On the sequence of models HOD_n

by

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Abstract. It is shown that iterating the HOD operation can lead to a strictly decreasing sequence of models HOD_n and that the limit model HOD_ω may or may not satisfy AC.

For a discussion of Myhill and Scott's notion of ordinal-definable set, the reader is referred to [7]. In Zermelo–Fraenkel set theory the class of hereditarily ordinal-definable sets, denoted HOD, is an inner model which satisfies ZFC. If we then iterate and consider the class of sets hereditarily ordinal definable in HOD, we obtain a second class HOD_2 which is again a model of ZFC. Continuing we can define the sequence HOD_n recursively as follows:

$$\mathrm{HOD}_0 = \mathrm{V}\,, \quad \mathrm{HOD}_{n+1} = \mathrm{HOD}^{\mathrm{HOD}_n} = \mathrm{HOD}_n^{\mathrm{HOD}}\,.$$

Unlike the passage from V to L which is idempotent since L is absolute, iterating the passage to HOD can produce a strictly decreasing sequence of models, as we shall see. For each standard $n \geqslant 1$, HOD_n is an inner model satisfying ZFC. It is not known whether the sequence $\langle \mathrm{HOD}_n, n \in \omega\rangle$ is "expressible" in ZF set theory. However, in recent work S. Grigorieff has shown that this sequence is "expressible" in the theory $\mathrm{ZF} + \exists x(\mathrm{V} = \mathrm{L}[x])$, where $\mathrm{L}[x]$ denotes the smallest transitive inner model of ZF containing all ordinals and $\{x\}$. Furthermore, in this theory the sequence can be extended to the transfinite so that at limit stages λ, $\mathrm{HOD}_\lambda = \bigcap_{\beta<\lambda} \mathrm{HOD}_\beta$ and HOD_λ also satisfies ZF; more precisely (cf. [2]),

THEOREM (Grigorieff). *There is a formula $E(\alpha, x, y)$ of the language of set theory such that the following are theorems of* ZF:

(i) $\forall\alpha\exists x E(\alpha, x, y) \wedge \forall x[E(0, x, y) \rightarrow \mathrm{L}[x] = \mathrm{L}[y]]$,

(ii) $E(\alpha, x, y) \wedge E(\alpha+1, z, y) \rightarrow \mathrm{L}[z] = \mathrm{HOD}^{\mathrm{L}[x]}$,

(iii) $\lambda = \bigcup\lambda \neq 0 \wedge E(\lambda, x, y) \rightarrow \mathrm{L}[x] = \{u\colon \forall\alpha < \lambda\forall z[E(\alpha, z, y) \rightarrow u \in \mathrm{L}[z]]\}$.

In this paper we shall construct — using the techniques of our [5] — two models of $ZF+\exists x(V=L[x])$, where the sequence HOD_n is strictly decreasing; in one example, the limit model HOD_ω will not satisfy AC and in the other the limit model HOD_ω will satisfy AC (but not $V=L$; cf. the proposition of § 5). In terms of relative consistency, we have

THEOREM 1. *If ZF is consistent, then so is* $ZF+\exists x(V=L[x])+\forall n(HOD_n\neq HOD_{n+1})\pm AC^{HOD_\omega}$.

As is to be expected, the forcing constructions involved yield the existence of complete Boolean algebras with special properties. Given a complete Boolean algebra B we denote by B', the *derivative* of B, the complete subalgebra of B of all elements which are left fixed by all automorphisms of B. P. Vopenka has established, cf. [4], the connection between the derivative B' and the class HOD in B generic extensions of the universe. Using this result, we obtain

THEOREM 2 (ZFC). *There exist complete Boolean algebras B such that the sequence $B^{(n)}$, $n\in\omega$, of successive derivatives is strictly decreasing.*

Considering Professor Mostowski's long established preeminence in the field of set theory, it is a great honor to contribute a paper on this subject to the volume dedicated to him on the occasion of his sixtieth birthday.

