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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Mostowski's collapsing function and the closed unbounded filter

by

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Abstract. We improve a result of Lévy by showing that if $P(x)$ is a Σ predicate of set theory and if $P(a)$ holds then $P(b)$ holds for almost every countable approximation b of a. Applications to model theory are discussed.

§ 1. Introduction. Our purpose here is to use Mostowski's collapsing function $c_s(x)$ to shed some light on an interesting Löwenheim-Skolem phenomenon in infinitary logic recently discovered by D. Kueker [4]. For each countable s, $c_s(x)$ can be thought of as a countable approximation x^s of x. Using the notion of "almost all" given by the closed unbounded filter we show that if P is a Σ predicate of set theory and $P(x, y)$ holds then $P(x^s, y^s)$ is true for almost all $\dot{s}$. Several of Kueker's results follow as well as Lévy's result that if a Σ predicate $P(x)$ has a solution a, then it has some hereditarily countable solution (namely a^s for almost all s).

For applications to model theory it is most natural for us to work in a universe of set theory which allows the existence of individuals. The universe of sets we have in mind can be described as follows. We are given a collection M of individuals (atoms, urelements) which can be used to form sets.

$$V_M(0) = 0 = \text{the empty set},$$
$$V_M(\alpha+1) = \text{the set of all subsets of } V_M(\alpha) \cup M,$$
$$V_M(\lambda) = \bigcup_{\alpha<\lambda} V_M(\alpha) \text{ for limit } \lambda,$$
$$V_M = \bigcup_{\alpha} V_M(\alpha).$$

The union in the last equation is taken over the class of all ordinals. A *set on M* is, by definition, an element of V_M. The reader who feels un-

(1) We are grateful to members of the logic seminar in Madison, Wisconsin, where we worked through Kueker's paper [4] together last spring (1972). Our research was partially supported by Grant NSF GP-27633.

comfortable with individuals can simply assume there are none, i.e., that $M = \emptyset$.

We use variables

$$p, q, r \text{ to range over } M,$$
$$a, b, c \text{ to range over } V_M, \text{ and}$$
$$x, y, z \text{ to range over } M \cup V_M.$$

We use ϵ for the membership relation on V_M, R, S, T for predicates on M and P, Q for predicates on $M \cup V_M$.

A set a is transitive if $x \epsilon y \epsilon a$ implies $x \epsilon a$ for all x, y. Individuals p are not considered transitive. For any x there is a smallest transitive set a such that $x \subseteq a$, called the *transitive closure* of a, $\mathrm{TC}(a)$. If x is an individual p then $\mathrm{TC}(x) = 0$. If x is a set then $\mathrm{TC}(x) = x \cup (\bigcup x) \cup \cup (\bigcup \bigcup x) \cup \ldots$ The support of a set a, $\mathrm{Sp}(a)$, is the set of individuals in $\mathrm{TC}(a)$. The *pure sets* are those sets with empty support. We use $\|a\|$ to denote the cardinality of $\mathrm{TC}(a)$; a is *hereditarily countable* if $\|a\| \leqslant \aleph_0$.

§ 2. The collapsing function. In Theorem 3 of Mostowski [8], it was shown that for any well founded relation R there is a unique map c_R of R onto a transitive set satisfying

$$c_R(x) = \{c_R(y)\colon\ y\, R\, x\}$$

for x in the field of R. This map c_R is called the *collapsing function*, or contraction function, for R. When R is $\epsilon \cap (s \times s)$ for some set s, we write c_s for c_R. In this case, however, there is no reason to restrict the domain of c_s to s. In our context with urelements we define $x^s (= c_s(x))$ for any x by 2.1.

2.1. Definition. For a fixed set s we define, for every x, an approximation x^s of x by recursion on ϵ as follows:

$$p^s = p\,,$$
$$a^s = \{x^s\colon\ x \epsilon s \cap a\}\,.$$

The reader unfamiliar with this function might want to read the collapsing lemma in § I.6 of Mostowski [9], though we will not need that lemma here. We need only the following.

2.2. Lemma. *Given sets a, s with s countable we have the following facts:*

(a) *If a is a set of individuals then $a^s = a \cap s$.*

(b) *The set a^s is hereditarily countable.*

(c) $\mathrm{TC}(a) \subseteq s$ *implies* $a^s = a$.

(d) *If a is transitive so is a^s; hence if α is an ordinal then α^s is a countable ordinal.*

(e) *If $s \cap \mathrm{TC}(a) = s' \cap \mathrm{TC}(a)$ for some other s' then $a^s = a^{s'}$.*

These facts are all easily verified; (b), (c) and (e) by induction on ϵ.

