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Ce volume paraît en 4 fascicules

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A new construction of a non-constructible Δ_3^1 subset of ω

by

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Abstract. A new proof is given of the fact that it is consistent with ZFC that $\omega_1 = \omega_1{}^L$ and there exists a non-constructible Δ_3^1 subset a of ω. In fact, a satisfies a Π_2^1 condition φ such that the following in provable in ZFC: If $\omega_1 = \omega_1^L$, then there is at most one $x \subset \omega$ satisfying φ. φ is defined from an ω-sequence of normal trees in L. Using subtrees, we force ω times with Souslin trees. The set a is defined from the sequence of generic branches.

1. Introduction. In this paper we present a new proof of the fact that it is consistent with ZF to assume that there is a non-constructible Δ_3^1 subset of ω. This was originally proved by Solovay, and improved (for the case that $\omega_1^L < \omega_1$) by Jensen and Solovay. See the papers by Jensen and Solovay [4] and Jensen [3]. The existence of a non-constructible Δ_3^1 set of integers also follows if one assumes the existence of a measurable cardinal, see Solovay [7]. For more historical details we refer to [3], [4] and Levy [6].

The theorem we prove will be a slight extension of Theorem 4.1 in [4], but the method of proof is completely different, Souslin trees and forcing with Souslin trees being our basic tools.

THEOREM. *There exists a Π_2^1 formula φ such that the following are provable in* ZF:

(a) $\varphi(x) \to x \subset \omega$,

(b) $V = L \to \neg \exists x \varphi(x)$,

(c) $\omega_1^L = \omega_1 \to \exists^{\leqslant 1} x \varphi(x)$,

(d) *if* ZF *is consistent, then so is*

$$\mathrm{ZF} + \mathrm{GCH} + \omega_1^L = \omega_1 + \exists a\big(\varphi(a) \wedge V = L^a\big), \tag{$*$}$$

(e) *if $M \models \mathrm{ZFC} + \omega_1^L = \omega_1 + \varphi(a)$, and N is a cardinal preserving extension of M, then $N \models \varphi(a)$.*

The assertion (c) is our improvement of [4, Th. 4.1]. The theorem has the following corollary:

Corollary. *If* ZF *is consistent, then so are*

(i) $ZF+GCH+\omega_1^L=\omega_1+V\neq L+\exists a\subset\omega(a\in\Delta_3^1\wedge V=L^a)$,

(ii) $ZFC+\neg CH+\omega_1^L=\omega_1+\exists a\subset\omega(a\in\Delta_3^1\wedge a\notin L)$.

If $\{a\}\in\Pi_2^1$ (i.e. a is implicitly Π_2^1-definable), then clearly $a\in\Delta_3^1$. Thus a model of (*) in the theorem is a model of (i). If $M\models(*)\wedge\varphi(a)$, then (c) and (e) give us that a will be implicitly defined by φ in every extension of M which preserves ω_1. This gives (ii) (use Cohen's conditions to destroy CH).

In section 2 we recall some basic material about Souslin trees, the proof of the theorem is given in section 3.

2. Some preliminaries. A *tree* is a partially ordered set $\langle T,\leqslant\rangle$ such that $\{y|\ y<x\}$ is well ordered for every $x\in T$. Let $T=\langle T,\leqslant\rangle$ be a tree, and define

$$|x|=\sup\{|y||\ y<x\}\,,$$

the *level* of x. ($|x|$ equals the order type of the predecessors of x.) Set

$$T|\alpha=\{x\in T|\ |x|<\alpha\}\,,$$
$$T_\alpha=\{x\in T|\ |x|=\alpha\}\,,$$

and define, for $X\subset T$

$$|X|=\sup\{|x||\ x\in X\}\,.$$

$|T|$ is called the *length* of T.

If $x\leqslant y$ or $y\leqslant x$, x and y will be called *comparable*; otherwise they are *incomparable*, and we write this $x|y$. A *chain* is a linearly ordered subset of T. A *branch* b is a chain which is $\leqslant$-closed (i.e. if $x\in b$ and $y\leqslant x$, then $y\in b$). If $|b|=\alpha$, b is called an α-*branch*. An *antichain* is a subset of T consisting of pairwise incomparable elements of T. An antichain is a *maximal antichain* if it is not included in a larger antichain. A tree $\langle T,\leqslant\rangle$ is called a *Souslin tree* if $|T|=\omega_1$ and every chain and antichain in T is countable.

It will often be important that the trees satisfy certain "normality" properties. We call a tree $\langle T,\leqslant\rangle$ a *normal tree of length* α if:

(i) $|T|=\alpha$,

(ii) T has a least point,

(iii) each non-maximal point has at least two immediate successors,

(iv) each point has successors at each level $<\alpha$,

(v) each branch of limit length has at most one immediate successor,

(vi) each level T_ν is countable.