1. Construction of the models. We shall religiously follow the notation and conventions of [5]. We have a fixed model M_0 which satisfies $ZF+V=L$ and a fixed set A which is $(\mathcal{Q}_{X_0},\Gamma_{X_0})$ generic over M_0. We now define

$$R=\{p\colon p \text{ is a finite function, } \operatorname{dom}p\subseteq(\omega_0-\{0\})\times\omega_1, \text{ and } \operatorname{rng}p\subseteq 2\}$$

and set the order relation $\leqslant$ on elements of R to be the inclusion ordering. Also we set

$$\Theta=\{\langle p,n,\beta\rangle\colon p\in R \text{ and } p(n,\beta)=0\}.$$

Let D be a fixed (R,Θ) generic set over $M_0[A^\omega]$.

For simplicity of notation, we denote M_0 henceforth by L.

The model $L[A^\omega][D]$ satisfies $ZF+GCH+V=L[D\cup A^\omega]$; moreover, the models L and $L[D\cup A^\omega]$ have the same regular cardinals.

We recall that A^ω is $(\mathcal{Q}_Y,\Gamma_Y)$ generic over L where $Y=X_0-\omega$. Consider the product set of conditions $S=R\times\mathcal{Q}_Y$. For $p\in S$ we have $p=\langle p_1,p_2\rangle$ with $p_1\in R$ and $p_2\in\mathcal{Q}_Y$. Set $\Lambda=\{\langle p,\alpha,\beta\rangle\colon p\in S \text{ and } (\langle p_1,\alpha,\beta\rangle\in\Theta \text{ or } \langle p_2,\alpha,\beta\rangle\in\Gamma_Y)\}$. The pair (S,Λ) is in L and so by [8, § I.2], the set $D\cup A^\omega$ is (S,Λ) generic over L. By [1] and/or [6], we have $L[D\cup A^\omega]\models P(\omega_1)=L[D]\cap P(\omega_1)$. We define an integer valued function h by

$$h(\alpha)=n\Leftrightarrow\langle 0,n\rangle\in\tilde{E}(\alpha).$$

Notice that $\langle\gamma,\delta\rangle\in\tilde{E}(\alpha)$ and $\gamma\neq 0$ imply $h(\gamma)=h(\alpha)$. We set, for $k\in\omega$,

$$B_k=\{\langle\alpha,\beta\rangle\in D\cup A^\omega\colon |\tilde{E}(\alpha)|\leqslant h(\alpha)-k \text{ and } \forall\gamma,\delta(\gamma\neq 0 \text{ and } \langle\gamma,\delta\rangle\in\tilde{E}(\alpha)\Rightarrow\langle\gamma,\delta\rangle\in D\cup A^\omega)\}.$$

We note that

$$B_{k+1}=\{\langle\alpha,\beta\rangle\in B_k\colon |\tilde{E}(\alpha)|<h(\alpha)-k\}$$

and

$$B_k=B_{k+1}\cup\{\langle\alpha,\beta\rangle\in D\cup A^\omega\colon |\tilde{E}(\alpha)|=h(\alpha)-k \text{ and } E(\alpha)\in B_{k+1}\}.$$

We now state

PROPOSITION 1. (i) *The models $L[B_k]$, $k\in\omega$, form a strictly decreasing sequence;*

(ii) $L[B_k]=HOD_k^{L[B_0]}$ *and hence* $\bigcap_{k<\omega}L[B_k]=HOD_\omega^{L[B_0]}$;

(iii) $\bigcap_{k<\omega}L[B_k]\models P(\omega)$ *is not well-orderable.*

Theorem 1 with $-AC^{HOD_\omega}$ follows from this proposition. Before proceeding to the proof, we describe the model for the $+AC^{HOD_\omega}$ part and state the corresponding proposition.

