Invariant sets in topology and logic

by

Robert Vaught (Berkeley, Cal.)

Abstract. A natural mapping is introduced which carries all subsets of a space acted on by a group onto all invariant subsets. Applications are made both in the general theory of such actions and in infinitary logic (where the space of all structures on ω acted on by $\omega!$ is such an action).

Let G be a topological group which is Polish (separable, complete metrizable). Suppose G acts on a (say, also Polish) topological space X, the action $(g, x) \mapsto gx$ being continuous. A subset B of X is invariant if $gx \in B$ whenever $x \in B$ and $g \in G$. The outer invariantization or saturation B^+ of B is the smallest invariant set including B and is given by $B^+ = \{x\colon \exists g (gx \in B)\}$. Dually, the inner invariantization

$$B^- = \{x\colon \forall g (gx \in B)\}\,.$$

If B is Borel, B^- and B^+ are respectively coanalytic and analytic, but usually not Borel.

We will prove that *there is a set* B^* (another "invariantization") *such that* B^* *is invariant*, $B^- \subseteq B^* \subseteq B^+$, *and* B^* *is Borel if* B *is*. Indeed, B^* preserves the level of B in hierarchies both below and above the Borel. B^* has the simple definition:

$B^* = \{x\colon \{g\colon gx \in B\}\}$ is comeager. A related transform was studied by P. S. Novikov (cf. [2]) — see § 1 below.

The transform $B \mapsto B^*$ is investigated in detail in § 1, under more general assumptions than those above. In the rest of the paper, the transform is used to obtain results about invariant sets (not involving the transform). For example, we can infer directly from the italicized statement above that: *if* Y *is an invariant subset of* X, *then* $B \subseteq Y$ *is invariant Borel in* Y *if and only if* B *is of the form* $B' \cap Y$ *for some* B' *invariant Borel in* X (cf. 2.3).

In § 2 the transform is applied to obtain invariant forms of some classical theorems of descriptive set theory. Thus (2.5): *every invariant analytic (or even* Σ_2^1*) set is the union of* $\aleph_1$ *invariant Borel sets.* (2.5 implies the result of C. Ryll-Nardzewski [22]: every orbit is Borel.) Likewise (2.7): *the reduction principle holds for invariant coanalytic and for invariant* Σ_2^1. (For 2.7 much simpler proofs have subsequently been obtained by Burgess and Miller [29].)

In § 3 the transform is applied to infinitary logic. Here X is, say, $2^{\omega\times\omega}$, and G is the group $\omega!$ of all permutations of ω. A well-known theorem of Lopez-Escobar [10] states that $(*)$ a subset B of X is invariant Borel if and only if B is the set $\mathrm{Mod}\,\sigma$ of all models of some sentence σ of the language $L_{\omega_1\omega}$. We obtain a new proof of $(*)$ and as a result the first proof of Lopez-Escobar's interpolation theorem from its classical antecedent, the first separation theorem. Our proof of $(*)$, unlike the old proof, shows further that: B *is invariant* Σ_α^0 (in the Borel hierarchy) *if and only if* $B = \mathrm{Mod}\,\sigma$ for some sentence σ *at the* α*-th level in the natural hierarchy of sentences.* Moreover, we can go above the Borel level: Let $\mathcal{A}$ be all sets obtainable from open sets by Borel operations plus the operation (A). $L_{\omega_1 G}$ is the enrichment of $L_{\omega_1\omega}$ by the game quantifier, studied for example in [4]. In 3.8 we show that: *invariant* $\mathcal{A} =$ *(expressible in)* $L_{\omega_1 G}$.

In the last section, § 5, we discuss the effective aspects of § 3. In particular we give the following effective form of $(*)$ above: If A is a transitive, primitive recursively closed set containing ω, and $\theta \in A$ is the name of an invariant Borel set, then $\mathrm{Mod}\,\theta = \mathrm{Mod}\,\sigma$ where σ is a sentence of $L_{\omega_1\omega}$ which belongs to A. The method of Lopez-Escobar and Barwise had established the same result assuming A is admissible.

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Some interesting later results of Burgess and Miller concerning invariant sets appear in [29].

§ 1. The $*$-transform. The following assumption is made hence forth: *G is a Baire topological space* — i.e., one in which no non-empty open set is meager (of first category); *and* $J\colon G\times X\to X'$. ($J(g, x)$ is written gx.) Sometimes we assume further that:

(0) *X and X' are topological spaces and J is continuous in each variable separately.*

For $B\subseteq X'$ we shall define $B^*\subseteq X$ in 1.1 below.

The letters g, h will always denote members of G; similarly $x, y\in X$.

There are two important special cases of our assumptions including (0). If G is also a (Baire) topological group, $X = X'$, and $(gh)x = g(hx)$ and $ex = x$ (e the identity element of G), then we say that *G acts on X*. (Thus for us this phrase implies that G is a Baire space and J is continuous in each variable.) This will be called the "action case" and is our primary concern. (However, quite a lot of the theory requires only the general assumptions above.) A second case, the "product case," is that in which G and X are given (as topological spaces, G Baire), $X' = G\times X$, and J is the identity function. Novikov (see [2]) dealt with the product case with $G = X =$ the unit interval. (The product case is practically the general case (pass from $B\subseteq X'$ to $J^{-1}B\subseteq G\times X$); but the notation of our general case specializes more readily to the action case.)

DEFINITION 1.1. Let $B\subseteq X'$. Put:

(a) $B^x = \{g\colon gx\in B\}$,

(b) $B^* = \{x\colon B^x$ is comeager$\}$, and, in general, if U is open in G, $B^{*U} = \{x\colon B^x\cap U$ is comeager in $U\}$ (so that $B^* = B^{*G}$),

(c) $B^\Delta = \sim(\sim B)^* = \{x\colon B^x$ is not meager$\}$.

We write B^- and B^+ as before, and $B^{-U} = \{x\colon (\forall g\in U)\, gx\in B\}$ and $B^{+U} = \{x\colon (\exists g\in U)\, gx\in B\}$. Note that $\sim(B^-) = (\sim B)^+$.

Remark. Let U be a non-empty open set in G and let J_U be J restricted to $U\times X$. Then U and J_U meet our assumptions regarding G and J, including (0) if (0) held (X and X' are fixed). Moreover, B^{*U} is just B^* for this U and J_U. (The same applies to B^- and B^{-U} and B^{+U}.) *Hence any theorem about B^* automatically implies a corresponding theorem for B^{*U}* (and likewise for B^-, B^+).

THEOREM 1.2. $B^-\subseteq B^*\subseteq B^\Delta\subseteq B^+$.

Proof. This is obvious, but $B^*\subseteq B^\Delta$ relies on our assumption that G is a Baire space.

The next several theorems, 1.3-1.7, will give us a kind of inductive definition for B^{*U} (induction over the formation of B).

THEOREM 1.3. *Assume* (0). *If B is closed in X' then B^* is closed in X and $B^* = B^-$. (Hence B^Δ is open if B is open.)*

Proof. B^x is closed (in G) because $g\mapsto gx$ is assumed continuous. Hence B^x is comeager if and only if $B^x = G$ (since G is a Baire space). Thus $B^* = \{x\colon G = B^x\} = B^-$. $B^- = \bigcap_g \{x\colon g\in B^x\}$ is closed, because each $\{x\colon g\in B^x\} = \{x\colon gx\in B\}$ is closed in view of the assumption that $x\mapsto gx$ is continuous.

Certain families are sometimes studied in place of the family of Borel sets (and often called by the too popular title "Baire sets"). For their sake we append to 1.3 an aside:

THEOREM 1.3′. *Assume* (0). *Then*

(a) *If B is clopen then B^* is a countable intersection of clopen sets, provided G is separable.*

(b) *If B is compact and closed then B^* is compact, provided we assume either the action case or else the product case with G a T_1-space.*

Proof. (a) If D is a countable dense subset of G, then $B^* = \{x\colon D \subseteq B^*\}$, so $B^* = \bigcap_{g \in D} \{x\colon gx \in B\}$ and each $\{x\colon gx \in B\}$ is clopen.

(b) $B^* = \bigcap_g \{x\colon gx \in B\}$. In either case each $x \mapsto gx$ is a homeomorphism of X onto a closed subset of X', so $\{x\colon gx \in B\}$ is compact.

THEOREM 1.4. $(\bigcap_n B_n)^* = \bigcap_n B_n^*$. (*Hence dually* $(\bigcap_n B_n)^\Delta = \bigcap_n B_n^\Delta$.)

1.4 is only the fact that a countable union of meager sets is meager.

We now need some familiar set-theoretical and topological notions. Let F range over maps $F\colon \omega \to \omega$ and $\underline{k}$ over the set Sq of all finite sequences of natural numbers. Write $F \restriction n = (F_0, \dots, F_{n-1})$. Recall that if $E = \bigcup_F \bigcap_n A_{F\restriction n}$ then E is said to be obtained by the operation (A) from $\underline{k} \mapsto A_{\underline{k}}$. (Of course we say E is obtainable by (A) from a class Q of sets if it is so obtained from some indexed family of members of Q.) Recall that (in a topological space) a set W has the Baire property if for some open O, $W \equiv O$ (modulo the ideal of meager sets). As is known [9], the sets with the Baire property are closed under complement, countable union, and the operation (A). We denote by $\mathfrak{C}$ the smallest family containing all Borel sets and closed under complement, countable union, and (A). (Thus $\mathfrak{C}$ implies the Baire property.)

For the remaining inductive conditions we must restrict the class of sets B being considered to the "normal" sets. $B \subseteq X'$ is said to be *normal* if each B^x (for $x \in X$) has the Baire property. Since inverse images preserve complement, countable union, and (A), it is clear that *the class $\mathfrak{N}$ of all normal sets is closed under complement, countable union, and* (A). Moreover, $\emptyset$ is normal. Under assumption (0), we saw that B^x is closed if B is closed, and thus every closed set is normal and so: *every $\mathfrak{C}$-set is normal.*

Call $\mathcal{K}$ a *weak basis for G* if $\mathcal{K}$ consists of non-empty open sets and every non-empty open set includes a member of $\mathcal{K}$. (If G has a countable weak basis, G is clearly separable and hence if it is metric, G simply has a countable basis.) We do not assume yet that G has a *countable* weak basis. But notationally we do now assume that $\mathcal{K}$ is a weak basis for G.

