Table des matières du tome LXXXII, fascicule 1

Les FUNDAMENTA MATHEMATICAE publient, en langues des congrès internationaux, des travaux consacrés à la *Théorie des Ensembles*, *Topologie*, *Fondements de Mathématiques*, *Fonctions Réelles*, *Algèbre Abstraite*

Ce volume paraît en 4 fascicules

Adresse de la Rédaction et de l'Échange:
FUNDAMENTA MATHEMATICAE, Śniadeckich 8, 00-950 Warszawa (Pologne)

Tous les volumes sont à obtenir par l'intermédiaire de
ARS POLONA-RUCH, Krakowskie Przedmieście 7, 00-068 Warszawa (Pologne)

Correspondence concerning editorial work and manuscripts should be addressed to:
FUNDAMENTA MATHEMATICAE, Śniadeckich 8, 00-950 Warszawa (Poland)

Correspondence concerning exchange should be addressed to:
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, Exchange
Śniadeckich 8, 00-950 Warszawa (Poland)

The Fundamenta Mathematicae are available at your bookseller or at
ARS POLONA-RUCH, Krakowskie Przedmieście 7, 00-068 Warszawa (Poland)

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

On hereditarily separable Hausdorff spaces in the constructible universe

by

Keith J. Devlin [1] (Manchester)

Abstract. We establish, from the assumption V = L, the existence of a hereditarily separable Hausdorff space of cardinality greater than the continuum. The proof, which uses Jensen's concept of a morass, depends upon a combinatorial reduction of the problem due to Hajnal and Juhász.

0. Introduction. We work in ZFC throughout, and use the usual notation and conventions. In particular, an ordinal is identified with the set of all smaller ordinals and a cardinal is an ordinal not equinumerous with any smaller ordinal. If X is a set, $|X|$ denotes its cardinality. If X is a set of ordinals, $\mathrm{otp}(X)$ denotes its order-type under the usual ordering. If X, A are sets ${}^{A}X$ denotes the set of all maps from A into X. For any set A, $H(A) = \{f \mid f \text{ is a function and } |\mathrm{dom}(f)| < \omega \text{ and } \mathrm{dom}(f) \subseteq A \text{ and } \mathrm{ran}(f) \subseteq 2\}$.

Problem 78 of [1] is the following. Does there exist a hereditarily separable Hausdorff space of cardinality greater than the continuum?

The aim of this paper is to prove that the answer is "yes" if we assume the axiom of constructibility, V = L, (i.e. *all* sets are constructible). To be a little more precise, what we shall do is establish from the assumption V = L, a certain combinatorial principle formulated by Hajnal and Juhász, which was proved by them to imply a positive answer to Problem 78.

1. The Hajnal–Juhász Principle. Let $\varkappa$ be any infinite cardinal. The *Hajnal–Juhász Principle* for $\varkappa$, $\mathrm{HJ}(\varkappa)$, says that there is a sequence $\langle f_\alpha \mid \alpha < 2^{(\varkappa^+)}\rangle$ such that each $f_\alpha \in {}^{\varkappa^+}2$, and for each $X \subseteq 2^{(\varkappa^+)}$, $|X| = \varkappa$, there is $\nu(X) < \varkappa^+$ such that:

$$\forall \varepsilon \in H(\varkappa^+ \dot{-} \nu(X))(\exists \alpha \in X)(\varepsilon \subseteq f_\alpha).$$

[1] The main result in this paper was proved during the Autumn of 1972, when the author was a visitor at the University of Oslo, Norway. He wishes to thank Professor J. E. Fenstad, who arranged this visit, and in so doing prevented said author from having to spend the three months concerned working for the British Postal Service in a somewhat arduous, perambulatory capacity.

$\mathrm{HJ}(\varkappa)$ is formulated in [3], where Hajnal and Juhász establish its consistency (by a forcing argument) for any $\varkappa$ such that $2^{\varkappa} = \varkappa^{+}$. In [3], they also prove that $\mathrm{HJ}(\varkappa)$ implies the existence of a hereditarily $\varkappa$-separable, 0-dimensional, normal Hausdorff topological space of cardinality $2^{(\varkappa^+)}$. We refer the reader to [3] and the references therein for further details on this matter. Our purpose here is to show that the principles $\mathrm{HJ}(\varkappa)$ hold in the constructible universe, L.

For a definition of L, we refer the reader to [2]. Our proof depends heavily upon recent work of Ronald Jensen. In § 2 we introduce the concept of a $\varkappa$-morass, a set-theoretical structure invented by Jensen. In § 3 we use a $\varkappa$-morass to establish the principle $\mathrm{HJ}(\varkappa)$. In § 4 we give a brief sketch of Jensen's proof that $V = L$ implies the existence of a $\varkappa$-morass for all $\varkappa$. The reader interested only in the principles $\mathrm{HJ}(\varkappa)$ can, of course, omit § 4, since all he needs to know about L is that the assumption $V = L$ is a consistent extension of ZFC which implies GCH and the existence of morasses. For the more demanding reader, we should perhaps warn that the sketch in § 4 is extremely brief compared with the proof itself (which is, as yet, unpublished), and that even the sketch requires a good acquaintance with Jensen's paper [4].

It is perhaps worth mentioning that the motivation for introducing morasses in the first place lies in model theory, and that there are, in L, much more powerful "morasses" which have very striking set-theoretical and model-theoretic consequences, in particular the "Gap-n Cardinal Transfer Property" for arbitrary $n \in \omega$ (for those who know what this means).

To aid the intuition, we shall restrict ourselves to the case $\varkappa = \omega$ throughout. In every instance, the proof for an arbitrary $\varkappa$ is entirely analogous, so there is no essential loss in this. The main benefit is that, as stated, the intuition can more easily follow the various induction argument for the case $\varkappa = \omega$.

