rem 3.3 one has to consider the reducts of the models of the full expansion of T to the original similarity type.

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The boundedness principle in ordinal recursion

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Abstract. An application of Spector's boundedness principle to ordinal recursion yields, for α recursively regular and $\beta < \alpha$: (1) Any α-recursive functional total on $\beta \times P(\beta)$ can be defined without the search operator. Let $|\alpha/\beta|$ be the closure ordinal of the class of α-recursive inductive operators over β. For example, an operator over ω is ω_1-recursive iff it is Δ_1^1. A new proof of the fact that $|\Delta_1^1|$ is recursively singular follows from the more general result (2) $|\alpha/\beta| > \alpha$ iff $|\alpha/\beta|$ is singular. Characterizations of closure ordinals are obtained in terms of projectibility. For example, (3) $|\alpha/\beta| \geq \alpha$ iff α is absolutely projectible to β.

1. Introduction. The boundedness principle, due to Spector [6], is basically that any Σ_1^1 set of well-orderings is bounded below ω_1. We apply this principle to ordinal recursion to obtain several results regarding functionals and specifically inductive definitions on sets of ordinals.

We assume that the reader is familiar with the concepts of inductive definitions and of ordinal recursion as outlined in [1].

Briefly, an inductive operator Γ over a set X is a map from $P(X)$ to $P(X)$ such that for all A, $A \subseteq \Gamma(A)$. Γ determines a transfinite sequence $\{\Gamma^\xi\colon\ \xi \in \mathrm{ORD}(\text{ordinals})\}$, where $\Gamma^\xi = U\{\Gamma^\sigma\colon\ \sigma < \xi\}$ for $\xi = 0$ or ξ a limit ordinal and $\Gamma^{\xi+1} = \Gamma(\Gamma^\xi)$. The closure ordinal $|\Gamma|$ of Γ is the least ordinal ξ such that $\Gamma^{\xi+1} = \Gamma^\xi$. The closure $\overline{\Gamma}$ of Γ is $\Gamma^{|\Gamma|}$, the set inductively defined by Γ.

The definition of the α-recursive functionals and the primitive ordinal recursive (p.o.r.) functionals is a natural generalization of standard recursion over the natural numbers. We list in § 3 some basic facts about ordinal recursion essential to this paper.

In this paper we consider the notion or ordinal recursive inductive operators on sets of ordinals. Given recursively regular ordinals α and β, $\beta \leq \alpha$, let $|\alpha/\beta|$ be the closure ordinal of the class of inductive operators over β which are α-recursive in parameters less than β; let $|(\alpha/\beta)|$ be the closure ordinal of inductive operators over β which are α-recursive in parameters less than α. (The latter are called weakly α-recursive.) For example, $|\omega/\omega| = |(\omega/\omega)| = \omega$. In this paper we consider only countable α and β.

The non-triviality of this concept can be seen by the fact that every arithmetic operator on ω is ω_1-recursive, so that $|\omega_1/\omega|$ is much larger than ω_1. (See [2] and [4] for a discussion of recursively large ordinals.) In § 3 we prove the following result, which implies that $|\Delta_1^1| = |\omega_1/\omega|$.

THEOREM 3.9. *An inductive operator Γ over ω is ω_1-recursive iff it is Δ_1^1.*

We obtain characterizations for the ordinals $|\alpha/\beta|$ and $|(\alpha/\beta)|$. We also present conditions regarding the relative sizes of these two ordinals and α and regarding the regularity of the two closure ordinals.

2. Summary of results. We need three notions of projectibility for recursively regular ordinals α and $\beta \leqslant \alpha$.

DEFINITION 2.1. (a) α is *absolutely projectible to* β iff there is an α-recursive function mapping α 1-1 into β;

(b) α is *projectible to* β iff there is a weakly α-recursive function mapping α 1-1 into β;

(c) α is *weakly projectible to* β iff for all $\bar{\alpha} < \alpha$, there is a weakly α-recursive function mapping $\bar{\alpha}$ 1-1 into β.

In § 4 we apply the boundedness principle to obtain two somewhat technical results which can be described informally as follows:

(1) Any *total* weakly α-recursive functional can be defined without using the search operator.

(2) Any weakly α-recursive inductive operator over α can be approximated by weakly α-recursive operators over ordinals less than α.

In § 5 we present the characterization theorem.