Moreover, henceforth, U, V range over $\mathcal{K}$, while W is used for arbitrary subsets of G. Note that $\{V\colon V \subseteq U\}$ is a weak basis for the subspace U.

THEOREM 1.5. *If B is normal then*

$$({\sim}B)^* = {\sim}\bigcup_U B^{*U}.$$

Note that 1.5 forces us to consider B^{*U} as well as B^*.

Proof. Let $x \in X$ and put $W = B^x$. Then 1.5 reduces to the following known proposition concerning a set W having the Baire property (in a Baire space):

(1) W is meager if and only if W is comeager nowhere (i.e., for each U, $W \cap U$ is not comeager in U).

From left to right, (1) is immediate from G being a Baire space. From right to left follows easily from the assumption that W has the Baire property.

For later convenience we write explicitly some other formulas implied by 1.4, 1.5:

$$(2)\qquad \begin{aligned} &(\bigcup_n B_n)^* = \bigcap_U \bigcup_{V \subseteq U} \bigcup_n B_n^{*V}, && \text{if each } B^* \text{ is normal},\\ &(B - C)^* = B^* - \bigcup_U C^{*U}, && \text{if } C \text{ is normal}. \end{aligned}$$

It follows at once from 1.3-1.5 that, assuming (0) and that G has a countable basis, if B is Borel then B^* is Borel. In the special product case he studied, Novikov (cf. [2]) established this fact and hence essentially 1.3-1.5. Before turning to such matters (in 1.8 below) we are going to add in 1.6 one more inductive condition, concerning the operation (A). Moreover 1.6 will involve us in a further result, 1.7, concerning (A) whose proof is closely related.

Several results in the rest of the paper do not involve (A) at all, and the reader can now skip ahead to those if he wishes.

A space satisfies the countable chain condition if any family of disjoint open sets is countable (and hence clearly if it is separable).

THEOREM 1.6. *Assume G satisfies the countable chain condition, $E = \bigcup_F \bigcap_n A_{F\restriction n}$, and each $A_{\underline{k}} \subseteq X'$ is normal. Then $x \in E^*$ if and only if*

$$(\forall U_0)(\exists V_0 \subseteq U_0)(\exists k_0)(\forall U_1 \subseteq V_0)(\exists V_1 \subseteq U_1)(\exists k_1) \dots \forall n [x \in A^{*V_n}_{k_0 \dots k_n}].$$

The last line of 1.6 signifies that the second (or (V, k)-) player has a winning strategy in the indicated ordinary infinite (closed) game.

It is possible to give a direct proof of (d), for example by making use of the Banach-Mazur game (see [18], [19]). However, the shortest proof we know establishes at the same time another proposition which we will use extensively in § 2, namely 1.7 below. 1.7 deals with the behaviour under * of the usual basic ordinal decompositions of a set obtained by (A) (as in [9]), which we now review:

Let α, β always range over *countable* ordinals.

Suppose $E = \bigcup_F \bigcap_n A_{F\restriction n}$ ($A_{\underline{k}}$ arbitrary sets). Then we put

(i) $A^0_{\underline{k}} = A_{\underline{k}}, \quad A^{\alpha+1}_{\underline{k}} = A^\alpha_{\underline{k}} \cap \bigcup_i A^\alpha_{\underline{k}\frown i}, \quad$ and $\quad A^\lambda_{\underline{k}} = \bigcap_{\alpha<\lambda} A^\alpha_{\underline{k}} \quad$ (α limit);

(ii) $E_\alpha = A^\alpha_\emptyset \quad$ and $\quad T_\alpha = \bigcup_{\underline{k}} (A^\alpha_{\underline{k}} - A^{\alpha+1}_{\underline{k}})$.

It follows (cf. [9]) that

(iii) $$E = \bigcap_\alpha E_\alpha = \bigcup_\alpha (E_\alpha - T_\alpha).$$

Finally, assume we are in a topological space satisfying the countable chain condition. (In [9], a countable basis is assumed but only the countable chain condition is used.) Then (cf. [9]):

(iv) If the $A_{\underline{k}}$ all have the Baire property, then for some α, $\sim T_\alpha$ is comeager (and hence $E_\alpha - T_\alpha$ is comeager in E).

Now we can state

THEOREM 1.7. *Assume the hypothesis of* 1.6. *Then*

$$E^* = \bigcap_\alpha E^*_\alpha = \bigcup_\alpha (E_\alpha - T_\alpha)^*.$$

Proof of 1.6 and 1.7. 1.6 and 1.7 can be trivially reduced to propositions (about comeager sets) taking place entirely in G (since $B \mapsto B^x$ preserves everything). However, since the only effect of this is to confuse the notation, we do not take this step.

Let D be the set of all x for which the last line of 1.6 (the game condition) holds. We first show that:

(I) $$E^* \subseteq D.$$

Let $x \in E^*$ so that $E^x = \bigcup_F \bigcap_n A^x_{F\restriction n}$ is comeager. Write $W_{\underline{k}} = \bigcup_{F \supseteq \underline{k}} \bigcap_n A^x_{F\restriction n}$, so that $W_\emptyset = E^x$. Let the first player choose U_0. Since $E^x \cap U_0$ is comeager in U_0, clearly at least one of the sets $W_j \cap U_0$ is not meager, say for $j = k_0$. Now $W_{k_0} \cap U_0$ has the Baire property, so we can choose $V_0 \subseteq U_0$ so that $V_0 \equiv W_{k_0} \cap U_0$. Thus $W_{k_0} \cap V_0$ is comeager in V_0 and our position is just as at the beginning. Repeating indefinitely, we have a way of playing the game which ensures that for each n, $W_{k_0\ldots k_n} \cap V_n$ is comeager in V_n — and hence the larger set $A_{k_0\ldots k_n} \cap V_n$ is comeager in V_n, i.e., $x \in A^{*V_n}_{k_0\ldots k_n}$. Thus (I) is proved.

Next we show that

(II) $$D \subseteq \bigcap_\alpha E^*_\alpha.$$

Given $V_0 \supseteq V_1 \supseteq \ldots \supseteq V_{j-1}$ and $\underline{k} = (k_0, \ldots, k_{j-1})$, we put

$$x \in D_{\underline{k}}(V_0, \ldots, V_{j-1})$$

if and only if the game can be played from there on, i.e.,

(3) $$(\forall U_j \subseteq V_{j-1})(\exists V_j \subseteq U_j)(\exists k_j)(\forall U_{j+1} \subseteq U_j) \ldots \forall n (x \in A^{*V_n}_{k_0\ldots k_n}).$$

(Take $V_{j-1} = G$ if $j = 0$.) We shall prove, more generally than (II), that:

(II′) $$D_{\underline{k}}(V_0, \ldots, V_{j-1}) \subseteq (A^\alpha_{\underline{k}})^{*V_{j-1}} \quad \text{for all } j, V_i, k_i \text{ as above}.$$

We proceed by induction on α. The case $\alpha = 0$ is immediate from the definition (3) of $D_{\underline{k}}$. If α is a limit, then $(A^\alpha_{\underline{k}})^{*V_{j-1}} = \bigcap_{\beta<\alpha} (A^\beta_{\underline{k}})^{*V_{j-1}}$ by (i) and 1.4, so (II′) holds. Now assume (II′), in general, for α, and let $V_0 \supseteq \ldots \supseteq V_{j-1}$ and $(k_0, \ldots, k_{j-1}) = \underline{k}$ be given. By (i) and (1.4), $(A^{\alpha+1}_{\underline{k}})^{*V_{j-1}} = (A^\alpha_{\underline{k}})^{*V_{j-1}} \cap \bigcup_i (A^\alpha_{\underline{k}\frown i})^{*V_{j-1}}$. But, by (2),

$$\begin{aligned}(\bigcup_i A^\alpha_{\underline{k}\frown i})^{*V_{j-1}} &= \bigcap_{U_j \subseteq V_{j-1}} \bigcup_{V_j \subseteq U_j} \bigcup_i (A_{k\frown i})^{*V_j} \\ &\supseteq \bigcap_{U_j \subseteq V_{j-1}} \bigcup_{V_j \subseteq U_j} \bigcup_i D_{\underline{k}\frown i} \quad \text{(by ind. hyp.)} \\ &= D_{\underline{k}} \quad \text{(see (3))}.\end{aligned}$$

Hence $(A^{\alpha+1}_{\underline{k}})^{*V_{j-1}} \supseteq D_{\underline{k}}$, as desired.

Now we will show that

(III) $$\bigcap_\alpha E^*_\alpha \subseteq \bigcup_\alpha (E^*_\alpha \cap (\sim T_\alpha)^*).$$

Suppose $x \in X$. Clearly (i) and (ii) are preserved under inverse image, so that E^x_α and T^x_α are the E_α and T_α for the decomposition $E^x = \bigcup_F \bigcap_n A^x_{F\restriction n}$. Now (using for the only time the countable chain condition for G), we can apply (iv), since the $A_{\underline{k}}$ are normal. We obtain an ordinal β such that $\sim T^x_\beta = (\sim T_\beta)^x$ is comeager, i.e. $x \in (\sim T_\beta)^*$. (III) follows at once.

The circle is completed with

(IV) $$\bigcup_\alpha (E_\alpha - T_\alpha)^* \subseteq E^*.$$

Suppose $x \in (E_\alpha - T_\alpha)^*$ so that $E^x_\alpha - T^x_\alpha$ is comeager. Applying (iii) to the above decomposition of E^x we see that $E^x_\alpha - T^x_\alpha \subseteq E^x$. Hence E^x is comeager and $x \in E^*$, as desired. Thus the proof of 1.6 and 1.7 is complete.

Remark. Assume $\mathcal{K}$ is countable. The game in 1.6 is thus an ordinary infinite game with countable choice sets. It was shown by Moschovakis

[14-17] (see also [27]) that every such game has its own decompositions analogous to but different from those in (i)-(iii) for the operation (A) (a one-man game!). Hence it is natural to ask whether the sets E_α^* and even $(A_{\underline{k}}^\alpha)^*$ correspond in some way to the game decomposition sets for the game in 1.6. That there is indeed such a correspondence was shown by John Burgess. (It is not at first clear just what correspondence to try.)