We set R^* to be the set $\{p\colon p\in L, p \text{ is a function}, \operatorname{dom}p\subseteq(\omega-\{0\})\times\omega_1, \operatorname{rng}p\subseteq 2 \text{ and } |p|<\omega_1\}$. We put the inclusion ordering on R^* and we set $\Theta^*=\{\langle p,n,\beta\rangle\colon p\in R^* \text{ and } p(n,\beta)=0\}$. Fixing a set D^* which is (R^*,Θ^*) generic over $L[A^\omega]$, we can then define, for $k\in\omega$,

$$B_k^*=\{\langle\alpha,\beta\rangle\in D^*\cup A^\omega\colon |\tilde{E}(\alpha)|\leqslant h(\alpha)-k \text{ and } \forall\gamma,\delta(\gamma\neq 0 \text{ and } \langle\gamma,\delta\rangle\in\tilde{E}(\alpha)\Rightarrow\langle\gamma,\delta\rangle\in D^*\cup A^\omega)\}.$$

PROPOSITION 1*. (i*) *The models $L[B_k^*]$ form a strictly decreasing sequence;*

(ii*) $L[B_k^*]=HOD_k^{L[B_0^*]}$ *and hence* $\bigcap_{k\in\omega}L[B_k^*]=HOD_\omega^{L[B_0^*]}$;

(iii*) $\bigcap_{k\in\omega}L[B_k^*]\models AC+V\neq L$.

The proofs of (i) and (i*) are identical as are those of (ii) and (ii*). So we shall do (i) and (ii). The disparity between (iii) and (iii*) is due essentially to the fact that $R^*\times\mathcal{Q}_Y$ is closed in L under countable monotone increasing sequences and so ${}^\omega L\cap L[B_0^*]={}^\omega L\cap L$, while ${}^\omega L\cap L[B_k]$ is strictly decreasing for $k\in\omega$.

2. Proof of (i). Let $D_k=D\cap(k+1\times\omega_1)$ and let $D^k=D-D_k$. We note that $h(\alpha)-k\geqslant|\tilde{E}(\alpha)|$ implies $h(\alpha)>k$ and so trivially

$$h(\alpha)-k\geqslant|\tilde{E}(\alpha)|\Rightarrow\forall\gamma,\delta[\langle\gamma,\delta\rangle\in\tilde{E}(\alpha) \text{ and } \gamma\neq 0\Rightarrow(\langle\gamma,\delta\rangle\in D\cup A^\omega\Leftrightarrow\langle\gamma,\delta\rangle\in D^k\cup A^\omega)].$$

Hence $L[B_k] \subseteq L[D^k \cup A^\omega]$. Moreover, by [1] and/or [6], we have

$$P(\omega_1) \cap L[D^k \cup A^\omega] = P(\omega_1) \cap L[D^k].$$

Hence, a fortiori, $L[B_k] \cap P(\omega_1) \subseteq L[D^k]$. On the other hand, by [8, § I.2], $\{n: \langle k+1, n\rangle \in D^k\}$ is a set which is $(\mathcal{S}_0, \Delta_0)$ generic over $L[D^{k+1}]$ and so is not an element of $L[D^{k+1}]$. Therefore, the sequence $L[B_k]$ is strictly decreasing.

3. Proof of (ii). We shall show that, for $k \in \omega$, $\mathrm{HOD}^{L[B_k]} = L[B_{k+1}]$. We remark that by the methods of [5, § 4], we have

$$\langle \alpha, \beta\rangle \in B_{k+1} \Leftrightarrow L[B_k] \models \mathcal{F}(F(\alpha, \beta))$$

and so the inclusion $L[B_{k+1}] \subseteq \mathrm{HOD}^{L[B_k]}$ follows. The opposite inclusion will require some further work. Recall that the set $D \cup A^\omega$ is (S, Δ) generic over L and that for $p \in S$ we have $p = \langle p_1, p_2\rangle$ with $p_1 \in R$ and $p_2 \in \mathfrak{Q}_Y$. To simplify notation, we set

$$p(\alpha, \beta) = \begin{cases} p_1(\alpha, \beta) & \text{if } \langle \alpha, \beta\rangle \in \operatorname{dom} p_1, \\ p_2(\alpha)(\beta) & \text{if } \alpha \in \operatorname{dom} p_2 \text{ and } \beta \in \operatorname{dom} p_2(\alpha), \\ \text{undefined otherwise} \end{cases}$$

and we write $\langle \alpha, \beta\rangle \in \operatorname{dom} p$ iff $p(\alpha, \beta)$ is defined.