THEOREM 5.1. *For any recursively regular ordinals α and $\beta \leqslant \alpha$:*

(a) *$|(\alpha/\beta)|$ is the least ordinal γ such that for any $\tau < \alpha$, $\gamma+1 \nsubseteq \{\sigma: \{\sigma\}$ is* p.o.r. *in τ and parameters less than $\beta\}$.*

(b) *For $|(\alpha/\beta)| \leqslant \alpha$, $|(\alpha/\beta)|$ is the least ordinal γ such that for any $\tau < \alpha$, $\gamma+1 \nsubseteq \{\sigma: \sigma$ is α-recursive in τ and parameters less than $\beta\}$, or equivalently, the largest ordinal which is projectible to β (by a weakly α-recursive function).*

(c) *For $|\alpha/\beta| \geqslant \alpha$, $|\alpha/\beta| = |(\alpha/\beta)|$.*

(d) *For $|\alpha/\beta| < \alpha$, $|\alpha/\beta|$ is the least ordinal γ which is not α-recursive in parameters less than β, or equivalently, the largest ordinal which is absolutely projectible to β (by an α-recursive function), or equivalently, the least α-stable ordinal greater than or equal to β.*

In § 6 we prove the following theorems.

THEOREM 6.1. *For any recursively regular ordinals α and $\beta \leqslant \alpha$:*

(a) *$|\alpha/\beta| \geqslant \alpha$ iff α is absolutely projectible to β;*

(b) *α absolutely projectible to β and $\{\alpha\}$ not* p.o.r. *in parameters less than β implies that $|\alpha/\beta| = \alpha$.*

THEOREM 6.2. *For any recursively regular ordinals α and $\beta \leqslant \alpha$:*

(a) *$|(\alpha/\beta)| \geqslant \alpha$ iff α is weakly projectible to β;*

(b) *α projectible to β and $\{\alpha\}$ not* p.o.r. *in parameters less than α implies that $|(\alpha/\beta)| = \alpha$;*

(c) *α weakly projectible to β and not projectible to β implies that $|(\alpha/\beta)| = \alpha$.*

When α is absolutely projectible to β, the notions of weakly α-recursive and α-recursive coincide, so that $|\alpha/\beta| = |(\alpha/\beta)|$. Conversely, when α is not absolutely projectible to β, the two ordinals are not equal.

It is known (see [1]) that $|\Delta_1^1|$ (which equals $|\omega_1/\omega|$ by Theorem 3.9) is recursively singular. This fact is a corollary to part (a) of the following result.

THEOREM 7.1. *For any recursively regular ordinals α and $\beta \leqslant \alpha$:*

(a) *$|\alpha/\beta| > \alpha$ implies that $|\alpha/\beta|$ is recursively singular;*

(b) *$|\alpha/\beta| < \alpha$ implies that $|\alpha/\beta|$ is recursively regular;*

(c) *$|\alpha/\beta|$ and $|(\alpha/\beta)|$ are both fixed points of the sequence $\{\omega_\tau: \tau \in \mathrm{ORD}\}$.*

3. Ordinal recursion. There are two approaches to ordinal recursion. First, there are the p.o.r. functionals, the smallest class containing the constant functions $0, 1, 2, \ldots$ and ω, the successore function, decision by cases, a supremum and an evaluation functional for functions, and closed under strong composition (or full substitution) and strong primitive recursion. These p.o.r. functionals are all total on total functions. Second, there are the α-recursive functionals $\{a\}_\alpha$ (indexed by $a < \omega$), defined by an inductive operator in a system with a search functional up to α and an enumeration functional. There is a set $\mathrm{POR} \subseteq \omega$, itself p.o.r., such that every p.o.r. functional is equal to $\{a\}_\infty$, or $\overline{U}\{\{a\}_\alpha: \alpha \in \mathrm{ORD}\}$, for some $a \in \mathrm{POR}$. A functional $\{a\}_\infty$ is strongly recursive iff $\{a\}_\infty = \{a\}_\alpha$ for all α. The usefulness of the p.o.r. functionals lies in the following two results.

PROPOSITION 3.1. *Every* p.o.r. *functional is strongly recursive.*

An ordinal α is recursively regular iff there is no weakly α-recursive function mapping a smaller ordinal cofinally to α; otherwise α is recursively singular.

PROPOSITION 3.2. *For each $l < \omega$, there is a* p.o.r. *relation T^l such that for all $a < \omega$, all $\underset{\sim}{\alpha}, \beta, \gamma$, and all $\underset{\sim}{f} = f_0, \ldots, f_{l-1}$:*

$$\{a\}_\gamma(\underset{\sim}{\alpha}, \underset{\sim}{f}) \simeq \beta \quad \text{iff} \quad \exists \xi \cdot T^l(\xi, \gamma, \langle a, \underset{\sim}{\alpha}, \beta \rangle, \underset{\sim}{f});$$

if γ is recursively regular, $\underset{\sim}{\alpha}, \beta < \gamma$, and each f_i γ-recursive, then

$$\{a\}_\gamma(\underset{\sim}{\alpha}, \underset{\sim}{f}) \simeq \beta \quad \text{iff} \quad \exists \xi < \gamma \exists \sigma < \gamma \cdot T^l(\xi, \sigma, \langle a, \underset{\sim}{\alpha}, \beta \rangle, \underset{\sim}{f}).$$

Since we are dealing in this paper with inductive definitions, it is important that our ordinal recursive functionals be closed under inductive definitions in the following sense. (See [2], Lemma 3.12, for details.)

