_m)).$$

The successive reduction of this disjunction as in the proof of 5.2 now shows that there exists an open subset of S, and hence a ball S' in S such that, for a particular k, $R \models Q(\eta_1, \ldots, \eta_m, p_{k1}(\eta_1, \ldots, \eta_m), \ldots, p_{k\mu}(\eta_1, \ldots, \eta_m))$ for all $(\eta_1, \ldots, \eta_m) \in S'$. From this we obtain 6.1 by renaming the polynomials $p_{k\mu}$ which correspond to the p_k and by absorbing the remainder in the f_k, while taking into account that a pointwise identity which holds between polynomials of $y_1, \ldots, y_m$ in an open set of $(y_1, \ldots, y_m)$-space must be a formal identity.

We have employed the method rather than the statement of 5.2 in order to obtain a more precise result.

An example to which the theorem applies is provided by the polynomial

$$p(x_1, x_2, y_1) = x_1^2 + y_1 x_1 x_2 + x_2^2.$$

This is a positive definite function of x_1 and x_2 for S: $|y_1| < 1$. A possible choice for S' is S': $|y_1 - 0.5| < 0.5$, and a corresponding representation of p is

$$p(x_1, x_2, y_1) = (1-y_1)x_1^2 + y_1(x_1+x_2)^2 + (1-y_1)x_2^2.$$

But we may also write

$$p(x_1, x_2, y_1) = (1+y_1)x_1^2 + (-y_1)(x_1+x_2)^2 + (1+y_2)x_2^2$$

which is appropriate for S': $|y_1 + 0.5| < 0.5$. A polynomial to which the theorem does not apply is

$$p(x_1, x_2, y_1) = x_1^2 + x_2^2 + y_1 x_1^3 + y_1 x_2^3,$$

which is positive definite for $y_1 = 0$ but not for any other y_1.

7. In section 5 we were concerned with the continuity of Skolem functions of $\forall\exists$ sentences. We now wish to show that, in certain circumstances, we can extend our result to arbitrary sentences in prenex normal form. Thus, suppose that the topological structure M has elimination of quantifiers in a language L and that the theory of M in L can be axiomatized by a set of universal sentences. This is the case, for example, if M is the additive group of real numbers, with the usual topology, and the vocabulary of L consists of $=$, $+$, 0 and a unary function symbol $f_n(x)$ for each integer $n \geq 2$, to denote multiplication by $1/n$. We obtain another example if we include the order relation, $<$, in L.

Now let X be a sentence in prenex normal form. We exemplify X by

$$(7.1)\qquad X = (\forall x)(\exists y)(\forall z)(\exists u)(\forall v)(\exists w)Q(x, y, z, u, v, w)$$

where Q is free of quantifiers. The corresponding Skolem open form is

$$(7.2)\qquad X' = Q(x, \varphi(x), z, \psi(x, z), v, \chi(x, z, w)).$$

Thus, we have to deal with Skolem functions of one, two or three variables. It will be clear what we mean by the *projections* of a set S in M^3 as (x, z, w)-space into M^2 as (x, z)-space and into $M = M^1$ as (x)-space. Conversely, by the *cylindrification* in (x, z)-space of a set S in (x)-space we mean the set of points of (x, z)-space whose first coordinate is in S, with similar definitions in other cases.

We are going to prove

7.3. *Suppose that X holds in M, where X is given by 7.1, and let S be a nonempty open subset of (x, z, w)-space where x, z and w range over M. Then there exists a nonempty open subset W of S so that the Skolem functions $\varphi(x)$, $\psi(x, z)$ and $\chi(x, z, w)$ can be interpreted by functions that are continuous on the projection of W into (x)-space, on the projection of W into (x, z)-space, and on W, respectively.*

Proof. By assumption, the theory of M is given by a set of universal axioms, K. Since M has elimination of quantifiers, there exists a quantifier-free predicate $Q_1(x, y)$ in L such that

$$K \models (\forall x)(\forall y)[Q_1(x, y) \equiv [(\forall z)(\exists u)(\forall v)(\exists w)Q(x, y, z, u, v, w)]]$$

and $K \vdash (\forall x)(\exists y)Q_1(x, y)$.

