Semi-Boolean algebras and their applications to intuitionistic logic with dual operations

by

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Abstract. The presented paper consists of two parts. Part one is devoted to the theory of semi-Boolean algebras. These algebras are used in an algebraic treatment, of intuitionistic logic with two additional connectives $\dot{-}$, $\ulcorner$. This logic is called H-B logic and it is examined in the second part. In order to develop semi-Boolean algebras a kind of lattice — to be called bi-topological Boolean algebras — is introduced and investigated. For the above algebras representation theorems are formulated and prove. For semi-Boolean algebras with infinite joins and meets a representation theorem analogous to the Rasiowa-Sikorski lemma is also proved. In the second part the H-B logic is examined. The main results of that part are the proofs of the completeness theorem, the deduction theorem and a theorem which explain the connections between intuitionistic propositional tautologies and tautologies of the propositional calculus of the H-B logic.

Lattice theory plays an important role in the algebra of logic. The connections between Boolean algebras and classical logic are well known. Analogous connections hold between pseudo-Boolean algebras — which are represented by algebras of open subsets of a topological space — and intuitionistic logic. Dual algebras to the pseudo-Boolean algebras are Brouwerian algebras (see [5]). They are isomorphic to algebras of closed subsets of a topological space. Brouwerian algebras can also be used for an algebraic interpretation of intuitionistic logic.

In this paper a class of lattices to be called semi-Boolean algebras is introduced and examined. Semi-Boolean algebras can be used in an algebraic treatment of intuitionistic logic with two additional connectives $\dot{-}$, $\ulcorner$ which are dual to the intuitionistic implication and to the intuitionistic negation, respectively. This logic is called the H-B logic. Semi-Boolean algebras play an analogous role for the above mentioned logic to that played by Boolean algebras for classical logic.

This paper consists of two parts. Part I is devoted to the theory of semi-Boolean algebras. In order to develop this theory a kind of lattice —

(*) This paper constitutes a part of the doctoral dissertation represented at the Warsaw University Institute of Mathematics in February 1971. The author wishes to thank Professor Helena Rasiowa for her valuable advice help in the preparation of the paper.

to be called bi-topological Boolean algebras — is also examined. There is established a relation between these algebras and semi-Boolean algebras analogous to that which holds between topological Boolean algebras (closure algebras) on the one hand and semi-Boolean algebras on the other hand. The main results of this part are certain representation theorems for these algebras. An example of a semi-Boolean algebra is given in § 6. In that example the construction of the Cantor discontinuum is used. Semi-Boolean algebras with infinite joins and meets are considered in § 7. For those algebras a representation theorem which is a weaker analogue of the Rasiowa-Sikorski lemma for Boolean algebras is formulated and proved.

In the second part the propositional calculi of the H-B logic are investigated. The set of axioms for that logic contains some axioms of intuitionistic propositional calculus and some formulas which characterize the operations $\dot{-}$ and $\ulcorner$. Two rules of inference are adopted, namely modus ponens and the rule (r) $\dfrac{a}{\neg\ulcorner a}$.

The main results of the second part are the proofs of the completeness theorem and the deduction theorem. A theorem which explain the connections between intuitionistic propositional tautologies and tautologies of the propositional calculus of H-B logic is proved in § 10. An analogous theorem for the formalized theories of the H-B logic is not true. This follows from the properties of semi-Boolean algebras with infinite joins and meets. Formalized theories of the H-B logic will be considered in a separate paper.

§ 1. Definition and some properties of semi-Boolean algebras. We shall say that an *abstract algebra* $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ is a semi-Boolean algebra provided that

(i) $(A, \cup, \cap, \Rightarrow)$ is a relatively pseudo-complement lattice,

(ii) $\dot{-}$ is a binary operation which satisfies the following condition:

$$a \dot{-} b \leqslant x \quad \text{if and only if} \quad a \leqslant b \cup x \quad \text{for any } a, b, x \in A .$$

The operation $\dot{-}$ will be called the *pseudo-difference*. This operation is dual to the relative pseudo-complement $\Rightarrow$.