PROPOSITION 3.3. *If Γ is an α-recursive (or* p.o.r.) *inductive operator on sets of ordinals, then* $\{\langle\tau,\xi\rangle\colon \tau\in\Gamma^{\xi}\}$ *is α-recursive* (p.o.r.).

The ordinal recursive functionals are connected with Π_1^1 relations by the next two results from [2], § 5.

PROPOSITION 3.4. *If $Q\subseteq\omega\times P(\omega)$ is Π_1^1, then there is a* p.o.r. *relation R such that for all m and A:*

$$Q(m,A)\quad\text{iff}\quad \exists\sigma\cdot R(\sigma,m,A)\quad\text{iff}\quad \exists\sigma<\omega_1^A\cdot R(\sigma,m,A)\,.$$

DEFINITION 3.5. $W(\varphi)$ iff φ is the characteristic function of a well-ordering of a subset of ω; $|\varphi|$ is the type of φ and $|u|^A$ the type of $\{u\}^A$.

PROPOSITION 3.6. *Let* $K[A]=\{\langle a,\underset{\sim}{m},n\rangle\colon T^1(\omega_1^A,\omega_1^A,\langle a,\underset{\sim}{m},n\rangle,A)\}$ *and let* $K_A[B]=\{\langle s,a,\underset{\sim}{u},v\rangle\colon T^1(|s|^A,|s|^A,\langle a,|\underset{\sim}{u}|^A,|v|^A,B)\}$, *for* $A,B\subseteq\omega$. *Then $K[A]$ is Π_1^1 uniformly in A and $K_A[B]$ is Π_1^1 uniformly in A and B.*

We need two lemmas to prove Theorem 3.9.

LEMMA 3.7. *There is a Π_1^1 relation L and Σ_1^1 relations M and M' such that*

(a) *for all* $\varphi,\psi\in{}^{\omega}\omega$, $L(\varphi,\psi)$ *iff* $W(\varphi)\wedge W(\psi)\wedge|\varphi|\leqslant|\psi|$;

(b) *if* $W(\psi)$, *then for all* φ, $M(\varphi,\psi)$ *iff* $W(\varphi)\wedge|\varphi|\leqslant|\psi|$;

(c) *if* $W(\psi)$, *then for all* φ, $M'(\varphi,\psi)$ *iff* $W(\varphi)\wedge|\varphi|<|\psi|$.

LEMMA 3.8. *For all* $A\subseteq\omega$,

(a) $W(A)=\{u\colon W(\{u\}^A)\}$ *is Π_1^1-A complete,*

(b) *for any $V\subseteq W(A)$, if V is Σ_1^1 in A, then there is a $\overline{u}$ such that for all* $v\in V$, $L(\{v\}^A,\{\overline{u}\}^A)$.

THEOREM 3.9. *An inductive operator Γ over ω is ω_1-recursive iff it is Δ_1^1.*

Proof. ($\rightarrow$) Suppose

$$m\in\Gamma(A)\quad\text{iff}\quad\{a\}_{\omega_1}(m,A)\simeq1\,,$$
$$m\notin\Gamma(A)\quad\text{iff}\quad\{a\}_{\omega_1}(m,A)\simeq0\,.$$

Then since $\{a\}_{\omega_1}^1$ is total, being an inductive definition, and for any A, $\omega_1\leqslant\omega_1^A$, we have as in Proposition 3.6:

$$m\in\Gamma(A)\quad\text{iff}\quad\langle a,m,1\rangle\in K[A]\quad\text{iff}\quad\langle a,m,0\rangle\notin K[A]\,,$$

proving that Γ is Δ_1^1.

($\leftarrow$) Suppose Γ is Δ_1^1. Then by Proposition 3.4 there are p.o.r. relations R and S such that for all m,A:

$$m\in\Gamma(A)\quad\text{iff}\quad\exists\sigma<\omega_1^A\cdot R(\sigma,m,A)\,,$$
$$m\notin\Gamma(A)\quad\text{iff}\quad\exists\sigma<\omega_1^A\cdot S(\sigma,m,A)\,.$$

(1) $$\forall m,A\,\exists\sigma<\omega_1^A[R(\sigma,m,A)\vee S(\sigma,m,A)]\,.$$

We want to show that $\sigma<\omega_1$ is always sufficient. Applying Proposition 3.6, we can rewrite (1) as

(2) $$\forall m,A\,\exists s[W(\{s\}^A)\wedge P(s,m,A)]\,,\qquad\text{where } P \text{ is } \Pi_1^1\,.$$

Let $V=\{v\colon \exists m,A\,\forall s[P(s,m,A)\rightarrow M(\{v\},\{s\}^A)]\}$. Then V is Σ_1^1 and $V\subseteq W(\emptyset)$ by (2), so by Lemma 3.8 there is a $\overline{u}$ such that for all $v\in V$, $|v|\leqslant|\overline{u}|<\omega_1$. It follows that

$$\forall m,A\,\exists\sigma<|\overline{u}|<\omega_1[R(\sigma,m,A)\vee S(\sigma,m,A)]\,,$$

so that Γ is ω_1-recursive.

