Definability in structures of finite valency

by

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Abstract. The notion of valency of simple graphs is generalized to elements of structures of arbitrary signature. Then the valency of a structure is defined in the obvious way. Relations definable in structures of finite valency are investigated. Every relation first-order definable in a structure of finite valency is almost symmetric or, equivalently, almost identity generated; these notions are defined below. Non-definability of addition, multiplication and ordering of integers in any structure of finite valency is shown.

A structure or graph $\mathfrak{A} = \langle A, R\rangle$ is said to be of finite valency if it satisfies $\forall x \exists^{<\omega} y\; xRy$ (R a binary symmetric relation on A, for generalization to arbitrary structures cf. Sec. 1). It has turned out that this class and even the subclass $\boldsymbol{G}_3$ of graphs satisfying $\forall x \exists^{<4} y\; xRy$ is rather complicated. The acting process of every Turing machine is describable [1] in the elementary theory $\mathrm{Th}\,\boldsymbol{G}_3$ and this implies $\mathrm{Th}\,\boldsymbol{G}_3$ to be of the highest recursively enumerable degree, hence $\boldsymbol{G}_3$ is a reduction class for first order predicate logic.

On the other hand it will be shown here that in no graph of finite valency an infinite linear ordering or non-trivial binary operation in an infinite set is definable. Relations definable in structures of finite valency have a special property which in the binary case amounts to saying that it is not antisymmetric on an infinite set.

It follows that there is no system of relations of finite valency to define the arithmetical basic operations. Various other examples of non-definability and non-interpretability will be given. The main theorem is based on the existence of homogeneous structures realizing special types. Other results are essential based on Ramsey's theorem.

We refer also to some classes of structures so as trees and n-separated graphs in which valencies can be reduced with respect to first order formulas. Thus no infinite linear ordering is definable by means of a unary function.

(1) A detailled proof is given in [3].

1. Definitions and notations. We are concerned here with relational structures $\mathfrak{A} = \langle A; R_1, \ldots, R_l\rangle$ of finite signature; $\mathfrak{A}$ determinates its first order language $L\mathfrak{A}$ the vocabulary of which contains besides the relational symbols and equality sign also constants denoting the elements of A. $(\mathfrak{A}, P)$ is the structure obtained from $\mathfrak{A}$ by adding a unary predicate P $(\subseteq A)$.

$F^k\mathfrak{A}$ denotes the set of formulas of $L\mathfrak{A}$ containing at most the variables x_i $(i < k)$ as free variables. By an abuse of language $F^k\mathfrak{A}$ is sometimes taken modulo equivalence of formulas in $\mathfrak{A}$; thus $F^k\mathfrak{A}$ becomes a Boolean algebra. $TF^k\mathfrak{A}$ is the set of all k-*types*, i.e. the set of all ultrafilters in $F^k\mathfrak{A}$. $a \in A^k$ *realizes* Δ $(\in TF^k\mathfrak{A})$ if $\varphi(x) \in \Delta \Leftrightarrow \mathfrak{A} \models \varphi(a)$ for all $\varphi \in F^k\mathfrak{A}$. $\varphi_{\mathfrak{A}}$ $(\varphi \in F^k\mathfrak{A})$ denotes the relation defined by φ in $\mathfrak{A}$, i.e. $\varphi_{\mathfrak{A}} = \{a \in A^k: \mathfrak{A} \models \varphi(a)\}$.

The binary relation $S = S_{\mathfrak{A}}$ is defined by

$xSy :\Leftrightarrow x = y$ *or* there is some element in $R_1 \cup \ldots \cup R_l$ containing both x and y as members.

$$xS^0a :\Leftrightarrow x = y; \quad xS^{n+1}y :\Leftrightarrow (\exists z \in A)(xS^nz \wedge zSy);$$

$$xS^\omega y :\Leftrightarrow xS^ny \text{ for some } n \in \omega.$$

The equivalence classes mod S^ω are called the *components* of A.