A normal tree certainly has length $\leqslant\omega_1$ (otherwise, by (iii) and (iv), $T\omega_1$ will be uncountable). It is also clear that a normal tree T of length

ω_1 is Souslin if every antichain in T is countable. (For, from an uncountable chain in T it is easy to construct an uncountable antichain (again using (iii)).)

The proof in section 3 uses "forcing". For basic facts we refer to Jech [1] and to Jensen [3, § 2]. A Souslin tree is a natural object to use as a set of forcing conditions. We reverse the ordering, and hence two conditions are incompatible iff they are incomparable in the tree. Since every antichain in the tree is countable, the set of conditions satisfies CCC. So cardinals are preserved in the extension. The generic set is simply an ω_1-branch through the tree, so in the extension the Souslin tree is "killed". (A detailed proof of this may be found in Solovay, Tennenbaum [8, § 2.2].) The reader should also observe that if M is a countable, transitive standard model of ZFC and $M\models T$ is Souslin, then every ω_1^M-branch of T is T-generic over M.

The proof of section 3 also uses iterated forcing, as developed by Solovay and Tennenbaum in [8]. The particular fact we need can be stated as follows: Suppose we iterate the forcing process ω times and then take the direct limit. (By the "direct limit" we mean the extension obtained by using the direct limit of the corresponding complete Boolean algebras as the set of conditions, see [8].) If CCC is satisfied at each level in the iteration, then the direct limit itself will be a CCC-extension of the ground model. Hence cardinals will be preserved.

When forcing with a Souslin tree, we principally add one generic branch. But other ω_1-branches may be definable from it, and hence in the extension, the generic branch may not be definable. Our proof is heavily based upon a construction due to Jensen of a tree T with the following properties:

(i) $V=L\to T$ is Souslin,

(ii) $\omega_1^L=\omega_1\to T$ has at most one ω_1-branch.

If we force with this tree, we add exactly one branch. The tree T_0 as constructed below will have these properties, so the construction of such a tree can easily be extracted from our proof.

The reader not familiar with Jensen's construction of a Souslin tree in L is advised to consult [1] or [2].

In conclusion a word about definability. Let H_{ω_1} denote the set of hereditarily countable sets. For $X\subset H_{\omega_1}$ we write $X\in\Pi_n(H_{\omega_1})$ if X is Π_n-definable in H_{ω_1} without parameters. (Similarly, we define $X\in\Sigma_n(H_{\omega_1})$ and $X\in\Delta_n(H_{\omega_1})$.) We note the well-known lemma:

Lemma. *Let* $X\subset\mathcal{P}(\omega)$ *and assume* $n\geqslant 1$. *Then*

$$X\in\Pi_{n+1}^1\leftrightarrow X\in\Pi_n(H_{\omega_1})\,.$$

(*We can also replace* Π *by* Σ *or* Δ.)

3. Proof of the theorem. We shall define in L an ω-sequence $\langle T_n |\ n \in \omega\rangle$ of normal trees of length ω_1^L with some special properties. In L, T_0 will be Souslin, and can be used as a set of forcing conditions (with reversed ordering). In the extension cardinals are preserved, and T_0 contains an ω_1-branch b_0. From b_0 we can define a subtree T_1^* of T_1 which will be Souslin. Forcing with T_1^* we get an ω_1-branch b_1 through T_1 and define $T_2^* \subset T_2$ etc. After ω such steps we take the direct limit. In this final extension, b_n will be the unique ω_1-branch through T_n for $n \in \omega$. This enables us to code $\langle b_n |\ n \in \omega\rangle$ as a subset a of ω which will be implicitly Π_2^1-definable.

The trees T_n will be defined by induction in the following way: For successor ordinals α, $T_n|\alpha+1$ will be defined simultaneously for each n; for limit ordinals α, we first define $T_0|\alpha+1$, then $T_1|\alpha+1$ etc., by forcing over larger and larger models. We first state nine properties which will be shown to hold for each $T_n|\alpha$ (by induction on α), and hence for each T_n. (Most of the properties will follow trivially from the definitions.)

(1) $$s \in T_n \wedge |s| = \alpha \rightarrow s \in \{n\} \times (Q^+)^\alpha ,$$

(2) $$s \in T_n \wedge q \in Q^+ \rightarrow s^*\langle q\rangle \in T_n ,$$

where Q^+ denotes the positive rational numbers (not including 0) and $*$ means concatenation of sequences. Hence, if $s \in T_n$ and $|s| = \alpha$, s will be a $(1+\alpha)$-sequence of the form $\langle n, s_1, s_2, \ldots\rangle$. The ordering $\leqslant_n$ of the tree T_n is the usual initial sequence ordering, i.e. $\leqslant_n = \subset \restriction (T_n)^2$ for $n \in \omega$.