Since we shall be assuming GCH (the existence of an ω-morass implies the CH) throughout, we shall thus mean, by HJ ($= \mathrm{HJ}(\omega)$), the following assertion:

HJ = There is a sequence $\langle f_\alpha |\ \alpha < \omega_2\rangle$ such that each $f_\alpha \in {}^{\omega_1}2$, and for each $X \subseteq \omega_2$, $|X| = \omega$, there is $\nu(X) < \omega_1$ such that:

$$\big(\forall \varepsilon \in \mathrm{H}(\omega_1 - \nu(X))\big)(\exists \alpha \in X)(\varepsilon \subseteq f_\alpha) .$$

Note that it is clearly sufficient to consider the principle HJ for sets X of *order-type* ω only.

Since we shall not have need to consider cardinal exponentiation any more, we shall write 2^α rather than ${}^{\alpha}2$ from now on.

For later use, and to fix ideas, we end this section by proving a very weak form of HJ-principle. We should mention, however, that it is the proof of this result, not the result itself, which we shall later use.

LEMMA 1. *Let γ, δ be countable, infinite ordinals, and let M be any countable collection of countable infinite subsets of δ. Then there is a sequence $\langle f_\alpha |\ \alpha < \delta\rangle$ such that each $f_\alpha \in 2^\gamma$ and $(\forall X \in M)\big(\forall \varepsilon \in H(\gamma)\big)(\exists \alpha \in X)(\varepsilon \subseteq f_\alpha)$.*

Proof. Assume first of all that $\delta = \omega$. Let $\langle \varepsilon_n |\ n < \omega\rangle$ enumerate $H(\gamma)$ in a one-one fashion. Let $\langle X_n |\ n < \omega\rangle$ be a one-one enumeration of M. (We may clearly assume that M is infinite.) For each $n < \omega$, let $\langle x_i^n |\ i < \omega\rangle$ be the monotone enumeration of X_n. We define, by induction on $n < \omega$, a strictly increasing $(n+1)$-tuple $p^n = \langle p_0^n, \ldots, p_n^n\rangle$ of integers, for each $n \in \omega$. The definition of p^n is by induction on $i \leqslant n$. Set $p_0^0 = 0$, $p^0 = \langle p_0^0\rangle$, and proceed thus:

p_0^{n+1} = the least p such that $x_p^0 > x_{p_n^n}^n$;

p_{i+1}^{n+1} = the least p such that $x_p^{i+1} > x_{p_i^{n+1}}^i$, $i \leqslant n$.

For each new $n < \omega$, let $f_{x_{p_i^n}^i}\colon \gamma \to 2$ be arbitrary such that $\varepsilon_{n-i} \subseteq f_{x_{p_i^n}^i}$, $i \leqslant n$.

If $m < \omega$ and $m \neq x_{p_i^n}^i$ for any $i \leqslant n < \omega$, let $f_m\colon \gamma \to 2$ be arbitrary.

Then $\langle f_m |\ m < \omega\rangle$ satisfies the lemma. For suppose $m, n < \omega$, and consider X_n, ε_m. By definition, $\varepsilon_m \subseteq f_{x^n_{p_n^{m+n}}}$, so we are done already.

Suppose now that $\delta > \omega$. Let $j\colon \delta \leftrightarrow \omega$. For each set $X \in M$, let $\bar{X} = j''X$. Set $\bar{M} = \{\bar{X} |\ X \in M\}$. By the above, pick $\langle \bar{f}_\alpha |\ \alpha < \omega\rangle$ such that each $\bar{f}_\alpha \in 2^\gamma$ and $(\forall \bar{X} \in \bar{M})\big(\forall \varepsilon \in H(\gamma)\big)(\exists \alpha \in \bar{X})(\varepsilon \subseteq \bar{f}_\alpha)$. Set $f_\alpha = \bar{f}_{j(\alpha)}$, for each $\alpha < \delta$. Then $\langle f_\alpha |\ \alpha < \delta\rangle$ clearly satisfies the lemma. ■

Note. If we had so desired, we could have arranged matters so that for each ε there are infinitely many $\alpha \in X$ with $\varepsilon \subseteq f_\alpha$, in the above lemma. This point will be important later on.

2. Definition of a morass. The structure which we define below, and call a "morass" is, in precise terms, a universal $(\omega_1, 1)$-morass. The analogous structure required to establish $\mathrm{HJ}(\varkappa)$ for $\varkappa > \omega$ (which we would here call a "$\varkappa$-morass") is a universal $(\varkappa^+, 1)$-morass.

Let $\mathcal{S}$ be a set of ordered pairs $\langle \alpha, \nu\rangle$ of primitive recursive closed ordinals $\alpha < \nu < \omega_2$ such that wherever $\langle \alpha, \nu\rangle, \langle \alpha', \nu'\rangle \in \mathcal{S}$, $\alpha < \alpha' \to \nu < \alpha'$.

Set

$S^0 = \{\alpha \in \omega_1 + 1 |\ (\exists \nu)[\langle \alpha, \nu\rangle \in \mathcal{S}]\}$;

$S^1 = \{\nu \in \omega_2 |\ (\exists \alpha)[\langle \alpha, \nu\rangle \in \mathcal{S}]\}$;

$S = S^0 \cup S^1$;

$S_\alpha = \{\nu \in S^1 |\ \langle \alpha, \nu\rangle \in \mathcal{S}\}$, for $\alpha \in S^0$;

α_ν = that unique ordinal $\alpha \in S^0$ such that $\langle \alpha, \nu\rangle \in \mathcal{S}$, for $\nu \in S^1$.