$\mathfrak{A}$ is said to be a *graph* if the R_i are all unary or binary. If $A = \langle A, R\rangle$ where R is binary, reflexive, and symmetric, then $\mathfrak{A}$ is a *simple* graph. Let $\mathfrak{A} = \langle A, R\rangle$ be simple. The *valency* of a point $a \in A$ is the number of points $b \in A$, $b \neq a$, such that aRb. The valency of the graph itself is the smallest cardinal number ν such that the valencies of the points of A are $< \nu$. These notions may be generalized to arbitrary structures $\mathfrak{A}$. Thus, $\mathfrak{A}$ is said to be of *finite valency* if $\mathfrak{A}_S = \langle A, S_{\mathfrak{A}}\rangle$ is of finite valency. This amounts to saying that all n-spheres $S_n(a) = \{x \in A: aS^nx\}$ are finite for every $a \in A$.

Let A be a non-empty set. For $a, b \in A^k$ put

$$a \sim b \quad \text{if} \quad (\forall i, j < k)(a_i = a_j \Leftrightarrow b_i = b_j)$$

and $a \approx b$ if $a \sim b$ and $\{a_i: i < k\} = \{b_i: i < k\}$. Thus, if $a, b, c \in A$ $a \neq b \neq c \neq a$ then

$$\langle a, a, b\rangle \approx \langle b, b, a\rangle \quad \text{and} \quad \langle a, a, b\rangle \sim \langle b, b, c\rangle.$$

A relation $R \subseteq A^k$ is said to be *symmetric* on A if $a \approx b \Rightarrow Ra \Leftrightarrow Rb$ for all $a, b \in A^k$. This is a generalization of the usual notion defined for binary relations.

A binary relation R is *fully anti-symmetric* on A iff $aRb \dot{\vee} bRa$ for all $a, b \in A$, $a \neq b$ ($\dot{\vee}$ denotes antivalence, *either — or*, but not both). R $(\subseteq A^k)$ is *almost symmetric* on A (R is a.s.) if for any infinite $B \subseteq A$ there is some infinite $C \subseteq B$ such that R is symmetric on C (speaking more precisely $R \upharpoonright C$ is symmetric on C).

R $(\subseteq A^k)$ is said to be *identity generated* (R is i.g.) if R is a boolean combination of the k-ary relations of the type $x_i = x_j$ $(i, j < k)$. It can easily be checked that R is i.g. iff R satisfies $a \sim b \Rightarrow Ra \Leftrightarrow Rb$ for all $a, b \in A^k$.

In the same way as above the notion of *almost identity generated* (a.i.g.) can be defined. Obviously every a.i.g. relation is a.s. Theorem 3.1 shows that the converse is also true.

Throughout this paper $A_{[k]}$ denotes the set of all k-element subsets of the set A. $U \cup V = A$ is sometimes called a *partition* of A.

2. Examples.

1. If for some binary relation R on A there is some infinite $B \subseteq A$ such that $\langle B, R\rangle$ (to be more precisely $\langle B, R \upharpoonright B\rangle$) is a linear ordered set then R is not a.s. on A. Theorem 3.3 shows that the converse also holds: for any n.a.s. (not almost symmetric) relation R on A there is an infinite $B \subseteq A$ such that $\langle B, R\rangle$ is linear ordered.

2. If $R \subseteq A^k$ and for every $a_0, \ldots, a_{k-2} \in A$, $i < k$ there are at most finitely many $x \in A$ such that $Ra_0 \ldots a_i x a_{i+1} \ldots a_{k-2}$ then R is a.i.g. (or a.s. what amounts to the same by Theorem 3.1). For it is easy to show that every infinite $B \subseteq A$ contains an infinite $C \subseteq B$ such that R equals $\emptyset$ or Id_k on C. Therefore the basic relations in every structure $\mathfrak{A}$ of finite valency are a.s. Theorem 4.1 will show that every relation definable in $L\mathfrak{A}$ is a.s.

3. Let $\mathfrak{J}_n$ denote the linear ordered n-element set and $\mathfrak{A} = \langle A, R\rangle$ the disjoint union of the $\mathfrak{J}_n$. If $\mathfrak{A}^*$ is an elementary extension of $\mathfrak{A}$ then $\mathfrak{A}^*$ contains an infinite subset such that R is a linear ordering on it. Hence to be a.s. is not expressible by first order formulas.