Simultaneously, for each $n \in \omega$ and $x \in T_n$, we define a subtree t_x of T_{n+1}, such that (for $n \in \omega$, $x, y, z \in T_n$):

(3) t_x is a normal tree of length $|x|$,

(4) $$x \leqslant_n y \rightarrow t_x = t_y||x| ,$$

(5) $x|y$ and z is the $\leqslant_n$-largest $z \leqslant_n x, y \rightarrow$
$(|z| \geqslant 1 \wedge t_x \cap t_y = t_z) \vee (|z| = 0 \wedge t_x \cap t_y = \{\langle n+1\rangle\})$
(the z above is well-defined since we have no splitting at limit levels),

(6) $$T_{n+1} = \bigcup_{x \in T_n} t_x .$$

To state the last three properties we need some more definitions. If $q_\nu \in Q^+ \cup \{0\}$ for $\nu < \lambda$, $\sum_{\nu\in\lambda} q_\nu$ denotes the supremum of all finite partial sums if this exists, ∞ otherwise. Let g: $Q^+ \leftrightarrow \omega$ and [ ]: $(Q^+)^2 \leftrightarrow Q^+$ be Σ_0-definable bijections, and let $(\ )_0$ and $(\ )_1$ be the projections of [ ] (i.e. $[(q)_0, (q)_1] = q$).

For $s, s' \in T_0$ and $\beta \in On$ we define

$$\Sigma_\beta(s, s') = \sum_{\beta\leqslant\nu<|s|,|s'|} |s_{1+\nu} - s'_{1+\nu}| ,$$

and for $s, s' \in T_n$, $n \geqslant 1$ and $\beta \in On$ we define

$$\Sigma_\beta(s, s') = \sum_{\beta\leqslant\nu<|s|,|s'|} |(s_{1+\nu})_1 - (s'_{1+\nu})_1| .$$

Also let

$$\Sigma(s, s') = \Sigma_0(s, s') .$$

When $T = T_0$ or $T = t_x$ for some x (with partial ordering $\leqslant$), we require:

(7) $$\Sigma(s, s') < \infty \quad \text{for} \quad s, s' \in T ,$$

(8) $\Sigma(s\restriction\beta, s'\restriction\beta) < \Sigma(s, s')$ when $s, s' \in T$, $|s| = |s'| = \alpha > \beta$ and $\lim(\alpha)$,

(9) if $m \in \omega$, $x_i \in T$ and $|x_i| = \alpha$ for $i \leqslant m$, $y_0 \in T$, $y_0 > x_0$ and $r \in Q^+$, then there exist $y_1, \ldots, y_m \in T$ such that $y_i > x_i$ and $|y_i| = |y_0|$ for $1 \leqslant i \leqslant m$, and $\Sigma_\alpha(y_i, y_j) < r$ when $i, j \leqslant m$.

(7) states that the "horizontal distances" shall be finite. The "vertical distances" $\sum_{\nu<|s|} s_{1+\nu}$ can (in fact: must) be infinite for some s.

We now turn to the construction of the trees T_n and the subtrees t_x. Assume $V = L$. As described above we define $T_n|\alpha$ for $n \in \omega$ by induction on α.

Case 1. $\alpha = 2$. For $n \in \omega$ set

$$T_n|2 = \{\langle n\rangle\} \cup \{\langle n, q\rangle |\ q \in Q^+\} ,$$
$$t_{\langle n\rangle} = \varnothing ,$$
$$t_{\langle n,q\rangle} = \{\langle n+1\rangle\} \quad \text{for} \quad q \in Q^+.$$

Case 2. $\alpha = \beta+3$. For $n \in \omega$ set

$$T_n|\alpha = T_n|\beta+2 \cup \{x^*\langle q\rangle |\ x \in T_n|\beta+2 \wedge |x| = \beta+1 \wedge q \in Q^+\} .$$

For $x \in T_n|\beta+2$, $|x| = \beta+1$ and $q \in Q^+$ we now set

$$t_{x^*\langle q\rangle} = t_x \cup \{y^*\langle[q, r]\rangle |\ y \in t_x \wedge |y| = \beta \wedge r \in Q^+\} .$$

Case 3. $\lim(\alpha)$. For $n \in \omega$, we have

$$T_n|\alpha = \bigcup_{\beta<\alpha} T_n|\beta .$$

In this case t_x is already defined for each $x \in T_n|\alpha$.

Case 4. $\alpha = \lambda+1$, where λ is a limit number. First we define $T_0|\alpha$. Let $\eta_{0,\lambda}$ be the least η such that

$$T_0|\lambda \in L_\eta , \quad L_\eta \models \mathrm{ZF}^- \quad \text{and} \quad \lambda \text{ is countable in } L_\eta .$$