Let $U_\nu \subseteq \nu \times \nu$, each $\nu \in S^1$, be such that

$$\nu \in S_{\alpha_\tau} \cap \tau \to U_\nu = U_\tau \cap (\nu \times \nu) \quad \text{for each } \tau \in S^1 .$$

Set

$$U_\alpha = \bigcup_{\nu \in S_\alpha} U_\nu , \quad \text{each} \quad \alpha \in S^0 ,$$

$$\overline{U}_\nu = \{U''_{\alpha_\nu}\{\tau\} \mid \alpha_\nu < \tau < \sup(S_{\alpha_\nu})\} , \quad \text{each} \quad \nu \in S_1 .$$

Let $\precsim$ be a tree on S^1 such that $\nu \precsim \tau \to \alpha_\nu < \alpha_\tau$.

Let $\{\pi_{\nu\tau} \mid \nu \precsim \tau\}$ be a commutative system of maps $\pi_{\nu\tau}\colon (\nu+1) \to (\tau+1)$.

Then, the structure $\mathfrak{I} = \langle S, \mathcal{S}, \precsim, (\pi_{\nu\tau})_{\nu \precsim \tau}, (U_\nu)_{\nu \in S}\rangle$ is a *morass* if it satisfies the following requirements:

(M0) (a) S_α is closed in $\sup(S_\alpha)$, all $\alpha \in S^0$;

(b) $\omega_1 = \max(S^0) = \sup(S^0 \cap \omega_1)$, $\omega_2 = \sup(S_{\omega_1})$.

(M1) (a) $\langle U''_{\omega_1}\{\nu\} \mid \omega_1 < \nu < \omega_2\rangle$ enumerates all bounded subsets of ω_2.

(b) If $\nu \precsim \tau$, then $\pi_{\nu\tau} \restriction \alpha_\nu = \mathrm{id} \restriction \alpha_\nu$ and

$$\pi_{\nu\tau}\colon \langle \nu+1, \varepsilon, \{\alpha_\nu\}, \{\nu\}, S_{\alpha_\nu} \cap \nu, U_\nu\rangle \underset{\Sigma_0}{\to} \langle \tau+1, \varepsilon, \{\alpha_\tau\}, \{\tau\}, S_{\alpha_\tau} \cap \tau, U_\tau\rangle$$

(where the symbol $\underset{\Sigma_0}{\to}$ means that all Σ_0-definable relations are preserved).

(M2) $\bar\tau \precsim \tau$ and $\bar\nu \in S_{\alpha_{\bar\tau}} \cap \bar\tau$ and $\nu = \pi_{\bar\tau\tau}(\bar\nu) .\to. \bar\nu \precsim \nu$ and $\pi_{\bar\nu\nu} \restriction \bar\nu = \pi_{\bar\tau\tau} \restriction \bar\nu$.

(M3) $\{\alpha_\nu \mid \nu \precsim \tau\}$ is closed in α_τ.

(M4) τ not maximal in $S_{\alpha_\tau} \to \{\alpha_\nu \mid \nu \precsim \tau\}$ is unbounded in α_τ.

(M5) $\{\alpha_\nu \mid \nu \precsim \tau\}$ unbounded in $\alpha_\tau \to \tau = \bigcup_{\nu \precsim \tau} \pi''_{\nu\tau}\nu$.

(M6) $\bar\tau$ a limit point of $S_{\alpha_{\bar\tau}}$ and $\bar\tau \precsim \tau$ and $\lambda = \sup_{\nu < \bar\tau} \pi_{\bar\tau\tau}(\nu) \to \bar\tau \precsim \lambda$ and $\pi_{\bar\tau\lambda} \restriction \bar\tau = \pi_{\bar\tau\tau} \restriction \bar\tau$.

(M7) $\bar\tau$ a limit point of $S_{\alpha_{\bar\tau}}$ and $\bar\tau \precsim \tau = \sup_{\nu < \bar\tau} \pi_{\bar\tau\tau}(\nu)$ and

$$\alpha \in \bigcap_{\nu \in S_{\alpha_{\bar\tau}} \cap \bar\tau} \{\alpha_\eta \mid \nu \precsim \eta \precsim \pi_{\bar\tau\tau}(\nu)\} .\to. (\exists \tau' \in S_\alpha)(\bar\tau \precsim \tau' \precsim \tau) .$$

This then, completes the definition of the morass. Intuitively, the sets S_α, $\alpha < \omega_1$, are countable approximations to the large set S_{ω_1}. Since $|S_{\omega_1}| = \omega_2$, the sense in which these approximations piece together to "give" S_{ω_1} is fairly complex, the tree $\precsim$ and the maps $\pi_{\nu\tau}$ along it defining the precise mode of approximation. Note that by (MO) (b), (M4) and (M5), every point in S_{ω_1} is included in the approximation procedure. The various morass axioms, in particular the "continuity" axioms (M6) and (M7), ensure that the approximations fit together smoothly. This will enable us to define, by an inductive procedure of length ω_1, an HJ-sequence $\langle f_\alpha \mid \alpha < \omega_2\rangle$, by defining successive approximations to such a sequence, following a morass. Since, at each stage in the induction, all the relevant concepts will be countable, we shall be able to use Lemma 1 to keep going.