We described 1.6 (at least for $\mathcal{H}$ countable) as an inductive condition (like 1.4, 1.5). This is justified (in (6) below) by the following well-known fact about infinite closed games over countable choice sets:

(5) Suppose $Q \in K$ if and only if

$$\forall j_0 \exists k_0 \forall j_1 \exists k_1 \ldots \forall n [Q \in K^{j_0 k_0 \ldots j_{n-1} k_{n-1}}].$$

Then K can be obtained by the operation (A) from the sets $K_{\underline{i}}$.

(5) is proved by applying familiar coding procedures to the statement "there is a winning strategy". The argument shows that in fact we obtain $K = \bigcup_F \bigcap_n L_{F \restriction n}$ where $L_{\underline{k}} = K^{H(\underline{k})}$, the function H being primitive recursive.

Using (5), we can establish

(6) If $\mathcal{H}$ is countable then in 1.6 the set E^* can be obtained by (A) from the sets $A_{\underline{k}}^{*V}(\underline{k} \in Sq,\ V \in \mathcal{H})$ plus $\emptyset$ and X.

To see (6), one first puts the game in 1.6 in the form in (5) — by putting the condition $U_i \subseteq V_{i-1}$, $V_i \subseteq U_i$ after $\forall n$ suitably. Then (6) follows at once from (5).

For the rest of § 1 we adopt assumption (0). We now consider consequences of 1.2-1.6. (1.7 will be applied in § 2.)

Σ_1^0 (Π_α^0), defined for $\alpha \geqslant 1$, will denote the Borel classifications. Thus Σ_1^0 (Π_1^0) means open (closed); $\Sigma_{\alpha+1}^0$ ($\Pi_{\alpha+1}^0$) means a countable union (intersection) of sets each of type Π_α^0 (Σ_α^0); and, for limit α, Σ_α^0 (Π_α^0) means a countable union (intersection) of sets each of type Σ_β^0 (Π_β^0) for some $\beta < \alpha$. (Unless every closed set is an F_σ, there are several possible (non-equivalent) definitions at limit ordinals.) Thus $F_\sigma = \Sigma_2^0$ and $G_\delta = \Pi_2^0$, etc.

COROLLARY 1.8. *Assume G has a countable weak basis $\mathcal{H}$. Then:*

(a) *B^* is Borel if B is.*

(b) *B^* (resp. B^Δ) is Π_α^0 (Σ_α^0) if B is.*

(c) *B^* is a $\mathcal{C}$-set if B is.*

1.8 (a) (and, in essence, (b)) was proved by Novikov (cf. [2]) in the special, but typical, case described earlier.

Proof. That (a) holds (for all B^{*U}) follows at once from 1.3-1.5, (written in ^{*U} form, by the remark after 1.1) by induction on B. This is because the union $\bigcup_{V \subseteq U}$ from 1.5 is countable. Similarly (c) follows from 1.3-1.6 in view of (6). (b) follows also from 1.3-1.5 with a little care (by induction on α). Indeed the case $\alpha = 0$ follows from 1.3 and its dual. Suppose B is $\Pi_{\alpha+1}^0$ so that $B = \bigcap_n B_n$ when each B_n is Σ_α^0. Then

$$B^* = \bigcap_n B_n^* = \bigcap_n (\sim \sim B_n^*) = \bigcap_n \bigcap_U \sim (\sim B_n)^{*U} = \bigcap_n \bigcap_U B_n^{\Delta U}.$$

But each $B_n^{\Delta U}$ is Σ_α^0 by inductive hypothesis, so B^* is $\Pi_{\alpha+1}^0$. On the other hand, if B is $\Sigma_{\alpha+1}^0$, then $\sim B$ is $\Pi_{\alpha+1}^0$, so $(\sim B)^*$ is $\Pi_{\alpha+1}^0$, and B^Δ is $\Sigma_{\alpha+1}^0$. Finally, if B is Π_α^0, α limit, then B^* is Π_α^0 by 1.4 and the inductive hypothesis.

Obviously (b) could be extended to a suitable classification of $\mathcal{C}$-sets.

Remark. 1.8 does not fully express the import of 1.3-1.6 because they can be applied to any particular $\mathcal{H}$. In applications to the actions of logic in § 3 we shall consider a specific $\mathcal{H}$ and obtain (in 3.1) additional information. It seems likely that some other actions have a particular $\mathcal{H}$ for which 1.3-1.6 will yield interesting information beyond 1.8.

Let $\mathfrak{I}_1$ (resp. $\mathfrak{I}_2$) be the smallest family containing all clopen sets (all compact G_δ's) and closed under countable union and complement (difference). Using 1.3′ in place of 1.3, and (2), we obtain

COROLLARY 1.8′. *B^* is $\mathfrak{I}_1$ ($\mathfrak{I}_2$) if B is.*

There is one more result, related to 1.6 and 1.7, and analogous to (iv):

THEOREM 1.9. *Suppose G has a countable weak basis, X satisfies the countable chain condition, $E = \bigcup_F \bigcap_n A_{F \restriction n}$, and each $A_{\underline{k}}$ is $\mathcal{C}$ in X'. Then there is an α such that $(\sim T_\alpha)^*$ is comeager (and hence $(E_\alpha - T_\alpha)^*$ is comeager in E^*).*

In Burgess and Miller [29], 1.9 is extended to arbitrary A_k having the Baire property, assuming the action case.

Proof. Here we shall not use directly the old results (iii), (iv) as we did in proving 1.7, but we shall imitate their proofs. By hypothesis we can take $\mathcal{H}$ to be countable. By (ii) and 1.4, 1.5,

$$(\sim T_\alpha)^* = \bigcap_{\underline{k}} \bigcap_U [\sim ((A_{\underline{k}}^\alpha)^{*U} \cap (\sim A_{\underline{k}}^{\alpha+1})^{*U})].$$

By 1.4, $(\sim B)^* \subseteq \sim B^*$. Using this and taking complements,

$$\sim (\sim T_\alpha)^* \subseteq \bigcup_{\underline{k}} \bigcup_U [(A_{\underline{k}}^\alpha)^{*U} - (A_{\underline{k}}^{\alpha+1})^{*U}]. \tag{7}$$

Let $\underline{k}$ and U be fixed. Clearly the sets $(A_{\underline{k}}^\alpha)^{*U}$ ($\alpha < \omega_1$) are descending; so the sets $S_\alpha = (A_{\underline{k}}^\alpha)^{*U} - (A_{\underline{k}}^{\alpha+1})^{*U}$ are pairwise disjoint. By 1.8 (c) each S_α is $\mathcal{C}$ and hence has the Baire property. Hence (cf. [9]), since X has the countable chain condition, only countably many S_α are non-meager.

So there is an ordinal $\beta(\underline{k}, U)$ such that S_a is meager for all $\alpha \geqslant \beta(\underline{k}, U)$. Since there are countably many $(\underline{k}, U)$, we can take α greater than all $\beta(\underline{k}, U)$. By (7), $\sim(\sim T_a)^*$ is meager, as desired.

We shall discuss the action case in the rest of the paper, so it may be worthwhile in ending § 1 to state some of its results specifically for the product case. Corollary 1.10 includes and extends the work of Novikov (cf. [2]).

COROLLARY 1.10. *Suppose G is a Baire topological space with a countable weak basis and X is a topological space. For $B \subseteq G\times X$ put $B^* = \{x \in X\colon \{g \in G\colon (g, x) \in B\}$ is comeager in $G\}$. Then*

(a) $\{x \in X\colon \forall g((g, x) \in B)\} \subseteq B^* \subseteq \{x \in X\colon \exists g((g, x) \in B)\}$.

(b) *If B is Π^0_α, Borel, or $\mathcal{C}$, so is B^*.*

(c) *If $E = \bigcup_F \bigcap_n A_{F\restriction n}$ where each $A_{\underline{k}}$ is $\mathcal{C}$ in $G\times X$, then $E^* = \bigcap_a E^*_a = \bigcup_a (E_a - T_a)^*$.*

§ 2. Action. We now assume the action case, so that G acts on $X = X'$. We also assume that G *has a countable weak basis* $\mathcal{K}$.

THEOREM 2.1. (a) *B^* and B^Δ are invariant (and $B^- \subseteq B^* \subseteq B^\Delta \subseteq B^+$, by* 1.2).

(b) *Hence $B = B^*$ if and only if B is invariant.*

(c) *If $x \in B^{*U}$ then $gx \in B^{*(Ug^{-1})}$.*

(d) *B^{*U} is invariant under the (induced) action of the group $G_U = \{g\colon Ug = U\}$ (a closed subgroup of G).*

(e) $(gB)^* = B^*$.

Proof. (b) follows from (a) and 1.2, while both (a) and (d) follow at once from (c). For (c) and (e), we begin by verifying the formulas

$$B^{gx} = B^x g^{-1} \quad \text{and} \quad (gB)^x = gB^x.$$

For the first, note the equivalence of: $h \in B^{gx}$, $hgx \in B$, $hg \in B^x$, and $h \in B^x g^{-1}$. For the second, note the equivalence of $h \in (gB)^x$, $hx \in gB$, $g^{-1}hx \in B$, $g^{-1}h \in B^x$, $h \in gB^x$.

Now for (c), if $x \in B^{*U}$ then B^x is comeager in U, so $B^x g^{-1} = B^{gx}$ is comeager in Ug^{-1}, since a right or left translation is an autohomeomorphism of G. Hence $gx \in B^{*(Ug^{-1})}$, as desired. For (e), consider the equivalent conditions: $x \in (gB)^*$, $(gB)^x = gB^x$ is comeager, B^x is comeager, $x \in B^*$.

In the action case, because of 2.1, we can draw inferences from § 1 which do not involve $*$. The first is a kind of interpolation:

THEOREM 2.2. *If B is Borel ($\mathcal{C}$, Π^0_α, Σ^0_α) then there exists an invariant set C such that $B^- \subseteq C \subseteq B^+$ and C is also Borel ($\mathcal{C}$, Π^0_α, Σ^0_α).*

Proof. By 1.8 and 2.1 we can take $C = B^*$ except in the Σ^0_α case, where we take $C = B^\Delta$.

Note that if B is closed, then we can take $C = B^-$, which is known to be closed. On the other hand, 2.2 for F_σ or G_δ already seems to be new.