4. If $\mathfrak{G} = \langle A, R\rangle$ is the graph of a unary function F: $A \to A$ then R is a.s. on A. Later it will be shown that every definable relation in $\mathfrak{G}$ is a.s. From example 2 follows that the graph of a binary function $\circ$ on A is a.s. provided $a \circ x = b$ and $b \circ x = a$ have finite many solutions only. By projection we can sometimes derive a binary n.a.s. relation (addition of natural numbers).

5. The class of k-ary a.s. relations of a set A is closed under complementation, union and meet. From Presburger's elimination procedure for the theory of the group $\mathfrak{J}$ of integers it follows that the set of definable binary relations is the Boolean algebra generated by binary relations Rxy of the following types: $x = n$, $y = n$, $kx + my = n$, $x \equiv k \bmod n$, $y \equiv k \bmod n$. All these relations are obviously a.s. hence every definable binary relation in $\mathfrak{J}$ is a.s. In view of Theorem 3.3 every definable relation is a.s.

3. Properties of a. s. and a. i. g. relations.

3.1. THEOREM. *The class of almost symmetric relations and the class of almost identity generated relations in a set A coincide.*

Proof. Obviously every a.i.g. relation R on A is a.s. Let now R be a.s. and first assume R binary. Let $B \subseteq A$ be a given infinite subset of A. For some infinite $B' \subseteq B$ R is either reflexive or irreflexive on B'. Now let $C \subset B'$ such that

$$(\forall x, y \in C)(x \neq y \to xRy \Leftrightarrow yRx) .$$

Define a partition $U \cup V = C_{[2]}$ of the 2-element subsets of C such that $\{a, b\} \in U$ iff aRb holds. By Ramsey's theorem either $D_{[2]} \subseteq U$ or $D_{[2]} \subseteq V$ for some infinite $D \subseteq C$. In both cases R is i.g. on D.

We now take $k = 3$ as typical for the general case. Let $C \subseteq B \subseteq A$ be given as before, R either reflexive or irreflexive on C (i.e. $(\forall x \in C)(Rxxx)$ or $(\forall x \in C)(\neg Rxxx)$ holds) and R a.s. on C. Let $U \cup V = C_{[3]}$ be a partition such that $\{a, b, c\} \in U$ iff $Rabc$ holds. From Ramsey's theorem it follows that for some infinite $D \subseteq C$ $Rabc \Leftrightarrow Ra'b'c'$ for $\{a, b, c\}$, $\{a', b', c'\} \in D_{[3]}$. Next construct a partition $U \cup V = D_{[2]}$ such that $\{a, b\} \in U$ if and only if $Raab$ holds (this is a correct definition since $Raab$ is equivalent with $Rbba$). There is an infinite $E \subseteq D$ such that $Raab \Leftrightarrow Ra'a'b'$ for all $\{a, b\}, \{a', b'\} \in E_{[2]}$. Continuing in this way with the two remaining cases (second and third argument, and first and third argument are identical, respectively) one gets an infinite set $P \subseteq B$ such that R is i.g. on P. Q.E.D.

3.2. LEMMA. *A binary fully antisymmetric relation on an infinite set A is a linear order on some infinite subset of A.*

Proof. Since R is either reflexive or irreflexive on some infinite subset of A it may be supposed from the beginning that R has this property on A. Let $U \cup V = A_{[3]}$ be a partition such that $\{a_1, a_2, a_3\} \in U$ if and only if $a_{i_1} R a_{i_2} R a_{i_3} R a_{i_1}$ for some permutation (i_1, i_2, i_3) of $(1, 2, 3)$. By Ramsey's theorem there is an infinite $B \subseteq A$ such that either $B_{[3]} \subseteq U$ or $B_{[3]} \subseteq V$. The first case is excluded since there is no 4-element subset $C \subseteq B$ such that $C_{[3]} \subseteq U$. Hence $B \subseteq V$ and it is easily seen that R is a linear ordering on B. Q.E.D.

3.3. LEMMA. *A relation $R \subseteq A^k$ is almost symmetric on A iff every infinite subset $P \subseteq A$ contains a k-element subset Q on which R is symmetric.*

Proof. The condition is obviously necessary. Assume now it holds and let P be an infinite subset of A, $U \cup V = P_{[k]}$ a partition of the k-element subsets of P such that $X \in U$ iff R is symmetric on X ($X \in P_{[k]}$). By Ramsey's theorem there is some infinite $Q \subseteq P$ such that either $Q_{[k]} \subseteq U$ or $Q_{[k]} \subseteq V$. The second case is excluded since Q should contain an k-element subset on which R is symmetric. Thus $Q_{[k]} \subseteq U$ and therefore R is symmetric on Q. Q.E.D.