3. The Main theorem.

THEOREM 2. *If there is a morass, then* HJ *holds.*

Proof. Let $\mathfrak{I}$ be as above. Let $\exists^*$ denote the quantifier "there exist infinitely many". We construct, by induction on $\nu \in S^1$, functions $f_\nu\colon \alpha_\nu \to 2$ such that

(1) $\nu \precsim \tau \to f_\nu = f_\tau \restriction \alpha_\nu$;

(2) if $\nu \precsim \tau$ and $x \in \overline{U}_\nu$, $x \subseteq S_{\alpha_\nu} \cap \nu$, $\mathrm{otp}(x) = \omega$, then

$$(\forall \varepsilon \in H(\alpha_\tau - \alpha_\nu))(\exists^* \eta \in x)(\varepsilon \subseteq f_{\pi_{\nu\tau}(\eta)}) .$$

Before we commence the construction, let us see how this will yield the required result. Since S_{ω_1} is a closed set of ordinals of order-type ω_2, we can identify it with ω_2. Consider, then $\langle f_\nu \mid \nu \in S_{\omega_1}\rangle$ as above. Let $x \subseteq S_{\omega_1}$, $\mathrm{otp}(x) = \omega$. Pick $\nu \in S_{\omega_1}$ with $x \subseteq \nu$, $x = U''_\nu\{\gamma\}$ for some $\gamma < \nu$. By (M0) (b), such a ν exists, and will not be maximal in S_{ω_1}. So, by (M1), (M4) and (M5), it is easily seen that we can find $\bar\nu \precsim \nu$ such that there are $\bar x \subseteq S_{\alpha_{\bar\nu}} \cap \bar\nu$, $\bar\gamma < \bar\nu$, with $\pi_{\bar\nu\nu}(\bar\gamma) = \gamma$, $\pi''_{\bar\nu\nu}\bar x = x$, and $\bar x = U''_{\bar\nu}\{\bar\gamma\}$. By condition (2), if $\varepsilon \in H(\omega_1 - \alpha_{\bar\nu})$ is given, we can find $\bar\eta \in \bar x$ such that $\varepsilon \subseteq f_{\pi_{\bar\nu\nu}(\bar\eta)}$. Setting $\eta = \pi_{\bar\nu\nu}(\bar\eta)$, we have $\eta \in x$ and $\varepsilon \subseteq f_\eta$. Hence $\langle f_\nu \mid \nu \in S_{\omega_1}\rangle$ satisfies HJ.

We turn now to the construction of the f_τ's. In fact, we must construct an auxiliary system of maps as we proceed. Suppose $\alpha, \bar\alpha \in S^0$. Say $\theta(\alpha, \bar\alpha)$ iff $\bar\alpha < \alpha$ and there are $\tau \in S_\alpha$, $\bar\tau \in S_{\bar\alpha}$ such that $\bar\tau$ immediately precedes τ in $\precsim$. (Note that, in this case, $\bar\tau$ and τ are uniquely determined by $\alpha, \bar\alpha$, by virtue of (M4).) The inductive construction proceeds as follows. We look at each ordinal in ω_2 in turn, starting with 0 and working upwards. If we come to a τ which lies in S^1, we define f_τ as described below. If we come to an α which lies in S^0 we look to see if there are any $\bar\alpha$ such that $\theta(\alpha, \bar\alpha)$. If so, then for each such $\bar\alpha$ we define (by induction on $\bar\alpha$) a sequence $\langle f^{\alpha\bar\alpha}_\nu \mid \nu \in S_{\bar\alpha}\rangle$ of maps $f^{\alpha\bar\alpha}_\nu\colon \alpha \to 2$ such that:

(3) $f_\nu = f^{\alpha\bar\alpha}_\nu \restriction \bar\alpha$;

(4) if $\alpha' < \alpha$ and $\theta(\alpha', \bar\alpha)$, then $f^{\alpha'\bar\alpha}_\nu = f^{\alpha\bar\alpha}_\nu \restriction \alpha'$;

(5) for each $\lambda \in S_{\bar\alpha}$, each $\bar\lambda \precsim \lambda$, if $x \in \overline{U}_{\bar\lambda}$, $x \subseteq S_{\alpha_{\bar\lambda}} \cap \bar\lambda$, $\mathrm{opt}(x) = \omega$,

$$(\forall \varepsilon \in H(\alpha - \alpha_{\bar\lambda}))(\exists^* \eta \in x)(\varepsilon \subseteq f^{\alpha\bar\alpha}_{\pi_{\bar\lambda\lambda}(\eta)}) .$$

The way we actually define the $f^{\alpha\bar\alpha}_\nu$ is described below. (It is not a *true* induction on $\bar\alpha$, since there is no connection between distinct $\bar\alpha$'s, but we must do the definition in *some* order.)

It is convenient to describe the definition of the $f_\nu^{\alpha\bar\alpha}$'s first. We check that all of our requirements are preserved as we proceed. There are three cases to consider.

Case 1°. $(\forall\beta<\alpha)(\exists\gamma>\beta)\,\theta(\gamma,\bar\alpha)$.

Set $f_\nu^{\alpha\bar\alpha}=\bigcup\{f_\nu^{\alpha'\bar\alpha}|\ \bar\alpha<\alpha'<\alpha$ and $\theta(\alpha',\bar\alpha)\}$. By (4), $f_\nu^{\alpha\bar\alpha}\colon\alpha\to 2$. It is easily seen that (3)-(5) hold, by the induction hypothesis.

Case 2°. $(\forall\beta<\alpha)\neg\theta(\beta,\bar\alpha)$.

Extend each f_ν, $\nu\in S_{\bar\alpha}$, to a map $f_\nu^{\alpha\bar\alpha}\colon\alpha\to 2$ so that (5) holds. This is really just a degenerate case of 3°, below, so rather than repeat ourselves we shall immediately proceed with the more general situation, Case 3°.

Case 3°. There is a maximal $\alpha'<\alpha$ such that $\theta(\alpha',\bar\alpha)$.