Semi-Boolean algebras can be characterized by a simple set of axioms. Namely,

1.1. *An abstract algebra* $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ *is a semi-Boolean algebra if and only if it satisfies the following axioms*

(l_1) $a \cup b = b \cup a$, $\quad a \cap b = b \cap a$,

(l_2) $(a \cup b) \cup c = a \cup (b \cup c)$, $\quad (a \cap b) \cap c = a \cap (b \cap c)$,

(l_3) $(a \cap b) \cup b = b$, $\quad (a \cup b) \cap a = a$,

(l_4) $b \cup (a \dot{-} b) = a \cup b$, $\quad a \cap (a \Rightarrow b) = a \cap b$,

(l_5) $(a \dot{-} b) \cup a = a$, $\quad (a \Rightarrow b) \cap b = b$,

(l_6) $(a \dot{-} c) \cup (b \dot{-} c) = (a \cup b) \dot{-} c$, $\quad (a \Rightarrow b) \cap (a \Rightarrow c) = a \Rightarrow (b \cap c)$,

(l_7) $(a \dot{-} a) \cup b = b$, $\quad (a \Rightarrow a) \cap b = b$,

The proof of 1.1 is by an easy verification. ■

We say that every semi-Boolean algebra $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ has the zero element

$$\Lambda = a \dot{-} a, \quad a \in A \tag{1}$$

and the unit element

$$V = a \Rightarrow a, \quad a \in A . \tag{2}$$

1.2. *In every semi-Boolean algebra*

(3) $a \Rightarrow b = V$ *if and only if* $a \leqslant b$; *if and only if* $a \dot{-} b = V$,

(4) $a \Rightarrow V = V$, $\quad \Lambda \dot{-} a = \Lambda$,

(5) $V \Rightarrow b = b$, $\quad a \dot{-} \Lambda = a$,

(6) *if* $a_1 \leqslant a_2$ *then* $a_2 \Rightarrow b \leqslant a_1 \Rightarrow b$ *and* $a_1 \dot{-} b \leqslant a_2 \dot{-} b$,

(7) *if* $b_1 \leqslant b_2$ *then* $a \Rightarrow b_1 \leqslant a \Rightarrow b_2$ *and* $a \dot{-} b_2 \leqslant a \dot{-} b_1$,

(8) $b \leqslant a \Rightarrow b$, $\quad a \dot{-} b \leqslant a$,

(9) $(a \Rightarrow c) \cap (b \Rightarrow c) = (a \cup b) \Rightarrow c$,

(10) $(c \dot{-} a) \cup (c \dot{-} b) = c \dot{-} (a \cap b)$,

(11) $a \Rightarrow (b \Rightarrow c) = (a \cap b) \Rightarrow c = b \Rightarrow (a \Rightarrow c)$,

(12) $(c \dot{-} b) \dot{-} a = c \dot{-} (a \cup b) = (c \dot{-} a) \dot{-} b$,

(13) $(a \Rightarrow b) \leqslant (b \Rightarrow c) \Rightarrow (a \Rightarrow c)$,

(14) $(a \dot{-} c) \dot{-} (a \dot{-} b) \leqslant b \dot{-} c$.

This theorem follows from the properties of the operations $\Rightarrow$ and $\dot{-}$. ■

An element $c \in A$ is said to be the $\cap$-*complement of an element* a *in* $\mathfrak{A}$ if c is the greatest element such that $a \cap c = \Lambda$. In the semi-Boolean algebra $\mathfrak{A}$ every element has the $\cap$-complement, namely

$$\neg a = a \Rightarrow \Lambda \tag{15}$$

is the $\cap$-complement of an element a in $\mathfrak{A}$.

An element $c \in A$ is said to be the $\cup$-*complement of an element* a *in* $\mathfrak{A}$ if c is the least element such that $a \cup c = V$. In the semi-Boolean algebra $\mathfrak{A}$ every element has the $\cup$-complement, namely

$$\ulcorner a = V \dot{-} a \tag{16}$$

is the $\cup$-complement of an element a in $\mathfrak{A}$.

Hence the definition of a semi-Boolean algebra given above is equivalent to the following one: An abstract algebra $(A, \cup, \cap, \Rightarrow, \dot{-}, \neg, \ulcorner)$

will be called a *semi-Boolean algebra* provided that $(A, \cup, \cap, \Rightarrow, \neg)$ is a pseudo-Boolean algebra [2] and $(A, \cup, \cap, \dot{-}, \Gamma)$ is a Brouwerian algebra [1]. We list the fundamental properties of the operations $\Rightarrow$, $\dot{-}$, $\neg$ and Γ in semi-Boolean algebras in the form of the following theorem:

1.3. *In every semi-Boolean algebra the following conditions are satisfied*

(17) *if* $a \leqslant b$ *then* $\neg b \leqslant \neg a$ *and* $\Gamma b \leqslant \Gamma a$,

(18) $a \cap \neg a = \Lambda, \quad a \cup \Gamma a = V,$

(19) $\neg \Lambda = V, \quad \neg V = \Lambda, \quad \Gamma \Lambda = V, \quad \Gamma V = \Lambda,$

(20) $a \leqslant \neg\neg a, \quad \Gamma\Gamma a \leqslant a,$

(21) $\neg\neg\neg a = \neg a, \quad \Gamma\Gamma\Gamma a = \Gamma a,$

(22) $\neg(a \cup b) = \neg a \cap \neg a, \quad \Gamma(a \cup b) \leqslant \Gamma a \cap \Gamma b,$