Let R be a ternary relation on A. $R_{12} = \{\langle a, b\rangle \in A^2\colon Raab\}$ and $R_c = \{\langle a, b\rangle \in A^2\colon Rabc\}$ are examples of binary relations derived from R by *identification of arguments* and *parametrization* respectively.

3.4. THEOREM. *If R ($\subseteq A^k$, $k \geq 2$) is a not almost symmetric relation on A then there is some binary relation S obtained from R by identification of arguments or parametrization such that S is a linear ordering on some infinite subset of A.*

The case $k = 2$ follows easily from Lemma 3.2 and Lemma 3.3. For the rest, the case $k = 3$ is typical so we restrict ourselves to this case. Let $B \subseteq A$ be an infinite subset such that R is not symmetrical on every infinite subset of B. It may be supposed that R is either reflexive or irreflexive on B. Let $U \cup V = B_{[2]}$ be a partition such that for all $\{a, b\} \in U_{[2]}$ $Raab \Leftrightarrow Rbba$ (the two other cases of argument identification are treated similarly). For some infinite $C \subseteq B$ either $C_{[2]} \subseteq U$ or $C_{[2]} \subseteq V$. In the second case $xSy :\!= Rxxy$ would be fully antisymmetric on P and hence by Lemma 3.2 there is some infinite subset $P \subseteq C$ such that S is a linear ordering on P. Thus we may suppose now that there is an infinite $C \subseteq B$ such that for all $\{a, b\} \in C_{[2]}$ the equivalences $Raab \Leftrightarrow Rbba$, $Raba \Leftrightarrow Rbab$ and $Rabb \Leftrightarrow Rbaa$ hold. Now construct a partition of $C_{[3]}$ according as

$$(*) \qquad Rabc \Leftrightarrow Racb \Leftrightarrow Rcba \quad \text{for all } \{a, b, c\} \in C_{[3]},$$

holds or not. There is no infinite $C \subseteq D$ such that $(*)$ holds for all $t \in D_{[3]}$, otherwise R would be symmetric on D. Hence there is some infinite $D \subseteq C$ such that for $\{a, b, c\} \in D_{[3]}$ either

$$Rabc \nabla Racb \quad \text{or} \quad Rabc \nabla Rbac \quad \text{or} \quad Rabc \nabla Racb .$$

Applying Ramsey's theorem again we may assume e.g. $Rabc \nabla Racb$, for all $\{a, b, c\} \in D_{[3]}$. Choosing any $a_0 \in D$ one has $Ra_0xy \nabla Ra_0yx$ for all $x, y \in D \setminus \{a_0\}$, $x \neq y$. Therefore $xSy :\!= Ra_0xy$ is fully antisymmetric on some infinite subset of A which proves the theorem by Lemma 3.2. Q.E.D.

This theorem can be of service if one wants to prove every definable relation in a structure to be a.s. One has to check this only for binary relations.

4. The main theorem. As usual, a structure $\mathfrak{A}$ will be called *homogeneous* if every partial elementary automorphism of $\mathfrak{A}$ of cardinality less than card $\mathfrak{A}$ can be extended to an automorphism of $\mathfrak{A}$, A homogeneous structure $\mathfrak{A}$ obviously has the following nice property: if $a, b \in \mathfrak{A}$ realize the same 1-type in $F^1 \mathfrak{A}$ then there exist an automorphism $\alpha\colon \mathfrak{A} \to \mathfrak{A}$ such that $\alpha(a) = b$.