ZF^- denotes ZF without the power set axiom. Set $T = T_0|\lambda$ and $\eta = \eta_{0,\lambda}$. $T_0|\alpha$ will be determined by forcing over L_η. The set $\boldsymbol{P}$ of conditions is defined as follows:

$$\langle f, h\rangle \in \boldsymbol{P} \leftrightarrow \langle f, h\rangle \in L_\eta,$$
$$f\colon u \to T \quad \text{for some finite } u \subset \omega,$$
$$|f(i)| = |f(j)| \quad \text{for } i, j \in u,$$
$$h\colon u^* \to \boldsymbol{Q}^+, \quad \text{where } u^* = \{\langle i, j\rangle|\ i, j \in u \wedge i \neq j\},$$
$$h(i, j) = h(j, i) \quad \text{for } \langle i, j\rangle \in u^* \quad \text{and}$$
$$\Sigma\big(f(i), f(j)\big) < h(i, j) \quad \text{for } \langle i, j\rangle \in u^*.$$

$$\langle f, h\rangle \leqslant \langle f', h'\rangle \leftrightarrow \mathrm{dom}(f) \supset \mathrm{dom}(f') \wedge h \supset h' \wedge$$
$$\wedge \forall i \in \mathrm{dom}(f')\ f(i) \geqslant_0 f'(i).$$

Let G be the $<_L$-least $\boldsymbol{P}$-generic set over L_η, where $<_L$ denotes the canonical well-ordering of L. Set

$$b_i = \{f(i)|\ \exists h \langle f, h\rangle \in G\} \quad \text{and} \quad s_i = \bigcup b_i \quad \text{for} \quad i \in \omega,$$
$$h = \bigcup \{h'|\ \exists f \langle f, h\rangle \in G\}.$$

Claim 1. (i) $h\colon \{\langle i, j\rangle \in \omega^2|\ i \neq j\} \to \boldsymbol{Q}^+$,

(ii) *each* b_i *is a* λ*-branch of* T,

(iii) *the branches* b_i *cover* T, *i.e.* $T \subset \bigcup_{i\in\omega} b_i$,

(iv) $b_i \neq b_j$ *when* $i \neq j$,

(v) $\Sigma(s_i, s_j) \leqslant h(i, j)$ *for* $i \neq j$,

(vi) *if* $C \in L_\eta$ *is a maximal antichain in* T, *then* $b_i \cap C \neq \emptyset$ *for all* $i \in \omega$.

Proof. Though somewhat technical, the proof is standard. As an illustration, we prove (ii). (Detailed proofs of the other parts can be found in Johnsbråten [5].)

Let $i \in \omega$. From the definitions it easily follows that the elements in b_i are comparable, hence b_i is a chain in T. Trivially, b_i is $\leqslant_0$-closed, and therefore a branch. Suppose now that $b_i \subset T_0|\beta$ for a $\beta < \lambda$. Set

$$\Delta = \{\langle f, h'\rangle|\ |f(i)| \geqslant \beta\}.$$

Δ is obviously L_η-definable, and we now prove that it is dense.

Let $\langle f_0, h_0\rangle \in \boldsymbol{P}$. We can trivially assume that $i \in \mathrm{dom}(f_0)$. Let $\gamma = |f_0(i)|$. Let $y_i \geqslant_0 f_0(i)$ with $|y_i| \geqslant \beta$. Let $r \in \boldsymbol{Q}^+$ such that

$$r < \min\{h_0(j, k) - \Sigma\big(f_0(j), f_0(k)\big)|\ j, k \in \mathrm{dom}(f_0) \wedge j \neq k\}.$$

By (9) there exist y_j for $j \in \mathrm{dom}(f_0)$ such that $y_j \geqslant_0 f_0(j)$, $|y_j| = |y_i|$ and $\Sigma\gamma(y_j, y_k) < r$ for $j, k \in \mathrm{dom}(f_0)$, $j \neq k$. Define $\langle f_1, h_1\rangle$ by: $\mathrm{dom}(f_1) = \mathrm{dom}(f_0)$, $h_1 = h_0$ and $f_1(j) = y_j$ for $j \in \mathrm{dom}(f_1)$. Then we get

$$\Sigma\big(f_1(j), f_1(k)\big) = \Sigma_\gamma(y_j, y_k) + \Sigma\big(f_0(j), f_0(k)\big)$$
$$< r + \Sigma\big(f_0(j), f_0(k)\big) < h_0(j, k) = h_1(j, k),$$

for $j, k \in \mathrm{dom}(f_1)$, $j \neq k$. Hence $\langle f_1, h_1\rangle \in \boldsymbol{P}$, $\langle f_1, h_1\rangle \leqslant \langle f_0, h_0\rangle$ and $\langle f_1, h_1\rangle \in \Delta$.

So Δ is dense, and thus $G \cap \Delta$ contains an element $\langle f', h'\rangle$. Now $f'(i) \in b_i$, so by the assumptions $|f'(i)| < \beta$, contradicting the fact that $\langle f', h'\rangle \in \Delta$. Q.E.D.

We set

$$T_0|\alpha = T_0|\lambda \cup \{s_i|\ i \in \omega\}.$$

(t_{s_i} for $i \in \omega$ will be defined below.)

Claim 2. (i) (9) *continues to hold*,

(ii) $\Sigma(s_i \restriction \beta, s_j \restriction \beta) < \Sigma(s_i, s_j)$ *for* $i, j \in \omega$, $i \neq j$ *and* $\beta < \lambda$.