Extend each $f_\nu^{\alpha'\bar\alpha}$, $\nu\in S_{\bar\alpha}$, to a map $f_\nu^{\alpha\bar\alpha}\colon\alpha\to 2$ so that (4) and (5) hold. This is possible by a slight modification of the argument used in Lemma 1. It would be unmanageably cumbersome to describe the definition in detail, but the crucial fact is, that at each stage, not only do we realise a given ε on certain ω-sequences x, we in fact realise that ε *infinitely many times* in x. (This is why we use $\exists^*$ rather than $\exists$ in (2) and (5).) This enables us to preserve (3) whilst obtaining (5). (Note also that there are at most a countable number of x's to consider each time.)

(3) will hold at α since it did at α'.

That disposes of the $f_\nu^{\alpha\bar\alpha}$ definitions. We now precede with the definition of f_τ for $\tau\in S^1$. There are three cases to consider.

Case 1. τ minimal in $\lesssim$.

Let $f_\tau\colon\alpha_\tau\to 2$ be arbitrary. There are no new cases of (1) and (2) to consider.

Case 2. τ is a limit point in $\lesssim$. (i.e. $\{\alpha_\nu|\ \nu\lesssim\tau\}$ is unbounded in α_τ.)

Set $f_\tau=\bigcup_{\nu\lesssim\tau}f_\nu$. By (1), $f_\tau\colon\alpha_\tau\to 2$. It is easily seen that (1) and (2) are preserved.

Case 3. τ immediately succeeds $\bar\tau$ in $\lesssim$.

Thus $\theta(\alpha_\tau,\alpha_{\bar\tau})$. Set $f_\tau=f_{\bar\tau}^{\alpha_\tau\alpha_{\bar\tau}}$. By (3), $f_{\bar\tau}=f_\tau\restriction\alpha_{\bar\tau}$, so (1) is preserved. It is a little more tricky to check that (2) is preserved, since the argument depends upon the position of τ in S_{α_τ}. There are three cases to consider.

Case 3.1. τ is minimal in S_{α_τ}.

Then, by (M1) (b), $\bar\tau$ will be minimal in $S_{\alpha_{\bar\tau}}$. Hence (2) is vacuously true.

Case 3.2. τ immediately succeeds γ in S_{α_τ}.

By (M1), $\bar\tau$ will immediately succeed $\bar\gamma$ in $S_{\alpha_{\bar\tau}}$, where $\pi_{\bar\tau\tau}(\bar\gamma)=\gamma$. By induction hypothesis, (2) holds for $f_{\bar\gamma}$ and (by (M1)) no new cases of (2) arise when we pass from f_γ to f_τ, so (2) holds for f_τ.

Case 3.3. τ is a limit point in S_{α_τ}.

By (M1), $\bar\tau$ will be a limit point in $S_{\alpha_{\bar\tau}}$. There are two subcases to consider.

Case 3.3.1. $\sup_{\nu<\bar\tau}\pi_{\bar\tau\tau}(\nu)=\lambda<\tau$.

By (M6), $\bar\tau\lesssim\lambda$ and $\pi_{\bar\tau\lambda}\restriction\bar\tau=\pi_{\bar\tau\tau}\restriction\bar\tau$. Hence, (2) for f_τ follows trivially from (2) for f_λ.

Case 3.3.2. $\sup_{\nu<\bar\tau}\pi_{\bar\tau\tau}(\nu)=\tau$.

Let $\nu\,\underline{\lesssim}\,\bar\tau$, $x\in\bar U_\nu$, $x\subseteq S_{\alpha_\nu}\cap\nu$, $\operatorname{otp}(x)=\omega$.

Suppose x were not cofinal in ν. Then $\lambda=\sup(x)\in S_{\alpha_\nu}\cap\nu$, and (2) for f_τ, x follows from (2) for $f_{\pi_{\nu\tau}(\lambda)}$, x. Hence we can assume $\sup(x)=\nu$.

Suppose now that $\lambda=\sup\pi''_{\nu\tau}x<\tau$. Then, again, $\lambda\in S_{\alpha_\tau}\cap\tau$ and, since $\lambda=\sup\pi''_{\nu\tau}\nu$, (M6) tells us that $\nu\lesssim\lambda$ and $\pi_{\nu\lambda}\restriction\nu=\pi_{\nu\tau}\restriction\nu$, so (2) for f_τ, x follows from (2) for f_λ, x. Hence we can assume $\sup\pi''_{\nu\tau}x=\tau$.

Set $\bar x=\pi''_{\nu\bar\tau}x$. (By convention, let $\pi_{\nu\bar\tau}=\mathrm{id}\restriction(\nu+1)$ if $\nu=\bar\tau$.) Thus $\bar x\subseteq S_{\alpha_{\bar\tau}}\cap\bar\tau$ and $\operatorname{otp}(\bar x)=\omega$. Since x is cofinal in ν and $\pi''_{\nu\tau}x$ is cofinal in τ, we must have $\sup(\bar x)=\bar\tau$.

Let $\langle\bar x_n|\ n<\omega\rangle$ be the monotone enumeration of $\bar x$, cofinal in $\bar\tau$. Let $\langle x_n|\ n<\omega\rangle$ be the monotone enumeration of x. In particular, therefore $\pi_{\nu\bar\tau}(x_n)=\bar x_n$ for each n.

For each $n<\omega$, let η_n be the $\lesssim$-least η such that $\bar x_n\lesssim\eta\,\underline{\lesssim}\,\pi_{\bar\tau\tau}(\bar x_n)$. Since $\pi_{\bar\tau\tau}(\bar x_n)$ is not maximal in S_{α_τ} (lying, as it does, in $S_{\alpha_\tau}\cap\tau$), (M4) tells us that, in fact, $\eta_n\lesssim x_{\bar\tau\tau}(\bar x_n)$ for each n. Set $\alpha(n)=\alpha_{\eta_n}$, each n.