For $p \in S$, we say that p is *compatible* with B_{k+1} iff

$$\langle \alpha, \beta\rangle \in \operatorname{dom} p \text{ and } \alpha \in \operatorname{dom} B_{k+1} \quad \text{imply} \quad p(\alpha, \beta) = 0 \Leftrightarrow \langle \alpha, \beta\rangle \in B_{k+1}.$$

LEMMA. *Let Φ be a closed formula with parameters in* L *and let $\mathcal{B}_k$ be a canonical forcing term for B_k. The following are equivalent:*

(1) $L[B_k] \models \Phi$,

(2) *there exists $p \in S$, p compatible with B_{k+1}, such that $p \Vdash \Phi^{L[\mathcal{B}_k]}$.*

Proof. The implication (1) ⇒ (2) is immediate by the Truth Lemma for forcing. For (2) ⇒ (1) let us argue by contradiction and suppose that there is a condition $q_0 \in S$ which is compatible with B_{k+1} such that

$$L[B_k] \models \Phi \quad \text{and} \quad q_0 \Vdash \neg\Phi^{L[\mathcal{B}_k]}.$$

Let G be a generic collection of conditions converging to $D \cup A^\omega$. There is a condition $p_0 \in G$ such that $p_0 \Vdash \Phi^{L[\mathcal{B}_k]}$. Clearly, p_0 is compatible with B_{k+1}; and since G is generic, we can suppose that $\operatorname{dom} p_0 \supseteq \operatorname{dom} q_0$. Set $Z = \{\langle \alpha, \beta\rangle \in \operatorname{dom} q_0: p_0(\alpha, \beta) \neq q_0(\alpha, \beta)\}$. We define an automorphism σ of S by

$$\sigma p(\alpha, \beta) = \begin{cases} 1 - p(\alpha, \beta) & \text{if } \langle \alpha, \beta\rangle \in Z, \\ p(\alpha, \beta) & \text{otherwise}. \end{cases}$$

Clearly Z and σ lie in L and $\sigma p_0 \supseteq q_0$. Now consider the set $\sigma G = \{\sigma p: p \in G\}$; this is a generic collection of conditions containing σp_0 which converges to $\sigma(D \cup A^\omega) = (D \cup A^\omega) \triangle Z$, where $\triangle$ denotes symmetric difference. We set

$$\sigma B_n = \{\langle \alpha, \beta\rangle \in \sigma(D \cup A^\omega): |\tilde{E}(\alpha)| \leqslant h(\alpha) - n \text{ and } \forall \gamma, \delta[\gamma \neq 0 \text{ and } \langle \gamma, \delta\rangle \in \tilde{E}(\alpha) \Rightarrow \langle \gamma, \delta\rangle \in \sigma(D \cup A^\omega)]\}.$$

By well known symmetry arguments (cf. [3], [6]), we have (since $\sigma^{-1} = \sigma$)

$$\sigma q_0 \Vdash \neg\Phi^{L[\sigma\mathcal{B}_k]}.$$

Since $p_0 \geqslant \sigma q_0$, we can suppose that $\sigma q_0 \in G$. Thus $L[D \cup A^\omega] \models \neg\Phi^{L[\sigma B_k]}$; that is, $L[\sigma B_k] \models \neg\Phi$. On the other hand, $L[B_k] \models \Phi$ and Φ has parameters restricted to L. To reach the desired contradiction, we shall show that $L[\sigma B_k] = L[B_k]$.