The proof of Claim 2 uses the same technics as above, and we omit it.

We now assume that $T_n|\alpha$ is defined, and proceed to define $T_{n+1}|\alpha$. First we define t_x for $x \in T_n|\alpha$, $|x| = \lambda$ by

$$t_x = \bigcup_{y <_n x} t_y.$$

Let $\eta_{n+1,\lambda}$ be the least η such that

$T_n|\alpha$, $T_{n+1}|\lambda$, g and [ ] are in L_η, $L_\eta \models \mathrm{ZF}^-$ and λ is countable in L_η.

$T_{n+1}|\alpha$ will be defined by forcing over $L_{\eta_{n+1,\lambda}}$. For $x \in T_n|\alpha$, $|x| = \lambda$ define $\boldsymbol{P}^x$ by the same formula as for $\boldsymbol{P}$ above, but let η now denote $\eta_{n+1,\lambda}$ and $T = t_x$, and replace $\leqslant_0$ by $\leqslant_{n+1}$. (The expression $\Sigma(s, s')$ now has a different definition than above. The distances are measured along the second projections of the rationals in the sequences s; the first projections determine the x's for which $s \in t_x$.)

Let G^x be the $<_L$-least $\boldsymbol{P}^x$-generic set over $L_{\eta_{n+1,\lambda}}$, and define b_i^x, s_i^x and h^x as above (with G^x instead of G). Then the modified versions of Claim 1 and Claim 2 are valid. So we set

$$T_{n+1}|\alpha = T_{n+1}|\lambda \cup \{s_i^x|\ i \in \omega \wedge x \in T_n|\alpha \wedge |x| = \lambda\}.$$

Case 5. $\alpha = \lambda + 2$, where λ is a limit number. For $n \in \omega$ we set

$$T_n|\alpha = T_n|\lambda \cup \{x^*\langle q\rangle|\ x \in T_n|\lambda+1 \wedge |x| = \lambda \wedge q \in \boldsymbol{Q}^+\},$$

as in Case 2. Now let $n \in \omega$ and $x \in T_n|\lambda+1$, $|x| = \lambda$. The branches b_i^x ($i \in \omega$) are exactly the λ-branches in t_x which were extended to s_i^x in T_{n+1}.

We split them into ω parts (each of which will be shown to cover t_x) and set, for $q \in Q^+$

$$t_{x*\langle q\rangle} = t_x \cup \{s^x_{g([q,r])} |\ r \in Q^+\}\,.$$

Claim 3. *$t_x \subset \bigcup_{r\in Q^+} b^x_{g([q,r])}$, for $q \in Q^+$.*

Proof. Let $y \in t_x$. Set

$$\Delta = \{\langle f', h'\rangle \in P^x |\ \exists r \in Q^+\ \ y \leqslant_{n+1} f'(g([q, r]))\}\,.$$

Then Δ is $L_{\eta_{n+1},\lambda}$-definable and dense in P^+. So let $\langle f', h'\rangle \in G^x \cap \Delta$. Then $y \leqslant_{n+1} f'(g[q, r])) \in b^x_{g([q,r])}$ for some $r \in Q^+$. Q.E.D.

The induction is complete, and for $n \in \omega$ we set

$$T_n = \bigcup_{\alpha<\omega_1} T_n|\alpha\,.$$

From the definitions, Claim 1 (ii) and (iii) and Claim 3 it follows that each T_n is a normal tree of length ω_1. (1)-(6) are trivially satisfied. (7) follows from (v) in Claim 1, (8) and (9) from Claim 2.

We have not motivated the introduction of the trees t_x. The main reason is that we can now define a certain function f. This function will be a basic tool in the rest of the proof. For $n \in \omega$, $x \in T_{n+1}$, $|x| \geqslant 1$ we define

$$f(x) = \text{the } \leqslant_n\text{-least } y \in T_n \text{ such that } x \in t_y\,.$$

This is well-defined by (5) and (6). ((6) gives the existence of such a y, (5) gives uniqueness.) By (4), $|f(x)| = |x|+1$.

Claim 4. *If $\omega_1^L = \omega_1$, then each tree T_n has at most one ω_1-branch.*

Proof. By induction on n. Suppose first that $b \neq b'$ are two ω_1-branches of T_0. By (8),

$$\langle \Sigma(\bigcup b \restriction \nu, \bigcup b' \restriction \nu)|\ \nu < \omega_1 \wedge \lim(\nu)\rangle$$

will be an uncountable, strongly increasing sequence of reals, impossible.

Now, assume the claim holds for n, and suppose $b \neq b'$ are two ω_1-branches of T_{n+1}. Let b'' and b''' be the images of b and b' under f, more precisely,

$$b'' = \{y|\ \exists x \in b\ \ |x| \geqslant 1 \wedge y \leqslant_n f(x)\}\,,$$

and similarly for b'''. Then both b'' and b''' are ω_1-branches of T_n, hence $b''' = b''$. But then both b and b' are included in $\bigcup_{x\in b''} t_x$, so, again by (8),

$$\langle \Sigma(\bigcup b \restriction \nu, \bigcup b' \restriction \nu)|\ \nu < \omega_1 \wedge \lim(\nu)\rangle$$

forms an uncountable, strongly increasing sequence of reals. Q.E.D.