4. An ordinal recursive boundedness principle. In this section we apply the techniques of ($\leftarrow$) in the proof of Theorem 3.9 to bound more general computations.

We need the following lemma, due to Sacks [5]. We refer the reader to [3] for a proof.

LEMMA 4.1. *For any countable recursively regular ordinal α, there is an $A\subseteq\omega$ such that $\alpha=\omega_1^A$.*

THEOREM 4.2. *For any countable recursively regular ordinal α, any $\tau,\beta<\alpha$, and any functional $\lambda\zeta,B\cdot\{a\}_\alpha(\tau,\zeta,B)$ total on $\beta\times P(\beta)$, there is a $\overline{\sigma}<\alpha$ such that for all $\xi<\alpha$, all $\zeta<\beta$, and all $B\subseteq\beta$:*

$$\{a\}_\alpha(\tau,\zeta,B)\simeq\xi\quad\text{iff}\quad T^1(\overline{\sigma},\overline{\sigma},\langle a,\tau,\zeta,\xi\rangle,B)\,.$$

Proof. Let $\omega_1^A=\alpha$ as in Lemma 4.1. Choose a well-ordering φ recursive in A of type β and choose t so that $|t|^A=\tau$. For any $B\subseteq\beta$, let $B^*=\{u\colon M'(\{u\}^A,\varphi)\}$. B^* is p.o.r. in B, A and B is p.o.r. in B^*, A. (See [2], Proposition 5.12.) We have, for any $\xi<\alpha$, any $\zeta<\beta$, and any $B\subseteq\beta$,

$$\{a\}_\alpha(\tau,\zeta,B)\simeq\xi\quad\text{iff}\quad\exists\sigma<\omega_1^{A\times B^*}\cdot T^1(\sigma,\sigma,\langle a,\tau,\zeta,\xi\rangle,B)\,.$$

For $B\subseteq\omega$, let $\overline{B}=\{|u|^A<\beta\colon u\in B\}$. The relation R, defined by

$$R(b,s,A,B)\quad\text{iff}\quad\exists x\cdot T^1(|s|^{A\times B},|s|^{A\times B},\langle a,|t|^A,|b|^A,|x|^A\rangle,\overline{B})$$

is Π_1^1 by Proposition 3.5. We have

(*) $$\forall b,B[M'(\{b\}^A,\varphi)\rightarrow\exists s\cdot R(b,s,A,B)]\,.$$

Let

$$V=\{v\colon\exists b,B[M'(\{b\}^A,\varphi)\wedge\forall s(R(b,s,A,B)\rightarrow M(\{v\}^A,\{s\}^{A\times B}))]\}\,.$$

Then V is Σ_1^1-A and $V\subseteq\{u\colon W(\{u\}^A)\}$ by (*). It follows from Lemma 3.7 that there is a $\overline{u}$ such that for all $\xi<\alpha$, all $\zeta<\beta$, and all $B\subseteq\beta$,

$$\{a\}_\alpha(\tau,\zeta,B)\simeq\xi\quad\text{iff}\quad\exists\sigma<|\overline{u}|^A\cdot T^1(\sigma,\sigma,\langle a,\tau,\zeta,\xi\rangle,B)\,.$$

Let $\sigma=|\overline{u}|^A$.

This is our formal version of result (1) as stated in the summary of results. We next state without proof the formal version of result (2), as it requires a similar, although more complicated, argument.

THEOREM 4.3. *For any countable recursively regular ordinal* a, *any* $\tau, \xi < a$, *and any functional* $\lambda\zeta, B\cdot\{a\}_a(\tau, \zeta, B)$ *total on* $a\times P(a)$, *there is a* $\bar{\beta} < a$ *such that for all* $\zeta < \xi$ *and all* $B \subseteq a$: $\{a\}_a(\tau, \zeta, B) \simeq \{a\}_a(\tau, \zeta, B\restriction\bar{\beta})$.

5. The characterization theorem. In this section we prove Theorem 5.1.

(a) $|(a/\beta)|$ *is the least ordinal* γ *such that for any* $\tau < a$, $\gamma+1 \nsubseteq \{\sigma: \{\sigma\}$ *is* p.o.r. *in* τ *and parameters less than* $\beta\}$.