Partition Z into three subsets:

$$Z_1 = \{\langle \alpha, \beta\rangle \in Z: |\tilde{E}(\alpha)| < h(\alpha) - k\},$$

$$Z_2 = \{\langle \alpha, \beta\rangle \in Z: |\tilde{E}(\alpha)| = h(\alpha) - k\},$$

$$Z_3 = \{\langle \alpha, \beta\rangle \in Z: |\tilde{E}(\alpha)| > h(\alpha) - k\}.$$

Consider $\langle \alpha, \beta\rangle \in Z_1$. Note that $\tilde{E}(\alpha) \cap Z = \tilde{E}(\alpha) \cap Z_1$. Since $Z_1 \subseteq \operatorname{dom} q_0$ and q_0 is compatible with B_{k+1}, it follows that $\alpha \notin \operatorname{dom} B_{k+1}$. Therefore, there exists $\langle \gamma, \delta\rangle \in \tilde{E}(\alpha)$ such that $\gamma \neq 0$ and $\langle \gamma, \delta\rangle \notin D \cup A^\omega$. If we take γ to be smallest possible, we have $\langle \gamma, \delta\rangle \notin Z$ and $\tilde{E}(\gamma) \cap Z = 0$. Thus, we conclude that $\tilde{E}(\alpha) < h(\alpha) - k$ implies either

$$\langle \alpha, \beta\rangle \notin Z \quad \text{and} \quad \tilde{E}(\alpha) \cap Z = 0$$

or

$$\exists \gamma, \delta[\langle \gamma, \delta\rangle \in \tilde{E}(\alpha) \text{ and } \langle \gamma, \delta\rangle \notin Z \text{ and } \tilde{E}(\gamma) \cap Z = 0 \text{ and } \langle \gamma, \delta\rangle \notin D \cup A^\omega].$$

Whence $|\tilde{E}(\alpha)| < h(\alpha) - k$ implies $\langle \alpha, \beta\rangle \in B_k \Leftrightarrow \langle \alpha, \beta\rangle \in \sigma B_k$. Since $B_{k+1} = \{\langle \alpha, \beta\rangle \in B_k: |\tilde{E}(\alpha)| < h(\alpha) - k\}$, we have $B_{k+1} \in L[B_k] \cap L[\sigma B_k]$; in fact, $\sigma B_{k+1} = B_{k+1}$. Now set

$$W_2 = \{\langle \alpha, \beta\rangle \in Z_2: E(\alpha) \in B_{k+1}\}.$$

We remark that $W_2 \in L[B_{k+1}]$. Since

$$B_k = B_{k+1} \cup \{\langle \alpha, \beta\rangle \in D \cup A^\omega: |\tilde{E}(\alpha)| = h(\alpha) - k \text{ and } E(\alpha) \in B_{k+1}\}$$

and

$$\sigma B_k = \sigma B_{k+1} \cup \{\langle\alpha, \beta\rangle \in (D \cup A^\omega) \triangle Z\colon E(\alpha) \in \sigma B_{k+1} \text{ and } |\widetilde{E}(\alpha)| = h(\alpha) - k\},$$

we conclude that $\sigma B_k = B_k \triangle W_2$ and $B_k = \sigma B_k \triangle W_2$, which completes the proof of the lemma.

To establish (ii) it suffices to check that every set of ordinals which is definable in $L[B_k]$ by means of a formula with parameters in L is in fact a set in $L[B_{k+1}]$. Suppose therefore that X is a set of ordinals which is defined in $L[B_k]$ by a formula $\Phi(v)$ with parameters in L; that is,

$$\alpha \in X \Leftrightarrow L[B_k] \models \Phi[\alpha].$$

By the above lemma we have

$$\alpha \in X \Leftrightarrow \exists p [p \in S,\ p \text{ compatible with } B_{k+1} \text{ and } p \Vdash \Phi(\alpha)].$$

The comprehension scheme holds in $L[B_{k+1}]$ and so $X \in L[B_{k+1}]$. Hence $\mathrm{HOD}^{L[B_k]} = L[B_{k+1}]$ and (ii) is proved.