§ 3. The closed unbounded filter. The following definition was given in Kueker [4] and Jech [3] for the case where A was an infinite cardinal but the results go through just as well in general.

3.1. Definition. Let A be a transitive set and let I be the set $P_{\omega_1}(A)$ of all countable subsets of A. The *closed unbounded filter* D "on" A consists of all $X \subseteq I$ such that for some $X^0 \subseteq X$,

(a) every $s \epsilon I$ is a subset of some $s' \epsilon X^0$, and

(b) X^0 is closed under unions of countable chains.

3.2. Lemma. (Kueker [4], Jech [3]). *Given A and D as in 3.1, we have the following*:

(a) *D is a countably complete proper filter.*

(b) *If $X_a \epsilon D$ for all $a \epsilon A_0 \subseteq A$ then the diagonal*

$$Y = \{s\colon\ s \epsilon X_a \text{ for all } a \epsilon A_0 \cap s\}$$

is in D.

(c) *A subset $X \subseteq I$ is in D iff player* I *has a strategy for the two person game G_X given by the rules*: I *and* II *alternately choose elements of A,* I *wins if the set of their choices is in X, otherwise* II *wins* (Kueker [4]).

(A hint for the harder half of (c): Given a strategy $\sigma = \{F_n(x_1 \ldots x_n) | n < \omega\}$ for X let X^0 be set of $s \epsilon X$ closed under the various F_n; then 3.1 (a), (b) are clear for X^0.)

3.3. Definition. Let Q be a predicate of sets and individuals. For given $x_1 \ldots x_n$, in a transitive set A, we say that $Q(x_1^s \ldots x_n^s)$ *holds almost everywhere* (a.e.) if the set $\{s \epsilon P_{\omega_1}(A)\colon\ Q(x_1^s \ldots x_n^s)\}$ is a member of the closed unbounded filter on A.

The following lemma is a simple extension of a remark in Kueker [4], but since it is basic to our theorem, we sketch a proof.

3.4. Lemma. *The notion of "almost everywhere" defined in 3.3 is independent of the particular transitive set A.*

Proof. Let Q be a 1 place predicate to simplify notation, let $x \epsilon A_1 \cap A_2$, $I_i = P_{\omega_1}(A_i)$, D_i the closed unbounded filter on A_i and

$$X_i = \{s \epsilon I_i\colon\ Q(x^s)\}\,.$$

Assume $X_1 \epsilon D_1$ and let us show $X_2 \epsilon D_2$. By Lemma 3.2 (c), player I has a winning strategy σ_1 for the game G_{X_1}. By Lemma 2.2 (e) we can assume that this strategy only picks elements from $A_1 \cap A_2$ since $\mathrm{TC}(x) \subseteq A_1 \cap A_2$, and the value of x^s depend only on $s \cap \mathrm{TC}(x)$. But then player I can use σ_1 to get a winning strategy σ_2 for G_{X_2}. He simply ignores any move of II outside $A_1 \cap A_2$, replacing it by x and uses σ_1. Any $s \in I_2$ which results from such a play will agree with an $s' \in X_1$, at least on $\mathrm{TC}(x)$, in which case $x^s = x^{s'}$ by 2.2 (e). Thus I has a strategy for G_{X_2} and $X_2 \in D_2$ by 3.2 (c). ■

The results of the next section depend only on 3.2 and 3.4.

§ 4. The result. The language of set theory has a membership symbol ϵ, denoting ϵ, an equality symbol $=$ and symbols $R, S, T, \ldots$ for any relations R, S, T on M. The *Δ_0-formulas*, defined in Lévy [6], form the smallest class Φ containing the atomic formulas closed under:

(i) If φ is in Φ so is $\neg\varphi$,

(ii) if φ, ψ are in Φ so are $(\varphi\wedge\psi)$, $(\varphi\vee\psi)$,

(iii) if φ is in Φ and u, v are any variables then $\forall u \in v\varphi$ and $\exists u \in v\varphi$ are in Φ.

The *Σ-formulas* form the smallest class Φ containing the Δ_0-formulas and closed under (ii), (iii) and (iv).

(iv) If φ is in Φ and u is any variable then $\exists u\varphi$ is in Φ.

A predicate Q on $V_M \cup M$ is Σ if it is definable by a Σ formula of the above language; it is Δ if it and its negation are both Σ. The reader unfamiliar with Σ predicates should consult Lévy [6], p. 6 for basics and § 10 for examples and the theorem mentioned in the introduction.