Also, by the result of Bernays quoted in section 5 there exist terms $t_k(x)$ such that $K \vdash (\forall x)\bigvee_{k=1}^{j} Q_1(x, t_k(x))$ and so

$$K \vdash (\forall x)\bigvee_{k=1}^{j}(\forall z)(\exists u)(\forall v)(\exists w)Q(x, t_k(x), z, u, v, w)$$

and, for each k,

$$K \vdash (\forall x)[Q_1(x, t_k(x)) \supset (\forall z)(\exists u)(\forall v)(\exists w)Q(x, t_k(x), z, u, v, w)].$$

Rewriting the last sentence in prenex normal form, we obtain

$$K \vdash (\forall x)(\forall z)(\exists u)(\forall v)(\exists w)[Q_1(x, t_k(x)) \supset Q(x, t_k(x), z, u, v, w)].$$

Again, since M has elimination of quantifiers there exists quantifier-free $Q_2^k(x, z, u)$ in L such that

$$\text{(7.4)} \quad K \vdash (\forall x)(\forall z)(\forall u)\big[Q_2^k(x, z, u) \equiv (\forall v)(\exists w)[Q_1(x, t_k(x)) \supset Q(x, t_k(x), z, u, v, w)]\big]$$

and so $K \vdash (\forall x)(\forall z)(\exists u)Q_2^k(x, z, u)$. It follows that there exist terms $r_{k,l}(x, z)$ such that

$$K \vdash (\forall x)(\forall z) \bigvee_{l=1}^{m} Q_2^k(x, z, r_{k,l}(x, z))$$

and so

$$K \vdash (\forall x)(\forall z) \bigvee_{l=1}^{m} (\forall v)(\exists w)[Q_1(x, t_k(x)) \supset Q(x, t_k(x), z, r_{k,l}(x, z), v, w)]$$

and, for each l,

$$\text{(7.5)} \quad K \vdash (\forall x)(\forall z)(\forall v)(\exists w)[Q_2^k(x, z, r_{k,l}(x, z)) \wedge Q_1(x, t_k(x)) \supset Q(x, t_k(x), z, r_{k,l}(x, z), v, w)].$$

Now let S_1 be the projection of S into (x)-space. Then S_1 is open. As in section 5, with Q_1 for Q there exists a nonempty open $W_1 \subset S_1$ such that one of the t_k can be interpreted as a function which is continuous on W_1. For the corresponding k, choose Q_2^k as in 7.4. Let S_2 be the intersection of S with the cylindrification of W_1 in (x, z, w)-space and let S_3 be the projection of S_2 into (x, z)-space. Again we may find a nonempty open $W_2 \subset S_3$ such that one of the $r_{k,l}(x, z)$ can be interpreted as a function which is continuous on W_2. Let S_4 be the intersection of S with the cylindrification of W_2 in (x, z, w)-space. We now consider 7.5 for the chosen k and l. Since this is an $\forall\exists$ sentence as it stands, we know from 5.2 that there exists an open subset W of S_4 such that the Skolem function which corresponds to $(\exists w)$ can be interpreted by a continuous function on W. The projections of W into (x)-space and into (x, z)-space are contained in W_1 and W_2 respectively where the remaining Skolem functions have already been interpreted as continuous functions. This completes the proof of 7.3.

Although we have stated our theorem for the particular sentence X, it is obvious that a corresponding result is true for an arbitrary sentence in prenex normal form.

8. As developed so far, our theory does not include the case of a topological field in which inversion (reciprocation) is defined as a basic operation, $r(x)$ say. We adhere to the convention of model theory according to which a function must be a total function, and we define $r(x) = 1/x$ for $x \neq 0$ and $r(0) = 0$. This makes $r(x)$ discontinuous at $x = 0$ in the usual topology of the real numbers or of the complex numbers, but the same would apply for any other definition. To take this case into account we introduce the following concept which is tailormade for it.