2.2 implies at once the statement about relativization mentioned in the introduction:

COROLLARY 2.3. *Suppose Y is an invariant subset of X, considered as a subspace, and $B \subseteq Y$. Then B is invariant Borel ($\mathcal{C}$, Π^0_α, Σ^0_α) in Y if and only if $B = B' \cap Y$ for some B' which is invariant Borel ($\mathcal{C}$, Π^0_α, Σ^0_α) in X.*

As usual, we call $(A, \mathfrak{F})$ a *Borel space* if $\mathfrak{F}$ is a non-empty family of subsets of A closed under complement and countable union. Clearly the invariant Borel subsets of X form a Borel space. 2.3, for Borel sets, says that the invariant Borel sets for Y are induced from the invariant Borel sets for X in the standard way for forming sub-Borel spaces (cf., e.g., [11]).

A familiar example of an action is obtained by taking $X = G$ and $J(g, h) = ghg^{-1}$. 2.2 and 2.3 may be of interest even in this case.

§ 1 also tells us something about invariant meager sets:

THEOREM 2.4. *If $Y \subseteq X$ is invariant meager, then Y is included in a countable union of closed, nowhere dense sets, each of which is G_U-invariant for some U.*

Proof. By hypothesis $Y \subseteq \bigcup_n C_n$, where each C_n is closed, nowhere dense. Hence $Y = Y^\Delta \subseteq \bigcup_n C^\Delta_n = \bigcup_n \sim(\sim C_n)^* = \bigcup_n \bigcup_U C^{*U}_n$ (by 1.2, 1.4, 1.5). Now some g belongs to U, and hence $C^{*U}_n = C^{-U}_n$ (by 1.3) $\subseteq gC_n$. But gC_n is nowhere dense as a translate of C_n. Thus each C^{*U}_n is closed, nowhere dense, and, by 2.1 (d), is G_U-invariant.

It is proved in [29] that if $B \subseteq X$ is meager (or is a set with the Baire property) then so is B^*.

In the remainder of § 2 we are going to apply 1.7 to invariant sets. In this way we shall obtain invariant forms of some classical theorems of descriptive set theory. These results — 2.5, 2.7, and 2.8 — were already obtained in the logic spaces (i.e., those in § 3) in work of Moschovakis [14-17] and the author [27].

A topological space E is *analytic* if it is metrizable and is a continuous image of a Polish space. If E is analytic, $E \subseteq X$ and X is metrizable, then E is obtainable by the operation (A) from Borel, and even closed subsets of X. (The converse holds if X is Polish.) The next theorem (and 2.5′ below), with invariant omitted, is a classical theorem (cf. [9]):

THEOREM 2.5. *If E is invariant and E is obtainable by the operation* (A) *from Borel sets, then E is the union of $\aleph_1$ invariant Borel sets.*

Proof. Let $E = \bigcup_F \bigcap_n A_{F\restriction n}$, where each $A_{\underline{k}}$ is Borel. By 1.7, $E = E^* = \bigcup_a (E_a - T_a)^*$. As is known (cf. (i), (ii)), $E_a - T_a$ is Borel, so by 1.8 and 2.1 (a), $(E_a - T_a)^*$ is Borel, for each α.

COROLLARY 2.6. *If X is metrizable and G is Polish, then every orbit $\{gx\colon g \in G\}$ is Borel.*

Proof. Let $E = \{gx\colon g \in G\}$. Then E is a continuous image (under $g \mapsto gx$) of G and hence (see above), since X is metrizable, E is obtainable by $\mathcal{A}$ from Borel sets. Therefore, by 2.5, E is a union of invariant Borel sets; but as an orbit E is a minimal invariant set, so must be one of these.

2.6 is due to C. Ryll-Nardzewski [22], whose proof used the quite different method of selectors. A related result by that same method was obtained by Dixmier [5]. For the logic spaces, 2.6 had been obtained by D. Scott [23, 24]. It appears that our method and the method of selectors yield overlapping results. Indeed the latter method does not seem to yield 2.5. On the other hand, Ryll-Nardzewski showed that in 2.6 the orbit is absolutely Borel (Borel in the completion of X) and our method does not seem to yield this.

We assume for the rest of § 3 that both G and X are Polish (hence Baire, cf. [9]). Such actions will be called Polish. We use the notations Σ_1^1, Π_1^1, Σ_2^1 for analytic, coanalytic, and PCA of [9].

THEOREM 2.5′. *Any invariant Σ_2^1 subset B of X is the union of $\aleph_1$ invariant Borel sets.*

Proof. By the classical version of 2.5′, $B = \bigcup\limits_{a} B_a$ where each B_a is Borel (but not invariant). Then $B = B^+ = \bigcup\limits_{a} B_a^+$. But B^+ is invariant analytic (see below) so we can apply 2.5 to each B_a^+, obtaining 2.5′.

We just used the well-known fact that if C is Borel (or analytic) then $C^+ = \{x\colon \exists g\; gx \in C\}$ is analytic. This is obvious if J is fully continuous. When J is only continuous in each variable, it is known (cf. [9]) that J is a Borel function; hence again C^+ is clearly analytic.

A family $\mathfrak{K}$ of sets has the *reduction property* if for any $A_1, A_2 \in \mathfrak{K}$ there exist $A_1', A_2' \in \mathfrak{K}$ which reduce A_1, A_2, i.e., $A_1' \subseteq A_1$, $A_2' \subseteq A_2$, $A_1' \cap A_2' = \emptyset$, and $A_1' \cup A_2' = A_1 \cup A_2$. The reduction property for $\mathfrak{K}$, quite generally, implies certain separation principles for $\mathfrak{K}$ (cf. [9]). There is also a stronger reduction property involving a sequence $A_1, A_2, \ldots, A_n, \ldots$ in place of A_1, A_2 (cf. [9]); everything below is easily improved to give the stronger reduction property.

THEOREM 2.7. *The reduction property holds for the class of invariant Π_1^1 sets, and also for the class of invariant Σ_2^1 sets.*

The author proved 2.7 by using 1.7 in place of (iii) in the classical proofs, relying on the fact that B^{*U} has an inductive definition (by 1.3-1.5). From these remarks the proofs are an exercise, but a somewhat tedious one. (The argument for Π_1^1 is sketched below.) Subsequently D. Miller found a much shorter proof for Π_1^1, still using the $*$-transform. Then J. Burgess found a short proof for Π_1^1, directly from the classical analogue, with no use of $*$. Finally, Miller found an even shorter proof for Σ_2^1. All three proofs are in [29]. These proofs are also much simpler than those of [14-17] and [27] establishing 2.7 for the logic spaces. They also yield results more general than 2.8 (see [29]).

Since these short proofs exist our original proof might well be altogether omitted. However there is some possibility that the method might yet yield information not otherwise obtainable, so we shall give a sketch of the proof for Π_1^1 (from which the full proofs for Π_1^1 and Σ_2^1 are easily obtainable).

Proof for Π_1^1. Let D_1 and D_2 be invariant Π_1^1. Put $E_i = {\sim} D_i$ $(i = 1, 2)$ and $E_i = \bigcup\limits_{F\cdot} \bigcap\limits_{n} {}^iA_{F\restriction n}$ (Borel), so that $D_i = \bigcup\limits_{a} {\sim}\, {}^iE_a^*$, by 1.7. Put

$$D_1' = \{x\colon\ x \in D_1 \ \text{and}\ \mu a x \notin {}^1E_a^* \leqslant \mu a x \notin {}^2E_a^*,\ \text{if the latter exists}\}$$

D_1' is defined symmetrically, but with $<$ in place of $\leqslant$. Clearly D_1' and D_2' are invariant and reduce D_1 and D_2. We show that D_1' is Π_1^1, the case of D_2' being analogous. By (i) and (2),

$$(8)\qquad x \in ({}^iA_{\underline{k}}^{a})^{*U} \leftrightarrow x \in {}^iA_{\underline{k}}^{*U} \wedge (\forall \beta < a)(\forall V \subseteq U)(\exists W \subseteq V)(\exists j)\, x \in ({}^iA^{\beta}_{\underline{k}\frown j})^{*W}.$$

Now if R well-orders ω, it determines an ordinal a, and hence (8) can be translated from a to R. Thus, letting Q range over the Polish space $2^{Sq\times\mathcal{K}\times\omega}$, we form the Borel sets:

$$\begin{aligned} Z_i = \{(R, Q, x)\colon\ & R \text{ orders } \omega \wedge (\forall k, U, a)[(k, U, a) \in Q \\ & \leftrightarrow x \in {}^iA_{\underline{k}}^{*U} \wedge (\forall b R a)(\forall V \subseteq U)(\exists W \subseteq V)(\exists j)(\underline{k}\frown j, W, b) \in Q]\}. \end{aligned}$$

The proof is completed by showing that

$$\begin{aligned} (9)\quad x \in D_1' \leftrightarrow x \in D_1 \wedge (\forall R, a, Q, Q')[(R, Q, x) \in Z_1 \wedge (R, Q', x) \in Z_2 \wedge \\ \wedge (\emptyset, G, a) \in Q \rightarrow (\emptyset, G, a) \in Q']. \end{aligned}$$

Clearly (9) shows D_1' is Π_1^1. From right to left in (9) is obvious. If $x \in D_1'$ $(\subseteq D_1)$, $(R, Q, x) \in Z_1$, $(R, Q', x) \in Z_2$, and $(\emptyset, G, a) \in Q$, then R must be a well-ordering below a (from which it clearly follows that $(\emptyset, G, a) \in D')$. Indeed, if not, it is easy to prove that $x \in E_1 = E_1^*$ (a contradiction) by showing (from the definition of Z_1) that the game in 1.6 can be played.

In invariant form the well-known first separation principle becomes:

(10) Disjoint invariant analytic sets E_1 and E_2 can be separated by an invariant Borel set.

This result (which follows from 2.7) was obtained by Ryll-Nardzewski about twelve years ago (cf. [10]). It was proved by the following argument: By classical separation, obtain B_1 Borel such that $E_1 \subseteq B_1 \subseteq {\sim} E_2$. Then

B_1^+ is invariant analytic and $B_1 \subseteq B_1^+ \subseteq \sim E_2$. Repeating, we get $B_1^+ \subseteq B_2 \subseteq B_2^+ \subseteq \sim E_2$; and so on. Then $\bigcup_n B_n = \bigcup_n B_n^+$ is invariant Borel and separates E_1, E_2. (In [29] it is shown that invariant reduction (2.7) for Π_1^1 can in fact be proved by a similar argument.)

The same type of argument gives, as is known, an invariant form of the covering theorem (in [9]). However, the following result may be a little stronger, as the sets E_a^* are formed in a simple way.