(23) $\neg a \cup \neg b \leqslant \neg(a \cap b), \quad \Gamma a \cup \Gamma b = \Gamma(a \cap b),$

(24) $\neg a \cup b \leqslant a \Rightarrow b, \quad a \dot{-} b \leqslant a \cap \Gamma b,$

(25) $a \Rightarrow b \leqslant \neg b \Rightarrow \neg a, \quad \Gamma a \Gamma \dot{-} b \leqslant b \dot{-} a,$

(26) $\neg a \leqslant \Gamma a,$

(27) $\neg\Gamma a \leqslant a,$

(28) $\Gamma\Gamma a \leqslant \neg\neg a,$

(29) $a \dot{-} b \leqslant \Gamma(a \Rightarrow b),$

(30) $\neg(a \dot{-} b) \leqslant a \Rightarrow b,$

(31) $\Gamma(a \Rightarrow b) \leqslant \Gamma a \Rightarrow \Gamma b,$

(32) $\neg b \dot{-} \neg a \leqslant \neg(b \dot{-} a),$

(33) $\neg\Gamma a \cup \neg\Gamma b \leqslant \neg\Gamma(a \cup b),$

(34) $\neg\Gamma a \cap \neg\Gamma b = \neg\Gamma(a \cap b),$

(35) $\neg\Gamma(a \Rightarrow b) \leqslant \neg\Gamma a \Rightarrow \neg\Gamma b.$

The proofs of (17)-(25) are either known or obvious (see [1] and [2]).

Proof of (26). By (18) we have

$$\neg a = \neg a \cap V = \neg a \cap (a \cup \Gamma a) = \neg a \cap a \cup \neg a \cap \Gamma a = \neg a \cup \Gamma a .$$

Thus (26) holds.

(27) follows from (20) and (26).

(28) follows from (20).

Proof of (29). We prove that $a \dot{-} \Gamma(a \Rightarrow b) \leqslant b$. Hence, by (3) and (12) follows (29). By (24), (20) and (l_4) we have

$$a \dot{-} \Gamma(a \Rightarrow b) \leqslant a \cap \Gamma\Gamma(a \Rightarrow b) \leqslant a \cap (a \Rightarrow b) = a \cap b \leqslant b .$$

Proof of (30) is similar to the proof of (29).

(31) follows from the definition of $\Rightarrow$, (22) and (24).

(32) follows from the definition of $\dot{-}$, (22) and (24).

Proofs of (33) and (34) follow from (22) and (26).

(35) follows from the definition of $\Rightarrow$, (34) and (l_4). ∎

§ 2. Filters in semi-Bolean algebras. Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-}, \neg, \Gamma)$ will be a semi-Boolean algebra. A non-empty set ∇ of elements of a semi-Boolean algebra $\mathfrak{A}$ is said to be a *semi-Boolean filter* (henceforth abbreviated to $\neg\Gamma$*-filter*) in $\mathfrak{A}$ provided that, ∇ is a filter and the following condition is satisfied: for every $a \in A$

(f) if $a \in \nabla$ then $\neg\Gamma a \in \nabla$.

The set composed of the element V only is an example of a $\neg\Gamma$-filter. Let a_0 be an element of A. The set of all elements $x \in A$ for which there exists a positive integer n such that $\overbrace{\neg\Gamma \ldots \neg\Gamma}^{2n} a_0 \leqslant x$ is another example of a $\neg\Gamma$-filter. This filter will be called the $\neg\Gamma$*-filter generated by* a_0. In the sequel we shall denote by $T_n a$ the element $\overbrace{\neg\Gamma \ldots \neg\Gamma}^{2n} a$. For every non-empty set A_0 of elements in A there exists a least $\neg\Gamma$-filter ∇ containing A_0. Namely the $\neg\Gamma$-filter ∇ is the intersection of all $\neg\Gamma$-filters containing A_0. The least $\neg\Gamma$-filter ∇ is said to be *filter generated by the set* A_0.

2.1. *The* $\neg\Gamma$*-filter generated by a non-empty set* A_0 *of a semi-Boolean algebra* $\mathfrak{A}$ *is the set of all elements* $x \in A$ *for which there exist positive integers* $n_1 \ldots n_m$ *such that*

$$T_{n_1} a_1 \cap \ldots \cap T_{n_m} a_m \leqslant x \quad \textit{for some } a_1 \ldots a_n \in A_0 .$$

The proof is by an easy verification. ∎

2.2. *For any fixed element* $a_0 \in A$ *and a* $\neg\Gamma$*-filter* ∇ *in* $\mathfrak{A}$ *the set of all elements* x *for which there exists a positive integer* n *such that* $T_n a \cap c \leqslant x$ *for any element* $c \in \nabla$ *is the least* $\neg\Gamma$*-filter containing* a_0 *and* ∇.