4. Proof of (iii). To simplify notation, we denote $\bigcap_{k<\omega} L[B_k]$ by HOD_ω. Recall from § 2 that L and $L[B_0]$ have the same cardinals and that GCH holds in both models. Therefore, since HOD_ω is an inner model of $L[B_0]$, if $P(\omega) \cap \mathrm{HOD}_\omega$ is well ordered in HOD_ω, there must exist a set $X_0 \in \mathrm{HOD}_\omega$, $X_0 \subseteq \omega_1$, such that

$$(1) \qquad \mathrm{HOD}_\omega \models \forall x (x \subseteq \omega \to x \in L[X_0]).$$

Recall from § 3 that $P(\omega_1) \cap L[B_k] = P(\omega_1) \cap L[D^k]$ and so $P(\omega_1) \cap \mathrm{HOD}_\omega = P(\omega_1) \cap \bigcap_{k<\omega} L[D^k]$. We will need the following

LEMMA. *Suppose that $X \subseteq On$ and that $X \in \bigcap_{n<\omega} L[D^n]$. Then there exists $\alpha < \omega_1$ such that $X \in L[D \cap (\omega \times \alpha)]$.*

Proof. We say that an automorphism σ of R, $\sigma \in L$, is *restricted* to D_k iff for all $p \in R$, $\mathrm{dom}\, p$ disjoint from $(k+1) \times \omega_1$ implies $\sigma p = p$. Let t be a forcing term such that the empty condition weakly forces $t \subseteq On$. We say that t is *independent* of D_k iff for every automorphism σ of R which is restricted to D_k we have $p \Vdash^* \alpha \in t \Leftrightarrow \sigma p \Vdash^* \alpha \in t$ for all p and α.

Since D_k is generic over $L[D^k]$, for every k there is a term t which is independent of D_k such that $\mathrm{Val}_D(t) = X$. The ordered set R satisfies the countable antichain condition in L, and so there exists an ordinal $\alpha < \omega_1$ and a sequence $\langle t_n \rangle_{n\in\omega}$ of terms satisfying

(a) the sequence $\langle t_n \rangle$ is an element of $L[D \cap (\omega \times \alpha)]$,

(b) for every n the term t_n is independent of D_n and $\mathrm{Val}_D(t_n) = X$,

(c) for every n there exists $q \in R$, $\mathrm{dom}\, q \subseteq \omega \times \alpha$, q compatible with D such that $q \Vdash t_n = t_{n+1}$.

We claim that

$$(2) \qquad \beta \in X \Leftrightarrow \exists n \exists p\, [\mathrm{dom}\, p \subseteq \omega \times \alpha,\ p \text{ is compatible with } D \text{ and } p \Vdash^* \beta \in t_n].$$

Since the comprehension scheme is valid in $L[D \cap (\omega \times \alpha)]$, the equivalence (2) will suffice to establish the lemma. We note that the direction $\Leftarrow$ is immediate. So suppose $\beta \in X$. There exists then a condition q compatible with D such that $q \Vdash \beta \in t_0$. Let k be an integer such that $\mathrm{dom}\, q \subseteq (k+1) \times \omega_1$. By (c) there is a condition q_0 with domain a subset of $\omega \times \alpha$ such that $q_0 \Vdash t_0 = t_1 = \ldots = t_k$. Define $q_\alpha = q \restriction (\omega \times \alpha)$. The condition $q_\alpha \cup q_0$ is compatible with D and its domain is a subset of $\omega \times \alpha$. We now want to show that

$$(3) \qquad q_\alpha \cup q_0 \Vdash^* \beta \in t_k.$$

Suppose that (3) fails to hold and that there is a condition $r \supseteq q_\alpha \cup q_0$ such that $r \Vdash \beta \notin t_k$. Let $W = \{\langle n, \gamma\rangle\colon r(n, \gamma) \neq q(n, \gamma)\}$. Note that $\langle n, \gamma\rangle \in W$ implies $n \leqslant k$ and $\alpha \leqslant \gamma$. So we define an automorphism τ of R by

$$\tau p(m, \gamma) = \begin{cases} p(m, \gamma) & \text{if } \langle m, \gamma\rangle \notin W, \\ 1 - p(m, \gamma) & \text{if } \langle m, \gamma\rangle \in W. \end{cases}$$

Clearly $\tau(q \cup q_0) = \tau q \cup q_0$ and $\tau q \cup q_0 \cup r = \tau q \cup r$ is also a condition. Since τ is the identity on all conditions with domain disjoint from $(k+1) \times \omega_1$, it is restricted to D_k. On the other hand, t_k is independent of D_k and so, since $q \cup q_0 \Vdash^* \beta \in t_k$, we have $\tau q \cup q_0 \Vdash^* \beta \in t_k$; and at the same time we have $\tau q \cup r \Vdash \beta \notin t_k$ which is a contradiction. This establishes (3) and the lemma.