THEOREM 2.8 (Covering). *Suppose E is invariant analytic and in fact $E = \bigcup_F \bigcap_n A_{F\restriction n}$, each $A_{\underline{k}}$ being Borel. If D is (invariant) analytic and $D \subseteq \sim E$ then for some a, $D \subseteq \sim E_a^*$.*

Proof. By the classical theorem, $D \subseteq \sim E_a$ for some a. Hence $D = D^* \subseteq (\sim E_a)^* \subseteq \sim E_a^*$.

2.5, 2.6 (of Ryll-Nardzewski), and 2.7 extend to all (or at least all Polish) actions results first proved for the logic spaces (in [24], [14-17], and [26]) by methods peculiar to those spaces. It is natural to ask how many other theorems about $L_{\omega_1\omega}$ can be extended to all Polish actions. In some cases the problem may be initially to obtain an equivalent statement in purely action-theoretic terms. Sometimes, however, that statement is quite clear, but it is not known whether it can be extended to all Polish actions. A perfect example is Morley's Theorem [13]: *In a logic action, if E is invariant analytic then the number of orbits included in E is at most $\aleph_1$ or else $2^{\aleph_0}$.* Does this hold in any Polish action?

Added in proof. Burgess and Miller [29] observed that the conjecture at the end of § 2 would fellow from an earlier conjucture of H. Friedman. Now, in his thesis (Berkeley, 1973), Burgess has established Friedman's conjecture. He makes use of a recent result of J. Silver concerning Π_1^1 equivalence relations.

§ 3. Logic. Suppose I and J are disjoint sets and $\varrho'\colon I \to \omega$. Then $\varrho = (\varrho', J)$ is called a *similarity type*. For any set $A \neq 0$, $X_{\varrho,A}$ is the topological space $\prod_{i \in I} 2^{A^{\varrho'_i}} \times A^J$. A pair $(A, \underline{S})$ where $\underline{S} \in X_{\varrho,A}$ is called a *ϱ-structure with universe* A. $\mathfrak{X}_\varrho$ is the class of all ϱ-structures (over all A). X_ϱ will be the space $X_{\varrho,\omega}$ (where $A = \omega$) with the product topology. Its members $\underline{S}$ are often identified with the structures $(\omega, \underline{S})$. One can consider the notationally simpler case $X_\varrho = 2^{\omega\times\omega}$ to be typical.

Let G be the group $\omega!$ of all permutations of ω. Under the product topology, G is a Polish topological group which acts continuously on X_ϱ, $g\underline{S}$ being the usual isomorph of $\underline{S}$ under g. If X is any invariant subspace of X_ϱ, the induced action of G on X (a "logic action") meets the conditions at the beginning of § 2. Moreover if X is G_δ in X_ϱ and $I \cup J$ is countable, then X is Polish.

Logic deals primarily with sets $B \subseteq X$ which are closed under isomorphism, i.e., invariant. What is special about the logic actions is the availability of an inductive method for forming invariant Borel sets, at least if we pass to the spaces $X_\varrho, X_\varrho \times \omega, X_\varrho \times \omega^2, \ldots$ (Note that $X_\varrho \times \omega^n$ is itself one of our logic actions, i.e. it is $X_{\bar\varrho}$ for a suitable $\bar\varrho$). This inductive method is just given by the infinitary language $L_{\omega_1\omega}$ which we shall now describe. In the unusual but not vacuous case that $I \cup J$ is uncountable, our treatment of $L_{\omega_1\omega}$ will differ from former treatments.

The symbols of our language are $\boldsymbol{R}_i$ (a ϱ'_i-ary relation symbol) for $i \in I$, c_j (an individual constant) for $j \in J$, variables v_n $(n < \omega)$, and $\approx$, $\sim$, $\bigwedge$, $\forall$. *Atomic formulas* are $\tau_1 \approx \tau_2$ and $R_i\tau_1 \ldots \tau_{\varrho'_i}$, where each τ_k is a v_n or a c_j. A *closed formula* is an arbitrary (possibly uncountable) conjunction $\bigwedge_{t \in T} \theta_t$ such that each θ_t is of the form $\forall u_0 \ldots u_{k-1} M$, where M is a finite disjunction of atomic formulas and their negations. Here and below *conjunctions can only be formed which have finitely many free variables.* The class of *formulas* is the smallest containing all closed formulas and containing $\sim\varphi$, $\forall v_k\varphi$, and $\bigwedge_n \varphi_n$ (a countable conjunction) whenever it contains $\varphi, \varphi_0, \varphi_1, \ldots$, ($\sim\varphi$, $\bigwedge_{t \in \tau} \theta_t$, etc. are our names for certain possibly infinite formal expressions. For just what those expressions might be, see, e.g., [3].) An *n-formula* is one whose free variables are among $v_0, \ldots, v_{n-1}$; clearly every formula is an n-formula for some n. θ, φ, ψ always denote formulas. Sentences (denoted by σ) are just 0-formulas. (Of course, the connectives $\wedge$, $\bigvee$, E, etc. are understood in the usual way.)

Let $(A, \underline{S})$ be a ϱ-structure. If φ is an n-formula, and $a_0, \ldots, a_{n-1} \in A$, then $(A, \underline{S}) \vdash \varphi[a_0, \ldots, a_{n-1}]$ means that $a_0, \ldots, a_{n-1}$ satisfy φ in (A, S) (in the obvious sense). Let $\mathfrak{X} \subseteq \mathfrak{X}_\varrho$, and write $\mathfrak{X}^{(n)}$ for $\{(A, \underline{S}, a_0, \ldots, a_{n-1})\colon (A, \underline{S}) \in \mathfrak{X}$ and $a_0, \ldots, a_{n-1} \in A\}$. The set of $\mathfrak{X}$-models of φ, denoted by $\mathrm{Mod}^n_{\mathfrak{X}}\varphi$, is the set of all $(A, \underline{S}, a_0, \ldots, a_{n-1}) \in \mathfrak{X}^{(n)}$ such that $(A, \underline{S}) \vdash \varphi[a_0, \ldots, a_{n-1}]$. We write $\mathrm{Mod}_{\mathfrak{X}}\sigma = \mathrm{Mod}^0_{\mathfrak{X}}\sigma$. If $X \subseteq X_\varrho$ we consider also that X (really $\{\omega\} \times X) \subseteq \mathfrak{X}_\varrho$, and we consider $\mathrm{Mod}^n_X\varphi \subseteq X \times \omega^n$.

We now fix an arbitrary class $\mathfrak{X} \subseteq \mathfrak{X}_\varrho$ which is weakly closed under isomorphism, that is, if $(A, \underline{S}) \in \mathfrak{X}$, $(A, \underline{S}) \cong (A', \underline{S}')$ and $(A', \underline{S}'') \in \mathfrak{X}$ then $(A', \underline{S}') \in \mathfrak{X}$. We write $\mathrm{Mod}^n\varphi$ for $\mathrm{Mod}^n_{\mathfrak{X}}\varphi$. (Later we shall usually consider $\mathfrak{X} = X$, an invariant subset of X_ϱ. Then action notions will also be available. But it is just an important feature of the logical notions that they can be applied to the whole $\mathfrak{X}_\varrho$ and not just to X_ϱ.)

Classes of the form $\mathrm{Mod}\,\sigma$ are called *Borel'* or $L_{\omega_1\omega}$ classes (over $\mathfrak{X}$). Note that the Borel' classes of $\mathfrak{X}^{(n)}$ can also be described as all sets $\mathrm{Mod}^n\varphi$. Classes $\mathrm{Mod}\,\sigma$ where σ is closed' are called *closed'* classes and their complements are called *open'*.

There is a natural classification Σ'^0_a (Π'^0_a) $(a \geqslant 1)$ of Borel' classes (over $\mathfrak{X}$). Π'^0_α is just all closed′ classes. Σ'^0_α is the family of complements of Π'^0_α classes. $B \subseteq \mathfrak{X}$ is $\Pi'^0_{\alpha+1}$ if $B = \mathrm{Mod}\bigwedge_n (\forall v_0 \dots v_{k_n-1})\varphi_n$ where each $\mathrm{Mod}^{k_n}\varphi_n (\subseteq X \times \omega^{k_n})$ is Σ'^0_α. For limit α, B is Π'^0_α if $B = \bigcap_n B_n$ where each B_n is Π'^0_β for some $\beta < \alpha$.

Closed′ classes are just the UC_Δ classes of Tarski [26]. The G'_δ (i.e., Π'^0_2) classes have been considered by Keisler [8].

We now assume that $\mathfrak{X} = X \subseteq X_\varrho$.

It is easy to check and known for forty years (cf. [9]) that

(11) Borel′ implies invariant Borel, and indeed $\Sigma'^0_\alpha(\Pi'^0_\alpha)$ implies invariant $\Sigma^0_\alpha(\Pi^0_\alpha)$.

It is also easy and known that invariant closed = closed′ (though this will follow from 3.2 below).

By now several primed notions — closed′, Borel′, etc. — have been introduced and more will be, below. There are two important features shared by all these notions. Firstly (over X), a set which is $(P)'$ is formed by some process which clearly yields in general only invariant (P) sets. Thus $(P)'$ can be read: inherently invariant (P). Secondly, the notion $(P)'$ over X is a special case of a natural notion "$(P)'$ over $\mathfrak{X}$", uniformly defined for arbitrary $\mathfrak{X}$. (To make this more precise one would have to consider all the $\mathfrak{X}^{(n)}$, taken as topological spaces under the open classes, together with many canonical projection maps: $\mathfrak{X}^{(n)} \to \mathfrak{X}^{(m)}$, to form a multispace in terms of which the definition of $(P)'$ can be given.) We are going in each particular case to establish (usually non-trivially) that, in fact, over X, invariant $(P) = (P)'$.

We shall be able to obtain information about $L_{\omega_1\omega}$ over X by using the methods of § 1 while taking a particular basis $\mathcal{K}$ for G. Let s, t always denote finite sequences $\underline{k}$ of natural numbers which are non-repeating. Put $[s] = \{g\colon s \subseteq g^{-1}\}$. For $\mathcal{K}$ we take the well-known basis consisting of all $[s]$.

It will be convenient to deal with the following variant of *. If $B \subseteq X$, put $B^{(*n)} = \{(\underline{S}, s) \in X \times \omega^n\colon \underline{S} \in B^{*[s]}\}$, and similarly for $B^{(\Delta n)}$. It is easy to verify that $B^{(*n)}$ is invariant in $\mathfrak{X} \times \omega^n$.