Claim 1. $m<n<\omega\to\alpha(m)<\alpha(\eta)$.

By choice of $\langle\eta_n|\ n>\omega\rangle$, $\langle\alpha(n)|\ n<\omega\rangle$ is nowhere decreasing. And by (M4), it is thus in fact strictly increasing.

Claim 2. $\sup_{n<\omega}\alpha(n)=\alpha_\tau$.

Suppose not, and let $\alpha=\sup_{n<\omega}\alpha(n)<\alpha_\tau$. Using (M3), it is easy to see that for each $n<\omega$ there is $\bar\eta_n\in S_\alpha$ such that $\bar x_n\lesssim\bar\eta_n\lesssim\pi_{\bar\tau\tau}(\bar x_n)$. Hence, by (M7), there is τ' such that $\tau'\in S_\alpha$ and $\bar\tau\lesssim\tau'\lesssim\tau$. But this contradicts the choice of $\tau,\bar\tau$. The claim is verified.

Note that $\theta\big(\alpha(n),\alpha_{\bar\tau}\big)$ holds for all $n<\omega$. Suppose $\varepsilon\in H(\alpha_\tau-\alpha_\nu)$ is given. By the claims, pick $n<\omega$ such that $\varepsilon\in H\big(\alpha(n)-\alpha_\nu\big)$. By (5) for $\alpha(n)$, $\alpha_{\bar\tau}$, x, $\bar\tau$, ν, we can find arbitrarily large $m>n$ such that $\varepsilon\subseteq f_{\bar x_m}^{\alpha(n),\alpha_{\bar\tau}}$. For each such m, $\varepsilon\subseteq f_{\bar x_m}^{\alpha(m),\alpha_{\bar\tau}}$, by (4). By definition, $f_{\eta_m}=f_{\bar x_m}^{\alpha(m),\alpha_{\bar\tau}}$, for such m, so by (1), $\varepsilon\subseteq f_{\eta_m}\subseteq f_{\pi_{\bar\tau\tau}(\bar x_m)}=f_{\pi_{\nu\tau}(x_m)}$. This proves (2) for f_τ.

The theorem is proved. ■

4. The existence of a morass in L. The following sketch should convey, if nothing else, the idea behind the construction of a morass in L. We assume familiarity with §§ 1–5 of [4], and use the notation of § 4 of [4]. The first step is to formulate a combinatorial principle, which we shall call Θ, which holds in L, and which embodies just sufficient of the fine structure of L to enable us to construct a morass using Θ. Assume $V = L$ from now on.

If we are given a structure $\langle J_\varrho, A\rangle$, we write $X \prec_Q \langle J_\varrho, A\rangle$ to mean that $X \subseteq J_\varrho$ and for all Σ_0-formulas $\varphi(v_0, v_1)$ with parameters from X,

$$\models_{\langle X, A\cap X\rangle} (\nabla\alpha)(\exists\beta > \alpha)\varphi(\beta, J_\beta) \quad \text{iff} \quad \models_{\langle J_\varrho, A\rangle} (\nabla\alpha)(\exists\beta > \alpha)\varphi(\beta, J_\beta)\,.$$

Clearly, if $\lim(\varrho)$, then $X \prec_Q \langle J_\varrho, A\rangle$ implies $X \prec_{\Sigma_1} \langle J_\varrho, A\rangle$. Conversely, if $X \prec_{\Sigma_0} \langle J_\varrho, A\rangle$ and $X \cap \varrho$ is cofinal in ϱ, then $X \prec_Q \langle J_\varrho, A\rangle$.

Set $\mathcal{S} = \{\langle\alpha, \nu\rangle \mid \alpha < \nu < \omega_2$ and ν is p.r. closed and $\alpha = \omega_1^{J_\nu}$ and $\models_{J_\nu}$ "α is the largest cardinal"$\}$. Define S, S^0, S^1, S_α, α_ν as in § 2. Note that $S_{\alpha_\nu} \cap \nu$ is uniformly $\Sigma_1^{J_\nu}(\{\alpha_\nu\})$ for all $\nu \in S^1$. For $\nu \in S^1$, set:

$\beta(\nu)$ = the least $\beta \geqslant \nu$ such that ν is not regular at β;

$n(\nu)$ = the least $n \geqslant 1$ such that ν is not Σ_n-regular at β;

$\varrho(\nu) = \varrho_{\beta(\nu)}^{n(\nu)-1}$;

$A(\nu) = A_{\beta(\nu)}^{n(\nu)-1}$.

Θ: There is a sequence $\langle C_\nu \mid \nu \in S^1\rangle$ such that, for each $\nu \in S^1$, C_ν is an unbounded subset of ν such that:

(1) $\langle J_\nu, C_\nu\rangle$ is amenable;

(2) Every $x \in J_\nu$ is Σ_1-definable in $\langle J_\nu, C_\nu\rangle$ from parameters in α_ν;

(3) $\langle C_{\bar\nu} \mid \bar\nu \in S_{\alpha_\nu} \cap \nu\rangle$ is uniformly $\Sigma_1^{\langle J_{\varrho(\nu)}, A(\nu)\rangle}$;

(4) Suppose $\bar\nu \leqslant \nu$, $\bar C \subseteq J_{\bar\nu}$, and $\sigma\colon \langle J_{\bar\nu}, \bar C\rangle \prec_Q \langle J_\nu, C_\nu\rangle$. Then $\bar\nu \in S^1$, $\bar C = C_{\bar\nu}$, and $\sigma(\alpha_{\bar\nu}) = \alpha_\nu$. Furthermore, there is a $\tilde\sigma \supseteq \sigma$ such that $\tilde\sigma\colon \langle J_{\varrho(\bar\nu)}, A(\bar\nu)\rangle \prec_{\Sigma_1} \langle J_{\varrho(\nu)}, A(\nu)\rangle$.