We now proceed to find a model of (*) in the theorem. Let M be a countable, transitive standard model of ZF$+V=L$, and let $\langle T_n|\ n \in \omega\rangle$ be the above described sequence, constructed inside M. M will be extended by forcing. The set of conditions is given by

$$P = \{\langle p_0, \ldots, p_n\rangle|\ n \in \omega \wedge p_0 \in T_0 \wedge \forall i < n\ p_{i+1} \in t_{p_i}\}\,,$$

$$\langle p_0, \ldots, p_n\rangle \leqslant \langle p'_0, \ldots, p'_m\rangle \leftrightarrow n \geqslant m \wedge \forall i \leqslant m\ \ p_i \geqslant_i p'_i\,.$$

Let G be P-generic over M. We will later show that cardinals are preserved in $M[G]$. Set, for $n \in \omega$

$$b_n = \{p_n|\ p \in G \wedge n \in \operatorname{dom}(p)\}\,.$$

The following claim is trivial.

Claim 5. (i) *b_n is an ω_1^M-branch of T_n for $n \in \omega$,*

(ii) *$M[\langle b_n|\ n \in \omega\rangle] = M[G]$.*

Now let $M_0 = M$ and $T^*_0 = T_0$. By induction, assume that M_n and $T^*_n \subset T_n$ are defined such that T^*_n is Souslin in M_n and $b_n \subset T^*_n$. (If not, stop the induction.) Then b_n is T^*_n-generic over M_n. Let $M_{n+1} = M_n[b_n]$ and set

$$T^*_{n+1} = \bigcup_{x\in b_n} t_x\,.$$

Trivially, $b_{n+1} \subset T^*_{n+1}$.

That the definition above is not interrupted will be shown in Claim 6. For this we need to code ω-sequences of rationals as subsets of ω. So let k: $(Q^+)^{\omega} \leftrightarrow \mathfrak{P}(\omega)$ be some reasonably defined bijective map.

Claim 6. *T^*_n is Souslin in M_n for $n \in \omega$.*

Proof. The proof that T^*_0 is Souslin in M_0 is just a slight simplification of the general case below (drop the A), so we omit it. Thus, assume that T^*_n is Souslin in M_n. Hence cardinals are preserved under the translation to M_{n+1}.

We work in M_{n+1}. It is obvious how, using the function k defined above, we can code b_n as a subset A of ω_1 such that $A \cap \alpha$ codes $b_n|\alpha$ for all limit numbers $\alpha < \omega_1$. Then $V = L^A$.

Set $T = T^*_{n+1}$, and assume $C \subset T$ is a maximal antichain. We want to show that C is countable. Of course, $C, T \in L^A_{\omega_2}$. So let

$$M \prec L^A_{\omega_2} \quad \text{such that} \quad C, T \in M \text{ and } M \text{ is countable}\,.$$

Then $\omega_1 \cap M \in On$. Set $\omega_1 \cap M = \alpha$. There exist uniquely determined π, β such that

$$\pi\colon M \xrightarrow{\sim} L^{A\cap\alpha}_\beta\,.$$

Then, by elementary equivalence,

$$\pi(\omega_1) = \alpha\,,$$
$$\pi(T) = T|\alpha\,,$$
$$\pi(C) = C \cap T|\alpha \quad \text{and}$$
$$C \cap T|\alpha \text{ is a maximal antichain in } T|\alpha\,.$$

Since α is countable in $L_{\eta_{n+1,\alpha}}$, but not in $L_\beta^{A\cap\alpha}$, we have that $\beta < \eta_{n+1,\alpha}$. Now $T_n|\alpha+1 \in L_{\eta_{n+1,\alpha}}$. Hence $b_n|\alpha$ and thus also $A \cap \alpha$ lie in $L_{\eta_{n+1,\alpha}}$. So

$$L_\beta^{A\cap\alpha} \subset L_{\eta_{n+1,\alpha}}\,.$$

Now, using (vi) in Claim 1, we have that $C \cap T|\alpha$ is maximal in $T|\alpha+1$, and hence also in T. So $C = C \cap T|\alpha$ and therefore C is countable. Q.E.D.

Let N be the direct limit of the M_n's. We know that cardinals are preserved, so $N \models \omega_1^L = \omega_1$. By Claim 4, in N, b_n is the unique ω_1^N-branch through T_n for $n \in \omega$. So $\langle b_n|\ n\in\omega\rangle \in N$, and hence

$$M[G] = M[\langle b_n|\ n \in \omega\rangle] \subset N\,.$$

(In fact, they are equal.) Hence cardinals are preserved also in $M[G]$.