A *quasi-topological structure* M is a topological structure augmented by an additional function $r(x)$ called the *singular function* such that for a specific element of M, m_0 (called the *singular element*) $r(m_0) = m_0$, and such that $r(x)$ is continuous in the topology of M accept, possibly, at the singular element. The language of M, L, also is augmented to include a function symbol, $\varrho(x)$, which denotes $r(x)$ and the constant "m_0" (to denote m_0). We shall suppose that M is a T_1-space so that every set that consists of a single point in M or M^n is closed. Let $F(x_1, \ldots, x_n)$ be any function which is the interpretation of a term $t(x_1, \ldots, x_n)$ in L. Thus, F is obtained by composition from the basic functions of M, including iteration, identification of variables, e.g., $s(x) = g(x, x)$ where $g(x, y)$ is in the set, and cylindrification, $h(x_1, \ldots, x_n) = g(x_1)$, where $g(x)$ is in the set. We also include the constant functions $y = a$ corresponding to individual constants that occur in L and the identity function $y = x$.

We shall prove

8.1. *Let S be a nonempty open subset of M^n and let $F(x_1, \ldots, x_n)$ be any function as above. Then there exists a nonempty open subset of S, W, such that $F(x_1, \ldots, x_n)$ is continuous on W.*

The proof proceeds according to the complexity of the terms $t(x_1, \ldots, x_n)$ which represent $F(x_1, \ldots, x_n)$ in the formal language. (It is quite possible that two terms represent the same function.) Variables and constants are said to be of complexity 0. If $g(y_1, \ldots, y_m)$ is a basic function symbol and the terms $t_j(x_1, \ldots, x_m)$, $j = 1, \ldots, m$, are of complexity not exceeding k but one of them is of complexity k then $g(t_1(x_1, \ldots, x_m), \ldots, t_m(x_1, \ldots, x_m))$ is of complexity $k+1$. There is some ambiguity in the interpretation of a term by a function since we have admitted cylindrification and we shall take this into account in the proof.

Clearly, 8.1 is satisfied for functions of complexity 0, i.e., for the identity function and for constant functions (if any). Also, if $g(x_1, \ldots, x_n)$ is a basic function symbol, $n \geq 2$, and another term $t = t(x_1, \ldots, x_l)$ is obtained from it by the identification of some of the variables $x_1, \ldots, x_n$, then the corresponding function in M^l is, like the function represented by g, still continuous (since $\varrho(x)$ cannot occur in this context). Since we may always assume that the identification of variables has been carried out on the basic function symbols, we may disregard this possibility from now on.

Suppose that we have proved our assertion for all terms of complexity $\leqslant k$ and let $s(x_1, \ldots, x_n) = g(t_1(x_1, \ldots, x_n), \ldots, t_m(x_1, \ldots, x_n))$ be of complexity $k+1$ where g is a basic function symbol and the $t_j(x_1, \ldots, x_n)$ are terms of complexity not exceeding k. If g is not ϱ (which denotes the singular function) then we select successively nonempty open subsets of S, $W_1 \supset W_2 \supset W_3 \supset \ldots \supset W_m = W$ such that $t_1(x_1, \ldots, x_n)$ is continuous on W_1, $t_2(x_1, \ldots, x_n)$ is continuous on W_2, ..., $t_m(x_1, \ldots, x_m)$ is continuous on W. This is possible by our inductive assumption. Since g is continuous on M^m, it follows that $s(x_1, \ldots, x_m)$ is continuous on W.