(5) Suppose $\bar\nu$ is a limit point of $S_{\alpha_{\bar\nu}}$ and $\sigma\colon \langle J_{\bar\nu}, C_{\bar\nu}\rangle \prec_Q \langle J_\nu, C_\nu\rangle$. If $\lambda = \sup\limits_{\nu' < \bar\nu} \sigma(\nu')$, then $\lambda \in S_{\alpha_\nu}$ and $C_\lambda = \lambda \cap C_\nu$.

Assuming Θ, we may define our morass as follows.

For $\bar\nu, \nu \in S^1$, set $\bar\nu \lessdot \nu$ iff $\alpha_{\bar\nu} < \alpha_\nu$ and $(\exists\pi)[\pi\colon \langle J_{\bar\nu}, C_{\bar\nu}\rangle \prec_Q \langle J_\nu, C_\nu\rangle$ and $\pi \restriction \alpha_{\bar\nu} = \mathrm{id} \restriction \alpha_{\bar\nu}]$.

By $\Theta(2)$, the map π in the above is unique, so we set $\hat\pi_{\bar\nu\nu}$ = that π.

For $\bar\nu \lessdot \nu$, define $\pi_{\bar\nu\nu}\colon (\bar\nu+1) \to (\nu+1)$ by $\pi_{\bar\nu\nu} = (\hat\pi_{\bar\nu\nu} \restriction \bar\nu) \cup \{\langle\nu, \bar\nu\rangle\}$.

It is immediate that $\lessdot$ is a tree and that the systems $\{\hat\pi_{\bar\nu\nu} \mid \bar\nu \lessdot \nu\}$, $\{\pi_{\bar\nu\nu} \mid \bar\nu \lessdot \nu\}$ are commutative systems of maps.

Let $U \subseteq \omega_2 \times \omega_2$ be the canonical $\Sigma_1^{J_{\omega_2}}$-predicate such that:

(i) $U''\{\nu\} \subseteq \nu$;

(ii) $\langle U''\{\nu\} \mid \omega_1 < \nu < \omega_2\rangle$ enumerates all bounded subsets of ω_2;

(iii) $\nu < \tau \to U''\{\nu\} <_J U''\{\tau\}$.

For each $\nu \in S_{\omega_1}$, set $U_\nu = U \cap (\nu \times \nu)$. Thus U_ν is $\Sigma_1^{J_\nu}$. If $\nu \in S_{\omega_1}$ and $\bar\nu \lessdot \nu$, then $\hat\pi_{\bar\nu\nu}\colon J_{\bar\nu} \prec_{\Sigma_1} J_\nu$, so U_ν induces a $\Sigma_1^{J_{\bar\nu}}$ set $U_{\bar\nu} \subseteq \bar\nu \times \bar\nu$.

Set $\mathfrak{I} = \langle S, \mathcal{S}, \lessdot, (\pi_{\bar\nu\nu})_{\bar\nu\lessdot\nu}, (U_\nu)_{\nu\in S^1}\rangle$.

Using Θ, it is not very hard to show that $\mathfrak{I}$ is an ω-morass. In particular, (M4) requires $\Theta(4)$, (M6) requires $\Theta(5)$, and (M7) requires $\Theta(4)$. (These are the most tricky cases.)

Θ is established as follows. Construct the Θ-sequence in two disjoint cases, which do not clash at $\Theta(3)$-$\Theta(5)$ *in any way*. (eg. if ν in $\Theta(4)$ is in case 1, so is the $\bar\nu$.)

Case 1. $\beta(\nu) = \gamma+1$, $n(\nu) = 1$. (Thus $\varrho(\nu) = \beta(\nu)$ and $A(\nu) = \emptyset$.)

Let $p(\nu)$ = the $<_J$-least $p \in J_\gamma$ such that:

(i) Every $x \in J_\gamma$ is J_γ-definable from parameters in $\alpha_\nu \cup \{p\}$;

(ii) α_ν is Σ_1-definable in J_γ from p.

For each $m < \omega$, set $X_{m+1} = \{x \in J_\gamma \mid x$ is Σ_{m+1}-definable in J_γ from parameters in $\alpha_\nu \cup \{p(\nu)\}\}$. Then $X_{m+1} \cap J_\nu$ is transitive, and $X_{m+1} \prec_{\Sigma_{m+1}} J_\gamma$. Let $\pi_{m+1}\colon X_{m+1} \cong J_{\gamma_{m+1}}$, and set $p_{m+1} = \pi_{m+1}(p(\nu))$. Then $\langle\gamma_{m+1} \mid m < \omega\rangle$ is cofinal in ν. Set $\gamma_0 = p_0 = \alpha(\nu)$. Let $E_\nu = \{\langle\gamma_m, p_m\rangle \mid m < \omega\}$. Let $C_\nu \subseteq \nu$ be a canonical p.r. coding of E_ν.

The proof that $\langle C_\nu \mid \nu \in S^1$ and $\nu \in$ Case $1\rangle$ is as in Θ is similar to (but much easier than) the proof of $\square$ in [4], using § 4 of [4] heavily.

Case 2. Otherwise.

Let $p(\nu)$ = the $<_J$-least $p \in J_{\varrho(\nu)}$ such that:

(i) Every $x \in J_{\varrho(\nu)}$ is Σ_1-definable in $\langle J_{\varrho(\nu)}, A(\nu)\rangle$ from parameters in $\alpha_\nu \cup \{p\}$;

(ii) α_ν is Σ_1-definable in $J_{\varrho(\nu)}$ from p.