We shall now pick out the sequence of points in the branches b_n of level 1 and code it as an $a \subset \omega$. By use of the function f we then show the curious fact that $\langle b_n|\ n\in\omega\rangle$ is constructible from a.

For $n \in \omega$ let S_n be the point in b_n such that $|S_n| = 1$. Let $S = \{S_n|\ n \in \omega\}$. Notice that each S_n is equal to $\langle n, q\rangle$ for some $q \in \boldsymbol{Q}^+$. This means that $S \in (\boldsymbol{Q}^+)^\omega$. So let $a = k(S)$. Also define, for $n \in \omega$ and $x \subset \omega$: $x_n = \langle n, (k^{-1}(x))_n\rangle$. (Then $S_n = a_n$ for $n \in \omega$.)

Claim 7. $M[a] = M[G]$.

Proof. In $M[a]$, by induction on $\alpha < \omega_1$, we define a sequence $\langle b'_n|\ n\in\omega\rangle$ of branches which we shall prove is just $\langle b_n|\ n \in \omega\rangle$.

(i) $\alpha = 0$: Set $x_n^0 = \langle n\rangle$ for $n \in \omega$.

(ii) $\alpha = 1$: Set $x_n^1 = a_n$ for $n \in \omega$.

(iii) $\alpha = \beta+1$ $(\beta > 0)$: Set $x_n^\alpha = f(x_{n+1}^\beta)$ for $n \in \omega$.

(iv) $\lim(\alpha)$: If each $\{x_n^\beta|\ \beta < \alpha\}$ is a branch in T_n with a successor at level α, let x_n be this successor. If not, interrupt the definition.

Thus the induction goes up to an ordinal $\gamma \leqslant \omega_1^M$. Let $b'_n = \{x_n^\alpha|\ \alpha < \gamma\}$. By induction on α we show that $x_n^\alpha \in b_n$ for $n \in \omega$, from which it follows that $b'_n = b_n$ for every n.

This is trivial for $\alpha = 0, 1$. So let $\alpha = \beta+1$ $(\beta > 0)$ and assume that $x_n^\beta \in b_n$ for $n \in \omega$. Let $n \in \omega$. Since $x_{n+1}^\beta \in T_{n+1}^*$, there is a $y \in b_n$ with $x_{n+1}^\beta \in t_y$. By (4) we may assume that $|y| = \alpha$. But then, by the definition of f, $y = f(x_{n+1}^\beta) = x_n^\alpha$. Hence $x_n^\alpha \in b_n$.

Now, suppose α is a limit number, and assume $\{x_n^\beta|\ \beta < \alpha\} \subset b_n$ for $n \in \omega$. But each b_n is an ω_1^M-branch in T_n, so $\{x_n^\beta|\ \beta<\alpha\}$ has a uniquely determined successor in T_n at level α. Hence the definition is never interrupted.

So $b'_n = b_n$ for $n \in \omega$, and hence $\langle b_n|\ n\in\omega\rangle$ is definable in $M[a]$. So $M[G] \subset M[a]$. The converse is trivially true by the definition of a. Q.E.D.

Now we define a formula φ which we prove to be $\Pi_1(H_{\omega_1})$. Since φ will define a subset of $\mathcal{P}(\omega)$, φ will be equivalent to a Π_2^1 formula, by the lemma given in section 2. So it is enough to prove that φ (itself) satisfies (a)-(e) in the theorem. φ is defined as follows:

$$\varphi(x) \leftrightarrow x \subset \omega \wedge \forall \alpha < \omega_1^L\, \exists p \in \bigtimes_{i\in\omega} T_i|\alpha+1\ \forall n \in \omega$$
$$\big(\alpha \neq 0 \rightarrow \big(p_n \in T_n|\alpha+1 \wedge |p_n| = \alpha \wedge p_n \geqslant_n x_n \wedge$$
$$\wedge \forall u <_{n+1} p_{n+1}\big(|u| \neq 0 \rightarrow f(u) \leqslant_n p_n\big)\big)\big)\,.$$

In words, $\varphi(x)$ states that on every level we can pick a point p_n from every tree T_n lying above x_n (the nth point in the decoding of x), such that for all u beneath p_{n+1}, $f(u)$ will lie under p_n. That $M[a] \models \varphi(a)$ is easy: Pick the points p_n from the branches b_n.