The following lemma easily follows from Theorem 20.5 in [4]:

4.0. LEMMA. *A structure $\mathfrak{A}$ possesses a homogeneous elementary extension which realizes a given k-type $\Delta \in TF^k\mathfrak{A}$.*

The existence of homogeneous extensions can be a useful tool in studying elementary properties of structures as shows the following

4.1. THEOREM. *Let $\mathfrak{A} = \langle A; R_1, \ldots, R_l\rangle$ be a structure of finite valency and $\varphi(x_0, \ldots, x_{k-1}) \in F^k\mathfrak{A}$ be a formula of the language of $\mathfrak{A}$. Then the relation $\varphi_{\mathfrak{A}}$ defined by φ on $\mathfrak{A}$ is almost symmetric on A.*

Proof. By Theorem 3.4 it is sufficient to prove this theorem for $k = 2$ only (though, the proof for $k > 2$ is similar). So let $\varphi(x_0, x_1)$ be a given formula. In view of Lemma 3.3 it will be sufficient to prove that for any infinite $P \subseteq A$ the structure $(\mathfrak{A}, P)$ satisfies the condition

$$(*)\qquad \exists x \exists y [Px \;\&\; Py \;\&\; x \neq y \;\&\; \varphi(x, y) \Leftrightarrow \varphi(y, x)]\,.$$

Let Δ be some 1-type of the language of $(\mathfrak{A}, P)$ such that for any $\Psi(x_0) \in \Delta$ there is an infinite number of elements $a \in P$ satisfying $\Psi(x_0)$ (Δ can be taken as a completion of the consistent set $\{\gg Px \;\&\; x \neq a \ll\colon a \in P\}$.

Let Γ be the following set of formulas from $F^2(\mathfrak{A}, P)$

$$\Gamma = \{\Psi(x_0),\, \Psi_{x_0}(x_1),\, \neg x_0 S^n_{\mathfrak{A}} x_1\colon\ \Psi \in \Delta,\ n \in \omega\}\,.$$

We claim that Γ is consistent. To see this it will be sufficient to show that every formula

$$\chi(x_0, x_1) := \Psi(x_0) \;\&\; \Psi(x_1) \;\&\; \neg x_0 S^n_{\mathfrak{A}} x_1 \qquad (\Psi \in \Delta,\ n \in \omega)$$

is consistent. Take any $a \in A$ such that $\mathfrak{A} \models \Psi(a)$. The set $S = \{x \in A\colon a S^n_{\mathfrak{A}} x\}$ is finite since $\mathfrak{A}$ is of finite valency. On the other hand $Q = \{y \in A\colon \mathfrak{A} \models \Psi(y)\}$ is infinite by the choise of Δ. Therefore $Q \setminus S \neq \emptyset$. Take any $b \in Q\setminus S$ then it easily seen that the pair a, b satisfies $\chi(x_0, x_1)$.

Thus Γ is extendible to some 2-type Θ. Lemma 4.0 tells us that there is a homogeneous structure $\mathfrak{B} \succeq (\mathfrak{A}, P)$ and elements $c, d \in B$ realizing Θ. Let $U = \{x \in B\colon c S^\omega_{\mathfrak{B}} x\}$, $V = \{y \in B\colon d S^\omega_{\mathfrak{B}} y\}$. By definition of Θ we have $U \cap V = \emptyset$. On the other hand c and d realize the same 1-type Δ in the homogeneous structure $\mathfrak{B}$ hence there is an automorphism $\lambda\colon \mathfrak{B} \to \mathfrak{B}$ over $\mathfrak{A}$ such that $\lambda(c) = d$. It is obvious that $\lambda(U) = V$. Define now $\mu\colon \mathfrak{B} \to \mathfrak{B}$ in the following way:

$$\mu(x) = \begin{cases} \lambda(x) & \text{if} \quad x \in U\,, \\ \lambda^{-1}(x) & \text{if} \quad x \in V\,, \\ x & \text{elsewhere}\,. \end{cases}$$

It can easily be checked that μ is an automorphism of $\mathfrak{B}$ over $\mathfrak{A}$ and that μ interchanges c and d. Furthermore $c, d \in P_B$ and hence $\mathfrak{B}$ satisfies the formula $(*)$. B is an elementary extension of $(\mathfrak{A}, P)$ therefore $(*)$ also holds in $(\mathfrak{A}, P)$ which completes the proof. Q.E.D.

5. Applications. It follows at once from Theorem 4.1 that there is no system of relations of finite valency on the set N of natural numbers which defines addition and multiplication or even one of these operations. There is also no such system in any superset of N. The same holds for the additive group $\mathfrak{J}$ of integers although all definable relations in this structure are a.s. For if not, add a unary predicate symbol denoting the set of natural numbers. Then in the extended structure the ordering of the natural numbers would be definable but the extended structure is still of finite valency.