4.1. THEOREM. *Let Q be an n-ary Σ predicate. If $Q(x_1 \ldots x_n)$ holds, then $Q(x_1^s \ldots x_n^s)$ holds* a.e.

Since the closed unbounded filter is proper we have the following consequence of the theorem.

4.2. COROLLARY. *Let P be a Δ predicate. For all $x_1 \ldots x_n$, $P(x_1 \ldots x_n)$ is true iff $P(x_1^s \ldots x_n^s)$ is true for almost all s.*

Extending some terminology of Kueker from classes of structures to arbitrary predicates we say that P is *closed downward* if for each $x_1 \ldots x_n$ with $P(x_1 \ldots x_n)$,

$$P(x_1^s \ldots x_n^s) \text{ holds a.e.}$$

Thus, the theorem states that all Σ predicates are closed downward. Note that "x is uncountable", whose negation is Σ, is not closed downward.

To prove the theorem first note that every Δ_0-formula is equivalent to one where all negations occur in front of atomic formulas. Atomic predicates are trivially closed downward (if $x \in y$ then $x^s \in y^s$ whenever $x \in s$) and negated atomic predicates of individuals are also trivially closed downward. The other negated atomic formulas fall under 4.3.

4.3. LEMMA. *The predicates $x \neq y$ and $x \notin y$ are closed downward.*

Proof. We prove $\forall x \forall y Q(x, y)$, where $Q(x, y)$ is the conjunction of

$$x \neq y \to x^s \neq y^s \text{ a.e.}$$
$$x \notin y \to x^s \notin y^s \text{ a.e.}$$
$$y \notin x \to y^s \notin x^s \text{ a.e.}$$

by a double induction over ϵ. Thus, given x_0, y_0 we prove $Q(x_0, y_0)$ assuming

$$\forall x \in x_0 \forall y Q(x, y) \tag{1}$$

and

$$\forall y \in y_0 Q(x_0, y)\,. \tag{2}$$

Case (i). Assume $x_0 \neq y_0$. If either x_0 or y_0 is an individual p then $x_0^s \neq y_0^s$ as long as $p \in s$, so we may assume both are sets. If there is an $x \in x_0$, $x \notin y_0$ then $Q(x, y_0)$ by (1) so $x^s \notin y_0^s$ a.e. whereas $x^s \in x_0^s$ a.e., so $x_0^s \neq y_0^s$ a.e. If there is a $y \in y_0$, $y \notin x_0$ then we use (2) similarly.

Case (ii). Assume $x_0 \notin y_0$. If y_0 is an individual then $x_0^s \notin y_0^s$ for all s so we assume y_0 is a set. Now for each $y \in y_0$ we have $x_0 \neq y$ and $Q(x_0, y)$ and hence the set $X_y \in D$, where

$$X_y = \{s\colon\ x_0^s \neq y^s\}\,.$$

By 3.2, the diagonal

$$Y = \{s\colon\ s \in \bigcap_{y \in s \cap y_0} X_y\}$$

is in D. Now let $s \in Y$ be fixed. For every $y \in s \cap y_0$, $s \in X_y$, i.e., $x_0^s \neq y^s$, but $y_0^s = \{y^s\colon\ y \in s \cap y_0\}$, so $x_0^s \notin y_0^s$. Thus $x_0^s \notin y_0^s$ a.e.

Case (iii). If $y_0 \notin x_0$, the proof is similar to (ii). ■

The following result, with earlier remarks and 4.3, completes the proof of Theorem 4.1. It is of interest in its own right, though, since there are predicates which are closed downward which are not Σ definable.

4.4. PROPOSITION. *The predicates closed downward are closed under conjunction, disjunction, bounded quantification and unbounded existential quantification.*

Proof. Conjunction follows from one property for filters ($X, Y \in D \Rightarrow X \cap Y \in D$) and disjunction from the other ($X \subseteq Y, X \in D \Rightarrow Y \in D$). Existential quantification is routine by induction, bounded existential follows from unbounded and Lemma 4.3. For bounded universal quantification, let $P(x_1 \ldots x_n, y)$ be closed downward and suppose

$$\forall y \in a P(x_1 \ldots x_n, y)\,.$$

We need to show that for almost all s we have

$$\forall y \in a^s P(x_1^s \dots x_n^s, y).$$

For $y \in a$ let

$$X_y = \{s\colon P(x_1^s \dots x_n^s, y^s)\}$$

(which is an element of D since P is closed downwards) and let Y be the diagonal

$$Y = \{s\colon s \in \bigcap_{y \in s \cap a} X_y\}$$

which is an element of D by 3.2 (b). We see that

$$s \in Y \quad \text{iff} \quad P(x_1^s \dots x_n^s, y^s) \quad \text{for all } y \in a \cap s$$