Proof. ($\leqslant$) Assume $\beta < a$, and suppose that $\gamma < |(a/\beta)|$. Then there is a weakly a-recursive inductive operator Γ over β such that $|\Gamma| > \gamma$. By Theorem 4.2 there is a $\tau < a$ such that Γ is p.o.r. in τ. For any $\sigma \leqslant \gamma$, there is $\zeta < \beta$ such that σ is the unique ordinal with $\zeta \in \Gamma^{\sigma+1} - \Gamma^{\sigma}$, and therefore p.o.r. in τ and ζ by Proposition 3.3.

If $\gamma < |(a/a)|$, then there is a weakly a-recursive inductive operator Γ over a and a $\xi < a$ such that $\xi \in \Gamma^{\gamma+1} - \Gamma^{\gamma}$. Applying Theorem 4.3, choose $\bar{\beta} < a$ such that at ξ, Γ depends only on ordinals less than $\bar{\beta}$. Let Γ_0 be the restriction of Γ to $\bar{\beta}$; it follows that $\xi \in \Gamma_0^{\gamma+1} - \Gamma_0^{\gamma}$. The proof now follows as above for $\beta < a$.

($\geqslant$) Suppose we have $\tau < a$ such that for any $\sigma \leqslant \gamma$, σ is p.o.r. in τ and parameters less than β. We want to define Γ so that for all $\sigma \leqslant \gamma$, Γ^{σ} is a pre-well-ordering of type σ. For any $\sigma \leqslant \gamma$, we have $a \in \mathrm{POR}$ and $\zeta < \beta$ such that σ is the least ordinal with $\{a\}(\tau, \zeta, \sigma) \simeq 1$. It follows by a rather technical argument that for any pre-well-ordering A of type σ, there is an $a \in \mathrm{POR}$ and $\zeta < \beta$ such that $\{a\}(\tau, \zeta, A) \simeq 1$ and for any initial segment B of A, $\{a\}(\tau, \zeta, B) \simeq 0$. Let

$$\langle a, \zeta\rangle \in \Gamma_0(A) \quad \text{iff} \quad \langle\langle a, \zeta\rangle, \langle a, \zeta\rangle\rangle \in A\,,$$

and let

$$\langle a, \zeta\rangle \in \Gamma_1(A) \quad \text{iff} \quad a \in \mathrm{POR} \wedge \zeta < \beta \wedge \{a\}(\tau, \zeta, A) \simeq 1 \wedge \langle a, \zeta\rangle \notin \Gamma_0(A)\,.$$

Now, let

$$\sigma = \langle\langle a_0, \zeta_0\rangle, \langle a_1, \zeta_1\rangle\rangle \in \Gamma(A)$$

$$\text{iff} \quad \sigma \in A \vee \langle a_0, \zeta_0\rangle \in \Gamma_0(A) \cup \Gamma_1(A) \wedge \langle a_1, \zeta_1\rangle \in \Gamma_1(A)\,.$$

(b) *For* $|(a/\beta)| \leqslant a$, $|(a/\beta)|$ *is the least ordinal* γ *such that for any* $\tau < a$, $\gamma+1 \nsubseteq \{\sigma: \sigma$ *is* a*-recursive in* τ *and parameters less than* $\beta\}$.

Proof. One direction is trivial since if $\sigma < a$ is the unique ordinal such that $\{a\}(\tau, \zeta, \sigma) \simeq 1$, then $\sigma = \text{least } \xi < a\cdot\{a\}(\tau, \zeta, \xi) \simeq 1$ and is therefore a-recursive in τ and ζ. In the other direction, suppose that all $\sigma \leqslant \gamma$ are a-recursive in τ and parameters less than β. Let $f(\sigma) \simeq$ least $\langle a, \zeta, \xi\rangle < a[\zeta < \beta \wedge T^0(\xi, \xi, \langle a, \tau, \zeta, \sigma\rangle)]$, and let $\bar{\xi} = \sup_{\sigma\leqslant\gamma} f(\sigma)$. Then

$\bar{\xi} < a$ by regularity and every $\sigma \leqslant \gamma$ is p.o.r. in $\bar{\xi}$, τ, and parameters (ζ) less than β.

(c) *For* $|a/\beta| \geqslant a$, $|a/\beta| = |(a/\beta)|$.

Proof. For any $\sigma < |a/\beta|$, there is an a-recursive inductive operator Γ over β and a $\zeta < \beta$ such that $\zeta \in \Gamma^{\sigma+1} - \Gamma^{\sigma}$. If $\sigma < a$, then $\sigma = \text{least } \xi < a\cdot\zeta \in \Gamma^{\xi+1} - \Gamma^{\xi}$ and is therefore a-recursive in parameters less than β. Now if $|a/\beta| \geqslant a$, then every $\tau < a$ is a-recursive in parameters less than a, so that any weakly a-recursive inductive operator is a-recursive in parameters less than β and $|a/\beta| = |(a/\beta)|$.