Let $k(\nu)$ = the least k such that $\alpha_\nu, p(\nu) \in J_{k+1}$.

Let h be the canonical Σ_1 Sholem function for $\langle J_{\varrho(\nu)}, A(\nu)\rangle$. For $\tau < \varrho(\nu)$, let h_τ be the restriction of h to $\langle J_\tau, A(\nu) \cap J_\tau\rangle$.

For $k(\nu) < \tau < \varrho(\nu)$, set $X_\tau = h_\tau''(\omega \times (J_{\alpha_\nu} \times \{p(\nu)\}))$. Providing we have chosen h a little more carefully than might be the case if we followed [4] literally, we have $X_\tau \prec_{\Sigma_1} \langle J_\tau, A(\nu) \cap J_\tau\rangle$ for *all* τ. Each $X_\tau \cap J_\nu$ is transitive. Set $J_{\nu_\tau} = X_\tau \cap J_\nu$. Let $\pi_\tau\colon \langle X_\tau, A(\nu) \cap X_\tau\rangle \cong \langle J_{\gamma_\tau}, A_{\gamma_\tau}\rangle$, and set $p_\tau = \pi_\tau(p(\nu))$. Set $p_k = \gamma_k = \nu_k = \alpha$, $A_\alpha = \emptyset$. Let $E_\nu = \{\langle\gamma_\tau, A_{\alpha_\tau}, p_\tau, \nu_\tau\rangle \mid k(\nu) \leqslant \tau < \varrho(\nu)\}$. Let $C_\nu \subseteq \nu$ be a canonical p.r. coding of E_ν. By choice of $p(\nu)$, $\langle\gamma_\tau \mid k \leqslant \tau < \varrho(\nu)\rangle$ is cofinal in ν, so we may assume C_ν is.

The proof that $\langle C_\nu \mid \nu \in S^1$ and $\nu \in$ Case $2\rangle$ is as in Θ is again similar to the proof of $\square$ in [4]. The argument turns on the following easily established fact. Suppose $x \in J_\nu$, $\vec y \in J_{\alpha_\nu}$. Then x will be Σ_1-definable from $\vec y, p(\nu)$ in $\langle J_{\varrho(\nu)}, A(\nu)\rangle$ just in case x is Σ_1-definable from $\vec y$ (alone) in $\langle J_\nu, E_\nu\rangle$.

That completes our sketch. We should remark that, as we have written this out, it has taken up $3\frac{1}{2}$ sides of paper. The full argument, as we have it written out (on the same kind of paper, etc.) takes up some 24 sides. The reader has been warned. (Though, indeed, most of the details required *are* buried in [4], at some point or other.)

Remark. Since we have stated that all our proofs generalize to arbitrary $\varkappa$, we should perhaps mention how the above generalizes, since the observant reader may have noticed that ω figured heavily in our definition of $\mathcal{S}$ (more precisely, ω^+ so figured). In order to construct a $\varkappa$-morass for $\varkappa > \omega$, one would work entirely above $\varkappa$ and carry $\varkappa$ as a constant of all structures concerned, whence it would be entirely analogous to speak of $\varkappa^+$ (since $\varkappa$ would, for the purpose in hand, be "absolute").

References

[1] P. Erdös and A. Hajnal, *Unsolved problems in set theory*, in: Axiomatic Set Theory, Proc. Symp. Pure Maths. 13 (1) (1971), pp. 17–48.
[2] U. Felgner, *Models of ZF-set theory*, Lecture Notes in Maths. 223 (1971).
[3] A. Hajnal, and I. Juhász, *A consistency result concerning hereditarily a-separable spaces*, Preprints of Math. Inst. of Hung. Acad. Sci. 5 (1972).
[4] R. B. Jensen, *The fine structure of the constructible hierarchy*, Annals of Math. Logic 4 (1972), pp. 229–308.

Reçu par la Rédaction le 2. 2. 1973

Degrees of dependence in the theory of semisets

by

Petr Hájek (Prague)

> Probably we shall have in the future essentially differen intuitive notions of sets just as we have different notion of space, and will base our discussions of sets on axiom that correspond to the kind of sets which we want to study
>
> *A. Mostowski* (*)

Abstract. The relation of dependence between semisets is studied through the special outlook of the theory of degrees of unsolvability. Degrees of dependence are compared with similar notions of Recursion theory and Set theory; the Kleene-Post-Spector theorem on bounds is transferred to the theory of semisets; some consequences concerning transitive models of set theory are deduced.

The aim of this paper is to study the relation of dependence between semisets through the special outlook of the theory of degrees of unsolvability. We shall not attempt to create any systematic theory of degrees of dependence; we only show that it is possible and perhaps useful to consider them. In Section 1 we give a summary of used notions of the theory of semisets and try (again) to formulate the relations of the theory of semisets to the set theory. In Section 2 we compare degrees of dependence with similar notions of Recursion theory and Set theory. In Section 3 we transfer a proof of a (Kleene-Post-Spector) theorem on upper and lower bounds of degrees of unsolvability into the theory of semisets and obtain a result on bounds of degrees of dependence. So we have an application of Recursion theory to the theory of semisets. In Section 4 we formulate some consequences concerning model-classes ("transitive models containing all the ordinals") in Set theory in the style of usual applications of the theory of semisets to Set theory. It is of some interest that we obtain a model-class without the axiom of choice by a construction that does not use any notion of symmetry.

(*) A. Mostowski, *Recent results in set theory*, in: Problems in the Philosophy of Mathematics, 1967, p. 24.