A simple graph $\langle A, R\rangle$ is called a *tree* if it contains no circles. $\langle A, R\rangle$ is said to be *n-separated* if any two circles have at most n points in common. The graph of a unary function is 0-separated (this graph is directed but it can easily be shown that it is interpretable in some simple 0-separated graph). In [1] it has been shown that any sentence satisfied in some point-coloured n-separated graph is satisfiable in some n-separated graph of bounded valency. In particular, this holds for trees.

5.1 THEOREM. *If $\mathfrak{A}$ is a coloured tree (or to be more general a n-separated coloured graph) then every definable relation in $\mathfrak{A}$ is almost symmetric.*

Proof. If some formula defines a n.a.s. relation in $\mathfrak{A}$ then by Theorem 3 some formula $\varphi(x, y)$ defines a linear order on some infinite set $P \subseteq A$. Now there is some sentence in the language of $(\mathfrak{A}, P)$ telling us that there is some infinite set linearly ordered by $\varphi(x, y)$. This sentence is satisfiable in some tree (or n-separated graph, respectively) of bounded valency in contradiction to Theorem 4.1. Q.E.D.

An immediate corollary is that no n.a.s. relation can be defined in any structure of a unary function since every unary function is interpretable in some tree. This extends a result of Taimanow [5].

We assume as known the usual notion of *interpretability* (or *definable embedding*) of a structure $\mathfrak{A}$ in a structure $\mathfrak{B}$: the universe and the basic notions of $\mathfrak{A}$ have to be defined by suitable formulas of $L\mathfrak{B}$. It can easily be checked that the group $\mathfrak{J}$ is interpretable in the additive semigroup $\mathfrak{N}$ of natural numbers. Thus it is possible to construct $\mathfrak{J}$ as an inner model in the frame *of the theory of* $\mathfrak{N}$. It is rather surprising that the converse does not hold. Example 5 shows that all definable relations in $\mathfrak{J}$ are a.s. and this is not the case for $\mathfrak{N}$.

There is an important generalization of the interpretability notion used hithertoo: the elements of $\mathfrak{A}$ can be represented as congruence classes of elements of $\mathfrak{B}$ modulo *a* suitable definable congruence relation with respect to the other defined notions. But also in this sense $\mathfrak{N}$ cannot be interpreted in $\mathfrak{J}$ as follows easily from the notions involved:

If $\mathfrak{A}$ is interpretable in $\mathfrak{B}$ (in the general sense) and all definable relations of $\mathfrak{B}$ are a.s. *then the same applies to $\mathfrak{A}$.*

Hence $\mathfrak{I}$ is infact properly weaker then $\mathfrak{N}$. From Theorem 4.1 follows

If $\mathfrak{B}$ is of finite valency and $\mathfrak{A}$ is interpretable in $\mathfrak{B}$ then all definable relations in $(\mathfrak{A}, P)$ are a.s., *P any additive unary predicate.*

Hence no infinite Abelian group can be interpreted in a any structure of finite valency. This follows from the fact proved in [2] that for any infinite Abelian group $\mathfrak{A}$ and every at most countable structure $\mathfrak{B}$ there is some unary predicate $P \subseteq A$ such that B is interpretable in $(\mathfrak{A}, P)$ (P can be choosen e.g. in such a way that some infinite linear ordered set is interpretable in $(\mathfrak{A}, P)$).

To give further applications we refer to the notion of model-interpretability which has been succesfully used by several authors to solve or to reduce decision problems.

If $\boldsymbol{K}, \boldsymbol{M}$ are (not necessarily elementary) classes of structures of signatures σ, τ, respectively then $\boldsymbol{K}$ is said to be *model-interpretable* in $\boldsymbol{M}$, $\boldsymbol{K} \underset{\text{m.i.}}{\rightarrow} \boldsymbol{M}$ if there is a finite set Φ of sentences in L_τ such that there is an uniformly definable embedding of every $\boldsymbol{K}$-structure in some $\boldsymbol{M}$-structure satisfying Φ and every $\boldsymbol{M}$-structure which satisfies Φ can be obtained in this way. Many important classes of structures have turned out to be *universal* with respect to model-interpretability in the following sense: every finite axiomatizable class of structures is model-interpretable in such a class. Universal in this sense are the class $\boldsymbol{G}$ of simple graphs, the class of structures of a distributive lattice order relation (a fortiori the class of order relations, the class of rings, the class of groups and rather special subclasses of these examples.