(d) *For* $|a/\beta| \leqslant a$, $|a/\beta|$ *is the least ordinal* γ *which is not* a*-recursive in parameters less than* β.

Proof. ($\leqslant$) This is direct from the proof of (c) above. ($\geqslant$) Let $\bar{\gamma}$ be the least ordinal not a-recursive in parameters less than β. We split the proof into two cases.

Case I. $\bar{\gamma} = \beta$. For any $\gamma < \beta$ and any $B \subseteq \beta$, let

$$\Gamma_\gamma(B) = \{\sigma < \gamma\colon\ \forall\tau < \sigma,\ \tau \in B\}\,.$$

It is clear that for any $\sigma \leqslant \gamma$, $\Gamma_\gamma^{\sigma} = \sigma$, and that $|\Gamma_\gamma| = \gamma$.

Case II. $\beta < \bar{\gamma} \leqslant a$. Given $\gamma < \bar{\gamma}$, let

$$\bar{a} = \sup_{\tau<\gamma}\ \text{least } \sigma < a[\exists a < \omega\exists\xi < \beta\cdot T^0(\sigma, \sigma, \langle a, \xi, \tau\rangle)]\,.$$

Then $\bar{a} < a$ by regularity and a is a-recursive in parameters less than β since β and γ are.

Let $C(a, \xi, \tau)$ iff $T^0(\bar{a}, \bar{a}, \langle a, \xi, \tau\rangle)$ and let $E(\tau, B)$ iff $\exists a < \omega\exists\xi < \beta[\langle a, \xi\rangle \in B \wedge C(a, \xi, \tau)]$. We now define Λ_γ such that for all $\tau \leqslant \gamma$, $\{\sigma\colon E(\sigma, \Lambda_\gamma^{\tau})\} = \tau$ and such that $|\Lambda_\gamma| = \gamma$.

$$\Lambda_\gamma(B) = \{\langle a, \xi\rangle\colon\ a < \omega \wedge \xi < \beta \wedge \exists\tau < \gamma[C(a, \xi, \tau) \wedge \forall\sigma < \tau\cdot E(\sigma, B)]\}\,.$$

6. Projectibility conditions. Before proving Theorems 6.1 and 6.2, we need the following equivalences for the three notions of projectibility defined in § 2. See [2], Proposition 4.6, for proofs of (a) and (b); (c) is similar.

PROPOSITION 6.3. *For any recursively regular ordinals* a *and* $\beta < a$:

(a) a *is absolutely projectible to* β *iff*

$$a \subseteq \{\sigma\colon \sigma \text{ is } a\text{-recursive in parameters less than } \beta\}.$$

(b) a *is projectible to* β *iff there is a* $\tau < a$ *such that*

$$a \subseteq \{\sigma\colon \sigma \text{ is } a\text{-recursive in } \tau \text{ and parameters less than } \beta\}\,.$$

(c) a *is weakly projectible to* β *iff for all* $\bar{a} < a$, *there is a* $\tau < a$ *such that*

$$\bar{a} \subseteq \{\sigma\colon \sigma \text{ is } a\text{-recursive in } \tau \text{ and parameters less than } \beta\}.$$

We now prove Theorem 6.1.

(a) $|\alpha/\beta| \geqslant \alpha$ *iff* α *is absolutely projectible to* β.

Proof. $(\to)$ Given $|\alpha/\beta| \geqslant \alpha$, we have as in the proof of Theorem 5.1(c) that $\alpha \subseteq \{\sigma\colon \sigma$ is α-recursive in parameters less than $\beta\}$. It follows from Proposition 6.3 that α is absolutely projectible to β.

$(\leftarrow)$ Given $|\alpha/\beta| < \alpha$, we have by Theorem 5.1(d) that $|\alpha/\beta|$ is the least ordinal γ which is not α-recursive in parameters less than β. This being less than α implies by Proposition 6.3 that α is not absolutely projectible to β.

(b) α *absolutely projectible to* β *and* $\{a\}$ *not* p.o.r. *in parameters less than* β *implies that* $|\alpha/\beta| = \alpha$.

Proof. $(\geqslant)$ This is direct from (a).

$(\leqslant)$ Suppose that $|\alpha/\beta| > \alpha$. Then by Theorem 5.1(c), $|\alpha/\beta| = |(\alpha/\beta)|$, so by Theorem 5.1(a), $\{a\}$ is p.o.r. in some $\tau < \alpha$. By (a) above, this τ is α-recursive in some $\zeta < \beta$. We have that $\tau = \{b\}_\alpha(\zeta)$ and that α is the unique ordinal such that $\{a\}(\tau, \alpha) \simeq 1$. Then α is the unique ordinal such that

$$\exists \sigma, \tau < \alpha[T^0(\sigma, \sigma, \langle b, \zeta, \tau\rangle) \wedge \{a\}(\tau, \alpha) \simeq 1], \text{ and is p.o.r. in } \zeta .$$

Next we prove Theorem 6.2.