Claim 8. *φ is Π_1 (H_{ω_1}).*

Proof. We claim that $\langle T_n|\alpha|\ \alpha < \omega_1^L \wedge n \in \omega\rangle$ and f are Δ_1 (H_{ω_1}). The sequence above is defined by recursion. The reader should check some details in the construction and convince himself that the induction clause is Δ_1 in $L_{\omega_1^L}$. (For instance: Show that the expression "G is the $<_L$-least $\boldsymbol{P}$-generic set over L_{η_α}" is Δ_1 in $L_{\omega_1^L}$ in the parameters T and α.) But then the sequence $\langle T_n|\alpha|\ \alpha<\omega_1^L \wedge n \in \omega\rangle$ is Σ_1 in $L_{\omega_1^L}$. Since ω_1^L and $L_{\omega_1^L}$ are $\Sigma_1(H_{\omega_1})$, the sequence above is $\Delta_1(H_{\omega_1})$. f is treated in the same way. We then replace

$$\text{"}\forall \alpha < \omega_1^L\, \exists p \in \bigtimes_{i\in\omega} T_i|\alpha+1 \ldots\text{"}$$

by

$$\text{"}\forall \alpha,\, T \in H_{\omega_1}(\alpha < \omega_1^L \wedge T = \bigtimes_{i\in\omega} T_i|\alpha+1 \rightarrow \exists p \in T \ldots)\text{"}.$$

"$\alpha < \omega_1^L$" is $\Sigma_1(H_{\omega_1})$, hence the expression inside the parenthesis is $\Pi_1(H_{\omega_1})$, and we are done. Q.E.D.

Claim 9. (i) *Assume $\varphi(x)$. Then, for $n \in \omega$, there exists an ω_1^L-branch $b_n \subset T_n$ such that $x_n \in b_n$.*

(ii) *Assume $\omega_1^L = \omega_1$. Assume $\varphi(x)$ and $\varphi(y)$. Then $x = y$.*

Proof. (i) For $n \in \omega$, set

$$b_n = \{\langle n\rangle\} \cup \{p_n |\ p \text{ has the property stated in the definition of } \varphi \text{ with respect to } x\}.$$

Then each b_n has points at every level $< \omega_1^L$, so it remains to prove that b_n is linearly ordered. Obviously, $x_n \leqslant_n y$ for $n \in \omega$ and $y \in b_n$.

So assume that p and p' have the properties given in φ with respect to x, and suppose that $p_n | p'_n$ for some $n \in \omega$. We seek a contradiction.

First we prove that $p_m | p'_m$ for $m \geqslant n$. If not, let $m \geqslant n$ be such that $p_m | p'_m$ but $p_{m+1} \leqslant_{m+1} p'_{m+1}$. Let z be the largest $z \leqslant_m p_m, p'_m$, and let $z' <_{m+1} p_{m+1}$, $|z'| = |z|$. (Notice that $|z| \geqslant 1$.) By our assumptions about p and p', $f(z') \leqslant_m p_m, p'_m$. But this is impossible, since $f(z') >_m z$.

So let z_m be the largest $z \leqslant_m p_m, p'_m$ for $m \geqslant n$. By the same argument we must have

$$|z_n| > |z_{n+1}| > |z_{n+2}| > \ldots,$$

which is impossible.

(ii) If $x \neq y$, then $x_n \neq y_n$ for some n. But then, by (i), T_n will contain two different ω_1^L-branches, which is impossible by Claim 4. Q.E.D.

Now, (a) in the theorem is trivially satisfied by φ. From (i) in Claim 9 we obtain $\exists x\varphi(x) \to V \neq L$, which is equivalent to (b). (c) is exactly (ii) in Claim 9. In $M[a]$, (*) holds, so (d) is clear. (GCH is implied by $V = L^a$.) (e) is clear from the absoluteness in the construction (or simply by Shoenfield's absoluteness theorem). The proof is complete.

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Reçu par la Rédaction le 5. 2. 1973

Banach spaces and large cardinals

by

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Abstract. The purpose of this paper is to introduce a new type of a basis-notion; sets of indiscernibles, for Banach spaces. A structural theory for Banach spaces generated by sets of indiscernibles is developed. It is shown that any Banach space of the cardinality of a Ramsey cardinal has a set of indiscernibles of the same cardinality and that consequently it has a big subspace admitting non-trivial projections. The behaviour of linear operators on spaces of large cardinality is also studied.

0. Introduction and notation. Our intent is to study the applications of the theory of large cardinals to Banach spaces. The cardinals we choose to work with, Ramsey cardinals, are of a fairly high order. It is shown that the notion of sets of indiscernibles, which usually arises in the theory of Ramsey cardinals, has a natural interpretation in the context of Banach spaces. Chapter 1 is devoted to the study of the structural theory of Banach spaces generated by sets of indiscernibles. No large cardinality assumptions are needed here except that we do require the density character of the spaces in question to be uncountable. It seems from the many counterexamples one can construct that the countable case has very little coherence. In the remainder of this paper we then invoke large cardinality assumptions in order to get sets of indiscernibles; the general idea behind all of our proofs being that every big enough Banach space has a big, fairly homogeneous, subspace.

The author wishes to thank Professors Haskell Resenthal and Per Enflo for many helpful discussions. This research was conducted during the author's stay as a Miller Fellow at the University of California.

The notation and terminology conforms to that used in [1] and [2]. For example, cardinals are initial ordinals. Ordinals are denoted by small Greek letters $\alpha, \beta, \ldots$ The cardinality of the set X is denoted by $|X|$. The finite linear span of the set X (if it makes sense) is denoted by $[X]$. Operator always means bounded linear operator.