Suppose, on the other hand, that g coincides with ϱ, so $s(x_1, \ldots, x_n) = \varrho(t_1(x_1, \ldots, x_n))$ where $t_1(x_1, \ldots, x_n)$ is continuous on a nonempty open subset W_1 of S. We now have two possibilities. Either $t_1(x_1, \ldots, x_n)$ takes the constant value m_0 on W_1 — in which case $s(x_1, \ldots, x_n)$ takes the same constant value on W_1 and is therefore continuous on W_1; or, for some $(a_1, \ldots, a_n) \in W_1$, $t_1(a_1, \ldots, a_n) = b \neq m_0$. In that case, let B be an open neighborhood of b which does not include m_0. Then $W_2 = t_1^{-1}(B)$ and $W = W_1 \cap W_2$ are open and $t_1(x_1, \ldots, x_n)$ is continuous on W and does not take the value m_0 in that domain. It follows that $\varrho(t_1(x_1, \ldots, x_n))$ is continuous on W.

We still have to show, for each k, that if $t(x_1, \ldots, x_n)$ is a term of complexity k, and represents a function $F(x_1, \ldots, x_n)$ which satisfies the assertion of 8.1, then if we cylindrify $F(x_1, \ldots, x_n)$ by the addition of a new variable, so $G(x_1, \ldots, x_n, x_{n+1}) = F(x_1, \ldots, x_n)$, then 8.1 still holds with G for F. So let S be a nonempty subset of M^{n+1} taken as $(x_1, \ldots, x_{n+1})$-space and let S_1 be the projection of S on M^n as $(x_1, \ldots, x_n)$-space. Then S_1 is open, so $F(x_1, \ldots, x_n)$ is continuous on a nonempty open subset W_1 of S_1, so $G(x_1, \ldots, x_n, x_{n+1})$ is continuous on the intersection of the cylindrification of W_1 from M^n to M^{n+1}, with the original set S. This completes the proof of 8.1.

In consequence, *we can still prove* 2.1 *for a quasi-topological structure* M.

Indeed, let S be a nonempty open subset of M^n. By the successive choice of appropriate open sets we can find a nonempty open subset S' of S such that all the terms which occur in Q represent functions that are continuous on S'. We now interpret both T and S in 1.5 as our present S' and we let A be the set of points of S' that satisfy $Q(x_1, \ldots, x_n)$. Then 1.5 shows that either A or the complement of A in S' contains interior points. This proves our assertion.

It now follows immediately that 2.2, 2.3, 4.3, 4.4 *and* 5.1 *remain valid for quasi-topological structures.*

9. Let $M = C$ be the field of complex numbers. If we include in L the relation of equality and the operations of addition, multiplication, subtraction and reciprocation (i.e., $\varrho(x)$, as in the previous section), we thereby obtain a quasi-topological structure. However, this language is not sufficient in order to axiomatize the theory of C as an algebraically closed field of characteristic 0 in terms of universal axioms. For this purpose, we require, in addition, function symbols $\psi_n(x_1, \ldots, x_n)$ to denote solutions of the monic equations $p(y) \equiv y^n + x_1 y^{n-1} + \ldots + x_n = 0$ for $n = 2, 3, 4, \ldots$ So the question arises whether these function symbols can be interpreted by continuous functions $f_n(x_1, \ldots, x_n)$ in C. This is indeed the case as we see by choosing $f_n(x_1, \ldots, x_n)$ for given $x_1, \ldots, x_n$ in C as the root of $p(y) = 0$ whose real and imaginary parts are as small as possible (see [1], p. 432). Accordingly, 5.1 and 7.3 are applicable to the field of complex numbers, for the vocabulary just specified. We observe that this still leaves the field of real numbers, $M = R$, as a real closed ordered field, outside our framework. For in order to axiomatize the theory of R by a set of universal axioms, we now require (i) a function symbol $\sigma(x)$ for the (positive) square root and (ii) the above function symbols $\psi_n(x_1, \ldots, x_n)$ for *odd* $n \geqslant 3$. $\sigma(x)$ can, in fact, be interpreted by the continuous function which equals the positive square root of a number a for $a \geqslant 0$ and equals 0 for $a < 0$. But as far as (ii) is concerned, Henriksen and Isbell have in the above mentioned paper [1] given an example which shows that $\psi_3(x_1, x_2, x_3)$ cannot in any way be chosen as a continuous function on M^3.

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