Write $(\exists v_m \dots v_{n-1})^{\neq}\varphi$ for $(\exists v_m \dots v_{n-1})(\varphi \wedge \bigwedge_{i<j<n} v_i \neq v_j)$.

We now obtain an improvement of 1.8 (a), (b), by repeating its proof using now our special $\mathcal{K}$.

LEMMA 3.1. *If $B \subseteq X$ is Borel than $B^{(*n)}$ is Borel′. Indeed, if B is $\Sigma^0_\alpha(\Pi^0_\alpha)$ then $B^{(\Delta n)}(B^{(*n)})$ is $\Sigma'^0_\alpha(\Pi'^0_\alpha)$ $(\alpha \geqslant 1)$.*

Proof. (a) Suppose $B = \{\underline{S}\colon (\omega, \underline{S}) \vdash \theta[0, 1, \dots, p-1]$, where θ is a finite disjunction of atomic p-formulas or their negations. (Thus B is a basic closed set.) Then $B^{(*n)}$ is closed′. Indeed, since $B^{*[s]} = B^{-[s]}$ by 1.3, it is an exercise to show that $B^{(*n)} = \mathrm{Mod}(\forall v_n \dots v_{p-1})^{\neq}\theta$.

If B is closed then $B = \bigcap_{t\in T} B_t$ where each B_t is like B just above, and hence $B_t^{(*n)}$ is closed′. Now, $(\underline{S}, s) \in B^{(*n)}$ if and only if $\underline{S} \in B^{*[s]} = B^{-[s]} = \bigcap_{t\in T} B_t^{-[s]}$, so $B^{(*n)} = \bigcap_{t\in T} B_t^{(*n)}$. Thus $B^{(*n)}$ is closed′.

(b) If $B^{(*n)} = \mathrm{Mod}^n\varphi_n$ for each $n \in \omega$, then, by 1.5,

$$(\sim B)^{*n} = \{(\underline{S}, s) \in X \times \omega^n\colon \underline{S} \in \sim \bigcup_{m\geqslant n} \bigcup_{s \subseteq t \in \omega^m} B^{*[t]}\}$$

$$= \mathrm{Mod}^n \sim \bigvee_{m\geqslant n} (\exists v_n \dots v_{m-1})^{\neq}\varphi_m = \mathrm{Mod}^n \bigwedge_{m\geqslant n} (\forall v_n \dots v_{m-1})^{\neq} \sim\varphi_m\,.$$

(c) Suppose $B = \bigcap_p B_p$. Then, by 1.4, $B^{(*n)} = \bigcap_p B_p^{(*n)}$. Moreover, if B is Π^0_α, α limit, and each B_p is $\Pi^0_{\beta_p}$, $\beta_\varrho < \alpha$, then (by inductive hypothesis) $B_p^{(*n)}$ is $\Pi'^0_{\beta_p}$ and $B^{(*n)}$ is Π'^0_α. On the other hand, suppose $\alpha = \beta+1$, and each B_p is Σ^0_β. Then (by inductive hypothesis) $(\sim B_p)^{(*n)} = \mathrm{Mod}^n\varphi_{pn}$ is Π'^0_β. Hence $B_p^{(*n)} = (\sim\sim B_p)^{(*n)} =$ (by (b)) $\mathrm{Mod}^n \bigwedge_{m\geqslant n} (\forall v_n \dots v_{n+1})^{\neq} \sim\varphi_{pm}$. Hence $B^{(*n)}$ is Π'^0_α.

From (a)-(c), everything in 3.1 follows.

Taking $B = B^*$ in 3.1 we obtain at once:

THEOREM 3.2. *Invariant $\Sigma^0_\alpha(\Pi^0_\alpha) = \Sigma'^0_\alpha(\Pi'^0_\alpha)$ $(\alpha \geqslant 1)$.*

COROLLARY 3.3. *Invariant Borel = Borel′ $(= L_{\omega_1\omega})$.*

Lopez-Escobar [10] proved that 3.3 holds — under the additional assumptions that $I \cup J$ is countable and X is analytic (call this 3.3′). His proof of 3.3′ depended on his Interpolation Theorem [10] (for $I \cup J$ countable):

(IT) If $\vdash \sigma(\underline{S}, \underline{T}) \to \sigma'(\underline{S}, \underline{T}')$ then, for some sentence $\theta(\underline{S})$, $\vdash \sigma \to \theta$ and $\vdash \theta \to \sigma'$. ($\vdash \theta$ means that $\underline{S} \vdash \theta$ for all $\underline{S} \in X_\varrho$.)

In fact before (IT) was proved, it was known that (IT) implies 3.3′ (almost at once), and, on the other hand, Ryll-Nardzewski had observed via (10) above that conversely 3.3′ implies (IT) (cf. [10]).

Thus our direct proof of 3.3 (via (10)) gives what seems to be the first proof of (IT) from its predecessor, the classical first separation principle.

The strong form of 3.3 given in 3.2 can apparently not be obtained by the (IT)-proof (since the interpolant formula is obtained in a very indirect way from the given formulas). 3.2 at the lowest new level says that invariant $G_\delta = G'_\delta$ (i.e., Π'^0_2).

Even 3.3 itself generalizes Lopez-Escobar's result by allowing any invariant $X \subseteq X_\varrho$ and by allowing $I \cup J$ to be uncountable. Since in either case, the first separation principle (and hence (IT)) may fail, it appears that the old proof cannot work. Incidentally, instead of dealing all along with arbitrary X we could have inferred 3.3 for any X from the case $X = X_\varrho$ by using Corollary 2.3.

We now give a further result closely related to 3.2 for G_δ. For a moment, consider again a class $\mathfrak{X} \subseteq \mathfrak{X}_\varrho$ as above. A class $\mathfrak{Y} \subseteq \mathfrak{X}$ is called *meager'* if $\mathfrak{Y}$ is closed under isomorphism (in $\mathfrak{X}$) and $\mathfrak{Y} \subseteq \mathrm{Mod} \bigvee_n \exists v_0 \ldots v_{k_n-1} \varphi_n$, where each $\mathrm{Mod}^{k_n}\varphi_n$ is closed' and includes no non-empty set open' in $\mathfrak{X}^{(k_n)}$. (In case $\mathfrak{X} = X \subseteq X_\varrho$, our condition on $\mathrm{Mod}^{k_n}\varphi_n$ clearly just means that $\mathrm{Mod}^{k_n}\varphi_n$ is closed and nowhere dense.) The notion meager' occurs implicitly in the Henkin-Orey ω-completeness or omitting types theorem (see below). Now return to $\mathfrak{X} = X \subseteq X_\varrho$. As was observed in [6] (for a less general case), it is clear that: meager' implies invariant meager. From the $*$-transform we obtain the converse:

THEOREM 3.4. *Invariant meager = meager'.*

Proof. By the proof of 2.4, if Y is invariant meager, then $Y \subseteq \bigcup_n \bigcup_p \bigcup_{s \in \omega^p} C_n^{*[s]}$, where each $C_n^{*[s]}$ is closed and nowhere dense. Now $C_n^{(*p)}$ is closed and hence can be written $\mathrm{Mod}\varphi_{np}$; moreover each of it projections $C_n^{*[s]}$ is nowhere dense, so it is nowhere dense. Thus we have $Y \subseteq \mathrm{Mod} \bigvee_{n,p} (\exists v_0 \ldots v_{p-1})^{\neq} \varphi_{np}$ where each $\mathrm{Mod}^p \varphi_{np}$ is closed and nowhere dense, as desired.

For certain special X, related to complete first order theories Suzuki [25] made a detailed study of which orbital sets are meager or comeager. His work yields at once certain special cases of 3.4.

Although it is not connected with the new direction in 3.4, we shall say a few words about the

ω-COMPLETENESS THEOREM. *If X is G'_δ in X_ϱ then X is not meager' in itself.*

This result, which has been successively generalized several times from the original Henkin-Orey version, was given in the above form, for $I \cup J$ countable in Keisler [8]. Since trivially meager' implies meager and G'_δ implies G_δ, it follows at once from the Baire category theorem for X. (This proof — in a less general case — was first given in [6].) Notice that the theorem is correct with our definitions even if $I \cup J$ is uncountable, since the space X_ϱ is compact Hausdorff so the Baire theorem still holds. However, this result may be deceptive, as we shall see in a moment, for a situation which ought to be mentioned anyway.

The ω-completeness theorem is usually applied in an apparently different form (see, e.g., [6]). We consider a set Σ of arbitrary first-order ϱ-sentences and sets Φ_n of arbitrary first-order k_n-formulas. Under a certain hypothesis (related to meager-ness) we assert that

$$\mathrm{Mod}\,\Sigma \not\subseteq \mathrm{Mod} \bigvee_n \exists v_0 \ldots v_{k_n-1} \bigwedge_{\Theta \in \Phi_n} \Theta .$$

However, as is well-known, if we pass to a larger similarity type by introducing many definitions of the form

$$\forall v_0 \ldots v_m [P v_0 \ldots v_m \leftrightarrow \exists v_m \ldots v_m M(v_0 \ldots v_m)]$$

where M is finite, quantifier free, then our problem exactly reduces to the situation in the ω-completeness theorem above. (For details, see, e.g., [8]). This reduction, when $I \cup J$ is uncountable, involves uncountably many $\forall\exists$ sentences as above, and thus fails, since it reduces to an X which is G'_Δ rather than G'_δ.

Again consider $\mathfrak{X}$. $\mathfrak{Y} \subseteq \mathfrak{X}$ is said to have the *Baire' property* if $\mathfrak{Y}$ is closed under isomorphism (in $\mathfrak{X}$) and $\mathfrak{Y}$ symmetrically differs from some closed' class by a meager' class. Now return to $\mathfrak{X} = X \subseteq X_\varrho$.

COROLLARY 3.5. *Invariant + Baire property = Baire' property.*

Proof. As always from right to left is obvious. If Y is invariant and has the Baire property then $Y \equiv D(Y)$ (mod meager sets). (See [9] for the notion $D(Y)$.) $D(Y)$ is closed. Moreover, the definition of $D(Y)$ ensures that (Y being invariant) $D(Y)$ is invariant (cf. [20]). Hence $D(Y)$ is closed'. Finally, $Y \Theta D(Y)$ is invariant meager, so meager' by 3.4; thus Y has the Baire' property.