so we have, for $s \in Y$

$$\forall y \in a^s P(x_1^s \dots x_n^s, y)$$

since $a^s = \{y^s\colon y \in a \cap s\}$. ■

If one is willing to talk about infinitary predicates over V_M, then we see that the predicates closed downward are also closed under countable conjunctions and arbitrary disjunctions; the second is trivial, the first follows from the countable additivity of the closed unbounded filter.

§ 5. Applications to infinitary logic. Let L be a first order language with at most a countable (²) number of symbols. We think of these symbols as individuals (elements of M), formulas are built up from them by set theoretic principles and so are in V_M. We assume the reader is familiar with the infinitary languages $L_{\omega_1\omega}$ and $L_{\infty\omega}$. If $\varphi \in L_{\omega_1\omega}$ then $\varphi^s = \varphi$ for almost all s (for all $s \supseteq \mathrm{TC}(\varphi)$ by Lemma 2.2 (c)); if $\varphi \in L_{\infty\omega}$ then $\varphi^s \in L_{\omega_1\omega}$ a.e. since $L_{\infty\omega}$ is a Δ class and $L_{\omega_1\omega} = \{\varphi \in L_{\infty\omega}\colon \|\varphi\| \leqslant \aleph_0\}$. To see more clearly what φ^s means, for $\varphi \in L_{\infty\omega}$, define $\varphi^{[s]}$, for all s and all $\varphi \in L_{\infty\omega}$ by recursion:

$$\varphi^{[s]} = \varphi \quad \text{if } \varphi \text{ is atomic},$$

$$(\neg\varphi)^{[s]} = \neg(\varphi^{[s]}),$$

$$(\exists v\varphi)^{[s]} = \exists v(\varphi^{[s]}),$$

$$(\bigwedge_{j \in J} \varphi_j)^{[s]} = \bigwedge_{j \in J \cap s} (\varphi_j^{[s]}).$$

A simple inductive proof shows that $\varphi^s = \varphi^{[s]}$ for almost all s; i.e., that our φ^s is almost always equal to φ^s as defined in Kueker [4].

(²) This requirement on L could be dropped by replacing L by L^s at certain points. Since L is countable, $L = L^s$ a.e.

An *L-structure* $\mathfrak{A}$ is of the form $\langle A, F\rangle$ where A is a set of individuals (³) and F is a function which assigns to each symbol in L an interpretation of the appropriate kind. Then, for almost all s, $\mathfrak{A}^s$ is just the substructure $\mathfrak{A}_0$ of $\mathfrak{A}$ with universe $A_0 = A \cap s$.

Theorem 4.1 and its corollary were inspired by two results in Kueker [4], 5.1 and 5.2 below.

5.1. THEOREM (Kueker). *Let $\varphi \in L_{\infty\omega}$ and let $\mathfrak{A}$ be an L structure. Then $\mathfrak{A} \models \varphi$ if and only if*

$$\mathfrak{A}^s \models \varphi^s \text{ a.e.}$$

Proof. $\models$ is a Δ relation so the result follows from 4.2. ■

The proof shows that the result holds for logics stronger than $L_{\infty\omega}$. In the terminology of [2], the result goes through with $L_{\infty\omega}$ replaced by any absolute logic L^*. If L^* is absolute and $\varphi \in L^*$ then $\varphi^s \in L^*_{\omega_1}$ a.e. and $\mathfrak{A} \models \varphi$ iff $\mathfrak{A}^s \models \varphi^s$ a.e. If $\varphi \in L^*_{\omega_1}$ then $\varphi = \varphi^s$ a.e. In particular these results for the logics $L^* = L_{\infty G}$ and $L^*_{\omega_1} = L_{\omega_1 G}$ which allow some infinite alternations of quantifiers. This gives a very useful Löwenheim-Skolem result when applied to the theory of inductive definitions as in Moschovakis [7]; see his § 8.D.

5.2. THEOREM (Kueker). *Let $\mathfrak{A}, \mathfrak{B}$ be L-structures.*

(a) $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ *iff* $\mathfrak{A}^s \cong \mathfrak{B}^s$ a.e.

(b) $\mathfrak{A} \not\equiv_{\infty\omega} \mathfrak{B}$ *iff* $\mathfrak{A}^s \not\cong \mathfrak{B}^s$ a.e.