We shall now show that (1) is impossible. Applying the above lemma, we find an ordinal $\alpha_0 < \omega_1$ such that

$$P(\omega) \cap L[X_0] \subseteq L[D \cap (\omega \times \alpha_0)].$$

Consider the set $x \subseteq \omega$ defined by

$$n \in x \Leftrightarrow \langle n, \alpha_0\rangle \in D.$$

The set x is Cohen generic over $L[D \cap (\omega \times \alpha_0)]$ and so $x \notin L[D \cap (\omega \times \alpha_0)]$. However, for every $k \in \omega$,

$$x - (k+1) = \{n\colon \langle n, \alpha_0\rangle \in D^k\}$$

is an element of $L[D^k]$; hence for every k, $x \in L[B_k]$. Thus $x \in HOD_\omega$ but $x \notin L[X_0]$. This contradicts (1) and completes the proof of (iii).

5. Proof of (iii*). We will establish a general result from which (iii*) easily follows. To conform with Boolean notation, cf. [9], we shall reverse the sense of the ordering on conditions and write $p \leqslant q$ to mean that p is an extension or refinement of q.

We say that a set of conditions P in L satisfies the *condition of decreasing sequences* (c.d.s.) iff every monotone decreasing sequence in L of elements of P has a lower bound.

PROPOSITION. *Let P satisfy* c.d.s. *in* L *and let G be P generic over* L. *Then* $HOD_\omega^{L[G]} \models AC$. *Furthermore, if for all $k \in \omega$,* $HOD_k^{L[G]} \neq L$, *then* $HOD_\omega^{L[G]} \neq L$.

Proof. The ordered set P can be embedded in L in a complete Boolean algebra B so that P is dense in B. Let $B^{(n)}$ denote the successive derivatives of B in L. We can suppose that the $B^{(n)}$ are all atom free. The P generic set G extends canonically to an L complete ultrafilter on B which we shall denote by G_0. We set $G_k = G_0 \cap B^{(k)}$ and we have, by Vopenka's result

$$HOD_k^{L[G]} = L[G_k].$$

We say that $p \in P$ is *compatible* with G_k iff $p \Vdash^* \boldsymbol{b} \in \mathcal{G}_k \Rightarrow b \in G_k$, where $\mathcal{G}_k$ is a canonical term for G_k. We set $X = \{p \in P: \exists k (p \text{ is compatible with } G_k)\}$. We claim that

$$HOD_\omega^{L[G]} = L[X] \neq L.$$

First we show that $X \in HOD_\omega^{L[G]}$. Remark that

$$p \text{ compatible with } G_n \text{ and } m \geqslant n \Rightarrow p \text{ compatible with } G_m.$$

Thus $X \in L[G_k]$ for all k and $L[X] \subseteq HOD_\omega^{L[G]}$. To establish the converse inclusion, it suffices to show that

$$Y_0 \subseteq On \quad \text{and} \quad Y_0 \in HOD_\omega^{L[G]} \Rightarrow Y_0 \in L[X].$$

(Basically this is a proof by induction; we can reduce to $Y_0 \subseteq On$ since $X \subseteq L$ and $L[X] \models AC$.) By the fact that P satisfies c.d.s. in L, there is a sequence $\langle t_n \rangle_{n\in\omega}$ of forcing terms, the sequence lying in L, such that $Val_{\mathcal{G}_n}(t_n) = Y_0$. Let $p_0 \in X$ be a condition such that $p_0 \Vdash Val_{\mathcal{G}_n}(t_n) = Val_{\mathcal{G}_{n+1}}(t_{n+1})$ for all $n \in \omega$. We assert that

$$a \in Y_0 \Leftrightarrow \exists p [p \in X, p \leqslant p_0 \text{ and } \forall_n (p \Vdash \boldsymbol{a} \in Val_{\mathcal{G}_n}(t_n))].$$