(a) $|(\alpha/\beta)| \geqslant \alpha$ *iff* α *is weakly projectible to* β.

Proof. This is immediate from Theorem 5.1(a) and Proposition 6.3(c).

(b) α *projectible to* β *and* $\{a\}$ *not* p.o.r. *in parameters less than* β *implies that* $|(\alpha/\beta)| = \alpha$.

Proof. $(\geqslant)$ This follows from (a) and the fact that projectibility implies weak projectibility.

$(\leqslant)$ This follows directly from Theorem 5.1(a).

(c) α *weakly projectible to* β *and not projectible to* β *implies that* $|(\alpha/\beta)| = \alpha$.

Proof. $(\geqslant)$ This follows from (a) above.

$(\leqslant)$ Suppose $|(\alpha/\beta)| > \alpha$. Then by Theorem 5.1(a), there is a $\tau < \alpha$ such that $\alpha \subseteq \{\sigma\colon \sigma$ is p.o.r. in τ and parameters less than $\alpha\}$. As in the proof of Theorem 5.1(b), each σ is α-recursive in τ and parameters less than β, so that by Proposition 6.3(b), α is projectible to β.

We now obtain the converse of Theorem 5.1(c).

PROPOSITION 6.4. *For any recursively regular ordinals* α *and* $\beta \leqslant \alpha$ *such that* α *is not absolutely projectible to* β*;* $|\alpha/\beta| < |(\alpha/\beta)|$.

Proof. $|\alpha/\beta|$ is the least ordinal not α-recursive in parameters less than β by Theorems 5.1(d) and 6.1(a). It is clear that for $\gamma = |\alpha/\beta|$, $\gamma+1 \subseteq \{\sigma\colon \sigma$ is α-recursive in γ and parameters less than $\beta\}$. It follows from Theorem 5.1(a) that $|\alpha/\beta| < |(\alpha/\beta)|$.

7. Regularity. In this section we prove Theorem 7.1.

(a) $|\alpha/\beta| > \alpha$ *implies that* $|\alpha/\beta|$ *is recursively singular.*

Proof. Let $\bar{\gamma} = |\alpha/\beta|$. For $\tau < \alpha$, let $f(\tau) \simeq$ least $\gamma < \bar{\gamma}$ $[\gamma+1 \not\subseteq \{\sigma\colon \{\sigma\}$ is p.o.r. in τ and parameters less than $\beta\}]$. It follows from Theorem 5.1(c) that $\bar{\gamma} = \sup_{\tau<\alpha} f(\tau)$. We can write $f(\tau)$ as

$$\text{least } \gamma < \bar{\gamma}\cdot \forall a \in \mathrm{POR}\ \forall \zeta < \beta[\{a\}(\tau, \zeta, \gamma) \simeq 1 \to \exists \sigma < \gamma\cdot\{a\}(\tau, \zeta, \sigma) \simeq 1].$$

Then f is weakly $\bar{\gamma}$-recursive and it follows that $\bar{\gamma}$ is recursively singular.

(b) $|\alpha/\beta| < \alpha$ *implies that* $|\alpha/\beta|$ *is recursively regular.*

Proof. By Theorem 5.1(d), $|\alpha/\beta|$ is the least ordinal γ not α-recursive in parameters less than β. Since the set of ordinals α-recursive in parameters less than β is an initial segment by Proposition 4.10 of [2], $|\alpha/\beta|$ is closed under all α-recursive functions, or α-stable. It follows that $|\alpha/\beta|$ is recursively regular, recursively inaccessible, and recursively Mahlo. (Proposition 4.14 of [2].)

Since $|\Pi_1^1|$ is the least ordinal γ which is γ^+-stable, (Theorem A of [2]), we have the following corollary.

COROLLARY 7.2. *For any recursively regular ordinals* α *and* β, $\beta \leqslant \alpha \leqslant |\Pi_1^1|$ *implies that* $|\alpha/\beta| \geqslant |\Pi_1^1|$.

(c) $|\alpha/\beta|$ *and* $|(\alpha/\beta)|$ *are both fixed points of the sequence* $\{\omega_\tau\colon \tau \in \mathrm{ORD}\}$.

Proof. For $|\alpha/\beta| < \alpha$, this is given in (b). For $|(\alpha/\beta)|$ in general, suppose that $\gamma < |(\alpha/\beta)|$ and prove that $\gamma^+ < |(\alpha/\beta)|$. We need the following lemma from § 3 of [2].

LEMMA 7.3. (a) *For any ordinal* ξ, ξ *is recursively regular iff*

$$\forall a < \omega\ \forall \sigma_1, \sigma_2 < \xi[T^0(\xi+1, \xi, \langle a, \sigma_1, \sigma_2\rangle) \to T^0(\xi, \xi, \langle a, \sigma_1, \sigma_2\rangle)];$$

(b) *the relation* RR, *defined by* RR(ξ) *iff* ξ *is recursively regular, is* p.o.r.