The class $\boldsymbol{G}_3$ of simple graphs of valency 3 for every point (and actually the class of structures of finite valency of any signature) is not universal by the results presented here although it can be shown to be a reduction class in the sense of the predicate calculus describing the acting process of any Turing-machine by those graphs (cf. [3]). Virtually no interesting class of structures (lattices, groups, rings etc.) is model-interpretable in $\boldsymbol{G}_3$ for if we add one or several unary predicates to an infinite algebraic structure it is easy to define n.a.s. relations in the extended structure.

Let $\boldsymbol{K}$ be a class of structures of signature σ, $\varphi \in L_\sigma$ a formula containing at least one free variable. $\boldsymbol{SK}[\varphi]$ denotes the class of all structures homomorphic to some $\langle A, \varphi_{\mathfrak{B}} \restriction A\rangle$ such that $\mathfrak{B} \in \boldsymbol{K}$ and $\emptyset \neq A \subseteq B$. $\mathfrak{I}_n$ is the linear ordered set with n elements.

5.2. Lemma. *Let $k \in \omega$ and $\boldsymbol{K}$ be a class of structures of valency at most k, φ a formula with exactly two free variables. Then there is some $n_0 \in \omega$ such that $\mathfrak{I}_n \notin \boldsymbol{SK}[\varphi]$ for any $n \geqslant n_0$.*

Proof. Assume the contrary: any given finite linear ordered set is homomorphic to some $\langle A, \varphi_{\mathfrak{B}} \restriction A\rangle$, $\mathfrak{B} \in \boldsymbol{K}$. Then the set $\mathrm{Th}\, \boldsymbol{K} \cup C$ where $C = \{\varphi(c_i, c_j) \,\&\, \neg\varphi(c_j, c_i)\colon i, j \in \omega, i < j\}$ and $c_0, c_1, \ldots$ are new individual symbols, is easily seen to be consistent and hence has a model $\mathfrak{A}$. The valency of $\mathfrak{A}$ is at most k since the formula expressing this fact belongs to $\mathrm{Th}\,\boldsymbol{K}$. This is a contradiction since by Theorem 4.1 the relation $\varphi_{\mathfrak{A}}$ is a.s. and hence it cannot be fully antisymmetric on the infinite subset $\{c_0, c_1, \ldots\}$ of A. Q.E.D.

5.3. Theorem. *The class $\boldsymbol{G}_\omega$ of simple graphs of finite valency is not model-interpretable in the class $\boldsymbol{G}_{\text{bound}}$ of simple graphs of bounded valency. $\boldsymbol{G}_{\text{bound}}$ is not model-interpretable in $\boldsymbol{G}_3$.*

Proof. Otherwise the structure $\mathfrak{I}_\infty$, the disjoint union of the $\mathfrak{I}_n$ (cf. Example 3) would be definably embeddable in some structure $\mathfrak{A}$ of valency m, $m \in \omega$. If φ is the formula defining the order relation then obviously $\mathfrak{I}_n\ \boldsymbol{S}\{\mathfrak{A}\}[\varphi]$ in contradiction to Lemma 5.2. To prove the second part it is sufficient to observe that the class of finite ordered sets is model-interpretable in $\boldsymbol{G}_{\text{bound}}$; however, by Lemma 5.2 this class is not model-interpretable in $\boldsymbol{G}_3$. Q.E.D.

This theorem can obviously be extended to arbitrary finite signature. (Of course, 3 can be replaced by any $k \in \omega$.) Finally, we mention a problem which is connected with the questions discussed here.

Problem. Is there an infinite Abelian group $\mathfrak{A}$ and a formula $\varphi(x, y)$ which defines a linear order on some (not necessarily definable) infinite subset of $\mathfrak{A}$?

The answer would be in the negative if it is shown that every definable binary relation in an Abelian group is a.s. (cf. example 5 in section 2) or the more stronger result that every infinite Abelian group is stable in some infinite cardinal.

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