The direction $\Rightarrow$ is immediate. So suppose $p \leqslant p_0$ and that $p \Vdash \boldsymbol{a} \in Val_{\mathcal{G}_n}(t_n)$ for all n. Let $b_n = [\![\boldsymbol{a} \in t_n]\!]^{B(n)}$. We have $p \Vdash^* \boldsymbol{b}_n \in \mathcal{G}_n$ for all n. Hence, since $p \in X$, there exists n_0 such that $b_{n_0} \in G_{n_0}$; thus, by the Truth Lemma for forcing, $a \in Val_{G_{n_0}}(t_{n_0}) = Y_0$.

It remains to show that $X \notin L$. Following Solovay–Tennenbaum [9], we define $(p)_k = \text{glb}\{b \in B^{(k)}: p \leqslant b\} = \text{glb}\{b \in B^{(k)}: p \Vdash \boldsymbol{b} \in \mathcal{G}_k\}$. We note the following fact: if $p, q \in P$, there are p', q' such that $p' \leqslant p$, $q' \leqslant q$ and $(p')_k$ and $(q')_k$ are disjoint. To check this we use the fact that $B^{(k)}$ is atom free to find $b_1 \leqslant (p)_k$, $b_2 \leqslant (q)_k$ such that b_1 and b_2 are disjoint non-zero elements of $B^{(k)}$. Since P is dense in B, there exist $p' \leqslant p$, $q' \leqslant q$ such that $p' \leqslant b_1$ and $p' \Vdash \boldsymbol{b}_1 \in \mathcal{G}_k$, $q' \leqslant b_2$ and $q' \Vdash \boldsymbol{b}_2 \in \mathcal{G}_k$.

Let $\mathcal{X}$ be a canonical term for X and let $p \in P$. Suppose that $X_0 \in L$ and that $p \Vdash \mathcal{X} = \boldsymbol{X}_0$. If $q \leqslant p$, then clearly $q \Vdash^* \boldsymbol{q} \in \mathcal{X}$ and so $q \in X_0$. In other words, $p \Vdash \mathcal{X} \in L \Rightarrow p \Vdash^* \forall q (q \leqslant \boldsymbol{p} \rightarrow q \in \mathcal{X})$. Now we can construct in L a pair of monotone decreasing sequences $\langle p_n \rangle_{n\in\omega}$ and $\langle q_n \rangle_{n\in\omega}$ satisfying

(a) $p_0 \leqslant p$ and $q_0 \leqslant p$,

(b) $p_n \Vdash q_n \notin \mathcal{G}_n$ and $q_n \Vdash p_n \notin \mathcal{G}_n$.

Let p' be a lower bound of $\langle p_n \rangle$ and q' a lower bound of $\langle q_n \rangle$. We have $p' \Vdash \boldsymbol{q}' \notin \mathcal{G}_n$ for all $n \in \omega$ and $q' \Vdash \boldsymbol{p}' \notin \mathcal{G}_n$ for all $n \in \omega$. Therefore, $p \Vdash^* \boldsymbol{p}' \in \mathcal{X} \rightarrow \boldsymbol{q}' \notin \mathcal{X}$ which means that p can not force $\mathcal{X} \in L$. Since p was arbitrary we conclude that $X \notin L$.

6. Remarks. (Added in proof). In recent (independent) work, T. Jech has

(a) For arbitrary $\lambda \in L$, to constructed models $L[G]$ where the sequence HOD_α is strictly decreasing for $\alpha < \lambda$; and $HOD_\alpha \models AC$ for $\alpha \leqslant \lambda$.

(b) Constructed models $L[G]$ where the sequence HOD_n, $n \in \omega$, is strictly decreasing and where $HOD_\omega = L$.

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