It now follows easily from the classical theorem "(A) preserves the Baire' property" that

COROLLARY 3.6. *The game operation preserves the Baire' property. That is, suppose for each n and $\underline{k} \in \omega^n$, $B^{\underline{k}}$ has the Baire' property in $X \times \times \omega^{2n}$. Let*

$$C = \{\underline{S}\colon \forall m_0 \exists p_0 \exists k_0 \forall m_1 \exists p_1 \exists k_1 \ldots$$
$$\ldots \forall_n \big((\underline{S}, m_0, p_0, \ldots, m_{n-1}, p_{n-1}) \in B^{k_0 \ldots k_{n-1}}\big)\} .$$

Then C has the Baire' property in X.

Proof. C is invariant and by (5) in § 1 plus the classical theorem clearly has the Baire property, and hence, by 3.5, the Baire' property.

We shall discuss 3.6 again at the end of this section.

When the game operation (formalized) is added to the language $L_{\omega_1\omega}$, the much richer language $L_{\omega_1 G}$ is obtained, which has been studied by Moschovakis, Barwise [4], and others. The language $L_{\omega_1 G}$ (as we take it) has the symbols of $L_{\omega_1\omega}$ plus a new variable-binding operator $\boldsymbol{G}$. The class of *formulas* (of $L_{\omega_1 G}$) is the smallest containing all closed' formulas and closed under the formation of $\sim\varphi$, $\forall v_k \varphi$, $\bigwedge_n \varphi_n$ (with the old restriction) and also of $G^{\underline{k}} u_0 u'_0 \ldots u_n u'_n \ldots \varphi_{\underline{k}}$ — where $u_0, u'_0, u_1, \ldots$ are distinct variables and there are variables $w_0, \ldots, w_q$ such that each $\varphi_{\underline{k}}$, for $\underline{k} \in \omega^n$, has its free variables among $w_0, \ldots, w_q, u_0, u_0', \ldots, u_{n-1}, u_{n-1}'$. The semantical interpretation in $(A, \underline{S})$ of $G^{\underline{k}}u_0 \ldots \varphi_{\underline{k}}$ is to be (crudely written):

$$\forall u_0 \exists u_0' \exists k_0 \forall u_1 \exists u_1' \exists k_1 \ldots \forall_n \varphi^{k_0 \ldots k_{n-1}} \tag{12}$$

where u_i, u_i' range over A, k_i over ω. (For more detail see [3].) Again each formula φ of $L_{\omega_1 G}$ is a $\underline{k}$-formula for some k, and the notion $\mathrm{Mod}^k\varphi$ is understood just as before. Classes of this form are called $\mathfrak{C}'$ classes or, sometimes, $L_{\omega_1 G}$ classes.

There is a minor technical question about $L_{\omega_1 G}$ which must be discussed. It involves at least implicitly what is called weak second order logic. Each structure $\mathfrak{A} = (A, \underline{S})$ can be expanded to the structure $\overline{\mathfrak{A}} = (A \cup A^\omega \cup \omega, A, \omega, \underline{S}, \mathrm{Sc}, \mathrm{Val})$ — where A^ω is the set of all finite sequences of members of A, the sets A, ω, A^ω are assumed disjoint, $\mathrm{Sc}(n) = n+1$, and $\mathrm{Val}(s, n) = s_n$. Now each language, e.g., $L_{\omega_1\omega}$, has a weak second order version whose definable classes are obtained by applying sentences of $L_{\omega_1\omega}$ of the larger similarity type to the structures $\overline{\mathfrak{A}}$ — to define a class of $\mathfrak{A}$'s. It is well-known and easy to prove that the expanded $L_{\omega_1\omega}$ has no new expressive power.

Barwise [3] defined $L_{\omega_1 G}$ by adding to $L_{\omega_1\omega}$ the operator:

$$\forall u_0 \exists u_0' \forall u_1 \exists u_1' \ldots \forall n \varphi_n(u_0 \ldots u_{n-1}') . \tag{13}$$

By considering propositional calculus, it is obvious that (13) is strictly weaker than (12). On the other hand a weak second order version of Barwise's language would clearly include (12). Thus his language is not invariant under passage to weak second order. (Moschovakis on the other hand, assumed this invariance outright.) It should be noted, however, that in many contexts (number theory or much less), Barwise's language does coincide with ours ($I \cup J$ being countable).

Our language is invariant under passage to weak second order. To show this reduces to proving that

$$\forall s^0 \exists t^0 \mathrm{A} s^1 \exists t^1 \ldots \forall n (\forall \underline{r}, \underline{r}' \in \omega^n) \tag{14}$$
$$[(\forall i < n)(l(s_i) = r_i \wedge l(t^i) = r_i' \rightarrow \varphi_n^{\underline{r}, \underline{r}'}(s^0, \ldots, t^{n-1}))]$$

is expressible in our language, if the φ's are. (Here s^i, t^i are finite sequences in the model.) The proof that (14) can be so expressed is tedious but involves a very simple kind of coding, and will be left to the reader. ((5) in § 1 was a special case — also left to the reader!) Alternatively, one could simply define $L_{\omega_1 G}$ in the strongest way.

We return to $\mathfrak{X} = X \subseteq X_\varrho$.

LEMMA 3.7. *If $B \subseteq X$ is $\mathfrak{C}$ then $B^{(*n)}$ is $\mathfrak{C}'$.*

Proof. It suffices to add to (a), (b), (c) of the proof of 3.1 the following proposition:

(d) Suppose $B = \bigcup_F \bigcap_n B_{F\restriction n}$ when each $B_{\underline{k}}$ is $\mathfrak{C}$ and $B_{\underline{k}}^{(*n)} = \mathrm{Mod}\varphi_{\underline{k}}^n$. Then $B^{(*n)}$ is $\mathfrak{C}'$.

To prove (d), we apply 1.6 to obtain at once: $(\underline{S}, s) \in B^{(*n)}$ if and only if $\underline{S} \in B^{*[s]}$ if and only if

$$(\forall s^0 \supseteq s)(\exists t^0 \supseteq s^0) \exists k_0 \ldots \forall p (\underline{S} \in B^{*[t^p]}_{k_0 \ldots k_p}) ,$$

that is,

$$(\forall s^0 \supseteq s)(\exists t^0 \supseteq s^0) \exists k_0 \ldots (\forall p, n)[l(t^p) = n \rightarrow \varphi^n_{k_0 \ldots k_p}(t_0^p, \ldots t_{n-1}^p)] . \tag{15}$$

Clearly (15) is expressible in the weak second order version of $L_{\omega_1 G}$ and hence in $L_{\omega_1 G}$, as desired.

THEOREM 3.8. *Invariant* $\mathfrak{C} = \mathfrak{C}'$ $(= L_{\omega_1 G})$.

Proof. If B is $L_{\omega_1 G}$ then as is easy using (5) and known, B is invariant $\mathfrak{C}$. If B is invariant $\mathfrak{C}$ then by 3.7, $B = B^*$ is $L_{\omega_1 G}$.

We close this section with some remarks about arbitrary classes of ϱ-structures. Many questions about $\mathfrak{X}_\varrho$ reduce to questions about X_ϱ by means of the Löwenheim-Skolem theorem, which is known to apply to $L_{\omega_1\omega}$ and even $L_{\omega_1 G}$ (by a result of Barwise [3] and Moschovakis):

LÖWENHEIM-SKOLEM THEOREM. *If $I \cup J$ is countable and $\mathfrak{Y}$ is a non-empty $L_{\omega_1 G}$ class of infinite ϱ-structures, then $\mathfrak{Y} \cap X_\varrho \neq 0$.*

For example, as is well-known, the Löwenheim-Skolem theorem easily implies that the ω-completeness theorem (see above) extends to any *class* $\mathfrak{X}$; i.e.: *if $I \cup J$ is countable and $\mathfrak{X}$ is G_δ' in $\mathfrak{X}_\varrho$ then $\mathfrak{X}$ is not meager' in itself.* However, the notions meager' and Baire' apply to classes which are not $L_{\omega_1 G}$ and hence the Löwenheim-Skolem theorem cannot always be applied (to answer questions about the general notions Borel', meager', etc., etc. over, say, $\mathfrak{X}_\varrho$) when $I \cup J$ is countable. Even so, the consideration of X_ϱ and the associated action may at least yield conjuctures about $\mathfrak{X}_\varrho$ to be proved in some other way.

In this direction the author has established the following result: 3.6 *holds in general*, i.e., *the game operation preserves the Baire' property-relative* to any $\mathfrak{X}$. A proof will be given in a later paper.

§ 4. **$\varkappa$-logic.** In this section, α, β, etc. range over arbitrary ordinals. We take $\alpha = \{\beta : \beta < \alpha\}$. Cardinals are initial ordinals. $\varkappa$ is a fixed infinite, regular cardinal. As is known, all the notions at the beginning of § 3 can be extended to arbitrary $\varkappa$, the case there being $\varkappa = \omega$. This is done as follows:

We consider $\varrho = (\varrho', J)$ where $\varrho' : J \to \varkappa$. As before, $X_{\varrho,A} = \prod_{i \in I} 2^{A^{\varrho_i}} \times A^J$. The class $\mathfrak{X}_\varrho$ of all ϱ-structures consists of all $(A, \underline{S})$ where $\underline{S} \in X_{\varrho,A}$. (Thus we consider α-ary relations for $\alpha < \varkappa$.) X_ϱ is $X_{\varrho,\varkappa}$ with the $\varkappa$-*topology*, which has as a basis all intersections of fewer than $\varkappa$ sets open in the product topology. If X is any subspace of X_ϱ, then, since $\varkappa$ is regular, X is clearly a *$\varkappa$-space*, i.e., the intersection of fewer than $\varkappa$ open sets is open. The *$\varkappa$-Borel* sets form the smallest family containing all open sets and closed under complement and $\varkappa$-unions (i.e., unions of at most $\varkappa$ sets). (Thus the $\varkappa$-Borel sets are the same whether the product topology or the $\varkappa$-topology is used.) A *$\varkappa$-meager* set (in any space) is a $\varkappa$-union of nowhere dense sets.

G is the group $\varkappa!$ of all permutations of $\varkappa$, with the $\varkappa$-topology (based on the product topology), and $g\underset{\sim}{S}$ is the usual isomorph. It is known that (X being any invariant subspace of X_ϱ):

(16) *G is a topological group acting continuously on X, and G is a $\varkappa$-Baire space (i.e., no non-empty open set is $\varkappa$-meager).*

(See, for example, [20] and [21], where a pioneering study of this action and some of its connections with logic was made.)