For regular $\gamma < |(\alpha/\beta)|$, we have by Theorem 5.1(a) $\tau < \alpha$ such that $\gamma+1 \subseteq \{\sigma\colon \{\sigma\}$ is p.o.r. in τ and parameters less than $\beta\}$. We prove by induction that for all $\xi \leqslant \gamma^+$, $\{\xi\}$ is p.o.r. in τ and parameters less than β.

(i) $\xi \leqslant \gamma$. This is given by hypothesis.

(ii) $\gamma < \xi < \gamma^+$. This implies that ξ is singular, so that as in Lemma 7.3, we have $\sigma_1, \sigma_2 < \xi$ such that ξ is the unique minimal ordinal such that T^0 $(\xi+1, \xi, \langle a, \sigma_1, \sigma_2\rangle)$. By the induction hypothesis, $\{\sigma_1\}$ and $\{\sigma_2\}$ are p.o.r. in τ and parameters less than β; it follows easily that ξ is also.

(iii) $\xi = \gamma^+$. By hypothesis, γ is unique such that $\{a\}(\tau, \zeta, \gamma) \simeq 1$. Then γ^+ is the unique minimal ordinal ξ such that $\mathrm{RR}(\xi) \wedge \wedge \exists \sigma < \xi\cdot\{a\}(\tau, \zeta, \sigma) \simeq 1$.

8. Open problems. The notion of weak projectibility introduced in § 2 is different from projectibility since δ_ω, the ωth stable ordinal, is weakly projectible, but not projectible, to ω. We would like to know whether the ordinal $|(\alpha/\beta)|$ in Theorem 5.1(a) is similarly different from the least ordinal γ such that $\{\gamma\}$ is not p.o.r. in parameters less than α. (It is always less than or equal to that ordinal.) The equality of these ordinals would imply, for example, that $|\Delta_1^1|$ is the least ordinal σ such that for all $a \in \mathrm{POR}$, $\{a\}(\sigma) \simeq 1 \to \exists \tau < \sigma \cdot \{a\}(\tau) \simeq 1$, a nice characterization and parallel to those given in [2] for $|\Pi_1^1|$ and $|\Sigma_1^1|$.

A related question is whether for any β and any $\alpha > \beta$ absolutely projectible to β, $\{\sigma\colon \{\sigma\}$ is p.o.r. in parameters less than $\beta\}$ is all of α. It is easy to see that this set will be cofinal in α, whereas $\{\sigma\colon \sigma$ is p.o.r. in parameters less than $\beta\}$ is always bounded below α.

Another problem is whether or not $|\alpha/\beta|$ (or $|(\alpha/\beta)|$) equals the closure ordinal of the class of (weakly) $\Sigma_1 - \alpha$ operators over β. Recall that $|\Sigma_1^0| = |\Delta_1^0| = \omega$.

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Langages à valeurs réelles et applications

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Résumé. Dans cet article, on définit les notions de formules et de modèles de langages dans lesquelles les valeurs de vérité sont prises dans $\boldsymbol{R}$, et non, comme d'habitude, dans $\{0, 1\}$. Il y a beaucoup de possibilités pour définir ces notions, possibilités qui correspondent aux divers choix pour les "connecteurs propositionnels". On étudie l'une d'elles ici, qui semble particulièrement intéressante. Un certain nombre de théorèmes classiques du calcul des prédicats peuvent être démontrés, avec des modifications convenables, pour ces langages: théorème de Herbrand, interpolation, définissabilité. On en donne ensuite des applications à la théorie des espaces normés (en particulier les espaces L^p); certaines d'entre elles sont énoncées dans [5].

Table des matières

I. Définitions générales

On suppose connues les notions de formules et de langages du premier ordre et de modèles d'une théorie du premier ordre. Nous utiliserons la terminologie courante sur ce sujet (voir par exemple [4], ou [8]).

On considère un langage $\mathfrak{L}$, avec symboles de relation et de fonction, ne comportant pas le symbole $=$; on supposera toujours que $\mathfrak{L}$ possède au moins un symbole de constante, et un symbole de relation à un argument distingué que l'on note N. On désigne par A (resp. A_0) l'ensemble des formules atomiques (resp. atomiques closes) de $\mathfrak{L}$. Les termes de $\mathfrak{L}$ sont définis comme d'habitude.

On définit maintenant, mais pas de la façon habituelle, les notions de modèle et de formule du langage $\mathfrak{L}$.

Un *modèle* $\mathcal{M}$ *de* $\mathfrak{L}$ est, par définition, constitué par: un ensemble non vide (l'ensemble de base du modèle) noté $|\mathcal{M}|$; pour chaque symbole