Proof. The relation $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ are models of the same sentences of $L_{\infty\omega}$. It is a Δ relation (see [1]). For countable $\mathfrak{A}, \mathfrak{B}$, $\mathfrak{A} \equiv_{\infty\omega} \mathfrak{B}$ iff $\mathfrak{A} \cong \mathfrak{B}$. ■

The reason for stating both (a) and (b) is that the closed unbounded filter is not an ultrafilter so (b) does not follow immediately from (a).

Let K be any class of L-structures closed under isomorphism and closed downward. Since Kueker [4], [5] give a number of interesting model theoretic results for such K, it is useful to note the extreme ease with which such K can be identified, given Theorem 4.1, Proposition 4.4 and just a little familiarity with Σ predicates. For example, we have the following.

5.3. PROPOSITION. *Let K, K' be a class of L structures closed under isomorphism and closed downward. Then the classes of structures $\mathfrak{A}$ defined in the following are all closed downward.*

(³) If one thinks of V_M as given in advance then the requirements $L \subseteq M$ and $A \subseteq M$ seem odd. If, on the other hand, one thinks of L and A as given and then forms V_M for some $M \supseteq L \cup A$, then one sees that they are really not restrictions at all. It is just the usual mathematical practice; when working with some structure $\mathfrak{A} = \langle A, \dots\rangle$ we ignore any structure on elements of A not given by $\mathfrak{A}$.

(a) $\mathfrak{A} \subseteq \mathfrak{B}$ *for some* $\mathfrak{B} \in K$,

(b) $\mathfrak{A}$ *is a homomorphic image of some* $\mathfrak{B} \in K$,

(c) $\mathfrak{A}$ *is a retract of some* $\mathfrak{B} \in K$,

(d) $\mathfrak{A}$ *is isomorphic to* $\mathfrak{B} \times \mathfrak{C}$ *for some* $\mathfrak{B} \in K$, $\mathfrak{C} \in K'$,

(e) $\mathfrak{A}$ *is isomorphic to a direct sum* $\sum_{j \in J} \mathfrak{B}_j$ *of* $\mathfrak{B}_j \in K$,

(f) $\mathfrak{A}$ *is a direct factor of some* $\mathfrak{B} \in K$.

Proof. In each case the result follows by just writing down the definition of the class and applying 4.1 and 4.4. For (d), to write one out, we have $\mathfrak{A}$ in the given class iff

$$\exists \mathfrak{B}, \mathfrak{C}[\mathfrak{B} \in K \wedge \mathfrak{C} \in K' \wedge \underbrace{\exists f(\mathfrak{A} \overset{f}{\cong} \mathfrak{B} \times \mathfrak{C})}_{\Sigma}]. \ \blacksquare$$

We would like to conclude with an example a little less obvious than the ones given in 5.3. One such comes from the class K_0 studied by Kueker and defined by $\mathfrak{A} \in K_0$ iff

> there is a finite set $p_1 \ldots p_n \in A$ such that every $q \in A$ is definable in $(\mathfrak{A}, p_1 \ldots p_n)$ by a formula $\varphi(x)$ of $L_{\infty\omega}$.

5.4. Proposition. *The class* K_0 *is* Δ *definable.*

Proof. Simply writing out the above condition gives a Σ definition of K_0. It is not quite so obvious how to write $\mathfrak{A} \notin K_0$ as a Σ condition. To do it we use the following observation of Nadel [10]: If an element q of a structure $\mathfrak{B}$ is definable by some formula $\varphi(x)$ of $L_{\infty\omega}$, then it is definable by a formula $\varphi \in L_{(\mathfrak{B})^+}$, where $(\mathfrak{B})^+$ is the smallest admissible set with $\mathfrak{B}$ an element and $L_{(\mathfrak{B})^+} = L_{\infty\omega} \cap (\mathfrak{B})^+$. We can now write $\mathfrak{A} \notin K_0$ iff

> $\exists x$ (x is admissible $\wedge \mathfrak{A} \in x \wedge$ for all finite sequences $p_1 \ldots p_n \in A$, there is a $q \in A$ such that q is not definable on $(\mathfrak{A}, p_1 \ldots p_n)$ by any formula $\varphi \in x$).

The part inside the parentheses is easily seen to be Δ. $\blacksquare$

From 5.4 and 4.2 we obtain Kueker's result that $\mathfrak{A} \in K_0$ iff $\mathfrak{A}^s \in K_0$ a.e., a result which Kueker puts to good use in [5].

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