$L_{\varkappa^+\varkappa}$ has the symbols $\boldsymbol{R}_i$ $(i \in I)$, c_j $(j \in J)$, variables v_α $(\alpha < \varkappa)$, $\approx$, $\sim$, $\bigwedge$, $\forall$. Atomic formulas are $\tau_1 = \tau_2$ or $\boldsymbol{R}_i\tau_0 \ldots \tau_\alpha \ldots (\alpha < \varrho_i')$ (where τ_β is a v_α or a c_j). A closed′ formula is an arbitrary conjunction $\bigwedge_{t\in T}\theta_t$ where the θ_t are of the form $(\forall u_0 \ldots u_\alpha \ldots)_{\alpha<\beta}M$ where $\beta < \varkappa$ and M is a disjunction of fewer than $\varkappa$ atomic formulas and their negations. Conjunctions are never allowed having at least $\varkappa$ free variables. The class of formulas is the smallest containing all closed′ formulas and containing

$$\sim\varphi,\ (\forall u_0 \ldots u_\alpha \ldots)_{\alpha<\beta}\varphi,\ \bigwedge_{\alpha<\varkappa}\varphi_\alpha$$

whenever it constains $\varphi, \varphi_0, \ldots, \varphi_\alpha, \ldots$, and $\beta < \varkappa$.

Let $\mathfrak{X} \subseteq \mathfrak{X}_\varrho$ be weakly closed under isomorphism. The notion $\mathrm{Mod}^{\alpha}_{\mathfrak{X}}\varphi$, for $\alpha < \varkappa$, is understood just as before. Classes $\mathrm{Mod}_{\mathfrak{X}}\sigma$ are called *$L_{\varkappa^+\varkappa}$ classes* (over $\mathfrak{X}$). Now fix $\mathfrak{X} = X \subseteq X_\varrho$.

We write $\lambda^{\check{\varkappa}} = \sum_{\mu<\varkappa}\lambda^\mu$. Under the G.C.H., $2^{\check{\varkappa}} = \varkappa$ for every infinite $\varkappa$.

THEOREM 4.1. *If $\varkappa$ is regular and $2^{\check{\varkappa}} = \varkappa$, then invariant $\varkappa$-Borel $= L_{\varkappa^+\varkappa}$.*

Proof. As always, from right to left is easy and essentially known — and is left as an exercise. However, this argument does make heavy use of the assumption $2^{\check{\varkappa}} = \varkappa$. On the other hand, that invariant $\varkappa$-Borel implies $L_{\varkappa^+\varkappa}$ is valid for any regular $\varkappa$, as we now show.

In view of (16), certain parts of § 1 and § 3 can be directly imitated so as to yield our theorem. Hence we shall only give a sketch.

If s maps $\alpha < \varkappa$ one-to-one into $\varkappa$, put $[s] = \{g\colon g \supseteq s\}$. The set $\mathfrak{K}$ of all $[s]$ is clearly a basis of non-empty open sets of G. (If $2^{\check{\varkappa}} = \varkappa$ then $\overline{\overline{\mathfrak{K}}} = \varkappa$, but we shall not need this.) For such s and $B \subseteq X$ put $B^{*[s]} = \{x\colon B^x \cap [s]$ is $\varkappa$-comeager in $[s]\}$ and $B^* = B^{*[\sigma]}$. (We use either x or $\underset{\sim}{S}$ for members of X.) Clearly $B^- \subseteq B^*$ and $B^* \subset B^+$ (by (16)).

Now we show:

(a) If B is closed then $B^{*[s]}$ is closed.

(b) $(\bigcap_{\alpha<\varkappa} B_\alpha)^{*[s]} = \bigcap_{\alpha<\varkappa} B^{*[s]}$.

(c) $(\sim B)^{*[s]} = \sim\bigcup_{t\supseteq s} B^{*[t]}$ if B is $\varkappa$-Borel.

(a), (b), and $\subseteq$ in (c) go exactly as in 1.3-1.5 (of course using (16)). If B is $\varkappa$-Borel, then each B^x is $\varkappa$-Borel and hence B^x is $\varkappa$-Baire (cf. [20]). Now $\supseteq$ in (c) can be argued just as in 1.5.

Now, for $\alpha < \varkappa$, put $B^{(*\alpha)} = \{(\underset{\sim}{S}, s) \in X \times x^\alpha\colon \underset{\sim}{S} \in B^{*[s]}\}$. From (a)-(c) we infer by induction, just as in 3.1, that:

(d) If B is $\varkappa$-Borel, then $B^{(*\alpha)}$ is $L_{\varkappa^+\varkappa}$ (in $X \times \varkappa^\alpha$).

Hence if B is invariant $\varkappa$-Borel, then $B = B^{(*0)}$ is $L_{\varkappa^+\varkappa}$ in X, as was to be proved.

Note that the union in (c), which may not be a $\varkappa$-union if $2^{\check{\varkappa}} \neq \varkappa$, has been converted in (d) into a quantification. Thus the existence of a basis of power $\varkappa$ for G is never needed.

Of course, no proof of 4.1 along the lines of Lopez-Escobar's proof of 3.3 seems to be possible since there is no adequate version of (IT) available (cf. [12]).

It is clear that various other parts of § 1 and § 3 can be reworded in a $\varkappa$-way and established by the rewording of the old proofs. Assuming that $\varkappa$ is regular and $2^{\check{\varkappa}} = \varkappa$, this appears to apply to 1.2-1.5, 1.8 (a), (b), 2.1, to 2.2 and 2.3 except for $\mathfrak{C}$, and to 3.1-3.5. As is known, the Löwenheim-Skolem theorem (reworded) also holds, but the ω-completeness theorem (reworded) does not hold in general. On the other hand, all results of § 1-§ 3 concerning $\mathfrak{C}$ are in an entirely different situation, since it is not even clear what the $\varkappa$-analogue of $\mathfrak{C}$ should be.

§ 5. Questions of effectiveness. We return to the topic (and notation) of § 3. Familiarity with the terminology of [7] and [3], concerning admissible sets and primitive recursive set functions, will be assumed. Thus $L_{\omega_1\omega}$ is now defined as in [3] to be a certain subclass of the class of hereditarily countable sets.

Let ϱ be a countable similarity type and let C be a countably infinite set of individual constants not in ϱ. We fix $X = X_{\varrho,C}$, which is acted on by the group $C!$ Thus, for a ϱ-sentence σ, $\mathrm{Mod}\,\sigma \subseteq X$. If θ is a sentence (without quantifiers) of type $\varrho + C$, write $M(\theta) = \{\underset{\sim}{S}\colon (C, \underset{\sim}{S}, c)_{c\in C}$ is a model of $\theta\}$.

THEOREM 5.1. *We can define a set function °, primitive recursive in C and ϱ such that, for each $L_{\omega_1\omega}$ sentence θ of type ϱ (without quantifiers),*

θ° is an $L_{\omega_1\omega}$ sentence of type ϱ and $M(\theta)^* = M(\theta^\circ)$. If θ is Π^0_a (literally) then θ° is $\Pi^{0\prime}_a$ (literally).

In other words *given a name of a Borel set $B \subseteq X_{\varrho,C}$ we can effectively find a name for B^*.*

Proof. For $C = \omega$, the proof of 3.1 has been deliberately written so as to give the proof of 5.1.

For other C, which may not primitive recursively be denumerable, it is only necessary to modify the proof of 3.1 in the right way. The notion $B^{(*n)}$ used there should be replaced by $B^{(*d_0\ldots d_{n-1})}$, where the d_k are distinct c's from C. $d_0, \ldots, d_{n-1}$ play a role that was played by $0, 1, \ldots, n-1$. In other words, $B^{(*d_0\ldots d_{n-1})} = \{(\underline{S}, e_0, \ldots, e_{n-1})\colon B^{\underline{S}}$ is comeager in $\{g\colon g \supseteq \{(d_0, e_0), \ldots, (d_{n-1}, e_{n-1})\}\}\}$. Now the proof of 3.1 can be imitated. For example, if $B^{(*e_0\ldots e_{q-1})} = \mathrm{Mod}_q\,\theta_{e_0\ldots e_{q-1}}$ (in general), then

$$(\sim B)^{(*d_0\ldots d_{n-1})} = \mathrm{Mod}^n \sim \bigvee_p \bigvee_{d_n,\ldots,d_p} (\exists v_n \ldots v_p)^{\neq}\,\theta_{d_0\ldots d_n\ldots d_p}\,.$$

In this way 5.1 is easily proved.

In exactly the same way, the proof of 3.7 can be used to establish the continuation of 5.1 below. $L_{\omega_1 A}$ is obtained from $L_{\omega_1\omega}$ (without quantifiers) by adding a formal version of the (infinitary propositional) operation (A).

THEOREM 5.1 (continued). *Moreover: if θ is $L_{\omega_1 A}$ then θ° is $L_{\omega_1 G}$ and $M(\theta)^* = \mathrm{Mod}(\theta^\circ)$.*

Of course 5.1 implies at once an effective version of 2.2, giving interpolation between B^- and B^+. In particular, the same $^\circ$ has the property that if $M(\theta)$ is invariant then $M(\theta) = \mathrm{Mod}\,\theta^\circ$. Thus:

COROLLARY 5.2. *If A is a transitive, primitive recursively closed, countable set containing ϱ, C, and θ, and $M(\theta)$ is invariant (θ a $\varrho + C$ sentence) then $M(\theta) = \mathrm{Mod}\,\sigma$ for some ϱ-sentence σ in A.*

5.2 improves a theorem of Barwise, in which it was assumed A was admissible. (For such, he obtained the Barwise interpolation theorem, which implies 5.2.)

It is natural to ask for a proof of the Barwise interpolation theorem [3] from Addison's effective separation principle [1] via the $*$-transform. However, the proof in [1], which resembles the classical proofs, applies only to admissible sets $\mathcal{A}$ of the form $L^A_{\omega_1^A}$, where $A \subseteq \omega$. For such $\mathcal{A}$, the $*$-transform gives at once Barwise interpolation. For other countable admissible $\mathcal{A}$, one must establish non-invariant separation as well as invariant. It is hard to say there is no such proof, but there does not appear to be a short, elegant one.

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