This follows easily from 2.1. ∎

A $\neg\Gamma$-filter is said to be *maximal in* $\mathfrak{A}$ provided it is proper and it is not any proper subset of a proper $\neg\Gamma$-filter in $\mathfrak{A}$. It is easy to prove that

2.3. *For every proper* $\neg\Gamma$*-filter* ∇ *in* $\mathfrak{A}$ *there exists a maximal* $\neg\Gamma$*-filter* ∇^* *in* $\mathfrak{A}$ *such that* ∇ *is contained in* ∇^*. ∎

2.4. *In a semi-Boolean algebra, each prime* $\neg\Gamma$*-filter* ∇ *is maximal.* (*A* $\neg\Gamma$*-filter* ∇ *is prime: if* $a \cup b \in \nabla$ *then* $a \in \nabla$ *or* $b \in \nabla$.)

Suppose that ∇ is a prime $\neg\Gamma$-filter and ∇ is not maximal. Let $\nabla \subsetneq \nabla_1$, i.e. suppose there exists an element b such that $b \notin \nabla$ and $b \in \nabla_1$, where ∇_1 is a proper $\neg\Gamma$-filter. By 1.3 (18) $V = b \cup \Gamma b \in \nabla$. Hence $\Gamma b \in \nabla$ because ∇ is a prime $\neg\Gamma$-filter and $b \notin \nabla$. Thus $\Gamma b \in \nabla_1$ and on account of condition (f) we infer that $\neg\Gamma\Gamma b \in \nabla_1$ and $\neg\Gamma b \in \nabla_1$. By 1.3 (17) and (28) we have $\neg\neg\Gamma b \in \nabla_1$. Hence $\neg\Gamma b \cap \neg\neg\Gamma b = \Lambda \in \nabla_1$ and ∇_1 is not a proper $\neg\Gamma$-filter. This proves that ∇ is a maximal $\neg\Gamma$-filter. ∎

The theorem converse to 2.4 is not true. As an example we consider the finite Boolean algebra $a\overset{\vee}{\underset{\wedge}{\Diamond}}b$. The set $\{\vee\}$ is a unique proper $\neg\Gamma$-filter in this semi-Boolean algebra. It is a maximal $\neg\Gamma$-filter but it is not prime because $a\cup b\in\{\vee\}$ but neither $a\notin\{\vee\}$ nor $b\notin\{\vee\}$.

A $\neg\Gamma$-filter ∇ is said to be a *semi-prime $\neg\Gamma$-filter* provided: if for every positive integer n, $T_n a\cup T_n b\in\nabla$ then $a\in\nabla$ or $b\in\nabla$.

2.5. *In a semi-Boolean algebra $\mathfrak{A}$, each maximal $\neg\Gamma$-filter is a semi-prime $\neg\Gamma$-filter.*

Suppose that ∇ is not a semi-prime $\neg\Gamma$-filter, i.e. that for every positive integer n $T_n a\cup T_n b\in\nabla$ but $a\notin\nabla$ and $b\notin\nabla$. Let ∇_1 be the $\neg\Gamma$-filter generated by a and ∇, i.e. let ∇_1 be the set of all x for which there exists positive integer n_1 such that $T_{n_1}a\cap c\leqslant x$, for some $c\in\nabla$. We observe that $b\notin\nabla_1$. In fact the hypothesis $b\in\nabla_1$ implies that $T_{n_1}a\cap c\leqslant b$ for any positive integer n_1 and for an element $c\in\nabla$. Thus by 1.3 (27)

$$b=T_{n_1}a\cap c\cup b\geqslant(T_{n_1}a\cup T_{n_1}b)\cap(b\cup c)\,.$$

Since $T_{n_1}a\cup T_{n_1}b\in\nabla$ and $b\cup c\in\nabla$ we infer that $b\in\nabla$, in contradiction to our assumptions. Obviously $\nabla\subset\nabla_1$ and $\nabla\neq\nabla_1$. Thus the $\neg\Gamma$-filter ∇ is not maximal.

A $\neg\Gamma$-filter ∇ is said to be *irreducible in* $\mathfrak{A}$ if it is not a product of two $\neg\Gamma$-filters in $\mathfrak{A}$ different from ∇.

2.6. *Every irreducible $\neg\Gamma$-filter in a semi-Boolean algebra is a semi prime $\neg\Gamma$-filter.*

Suppose that ∇ is an irreducible $\neg\Gamma$-filter and for every positive integer n, $T_n a\cup T_n b\in\nabla$ and $a\notin\nabla$ and $b\notin\nabla$. Let ∇_1 be the $\neg\Gamma$-filter generated by a and ∇, i.e. ∇_1 is the set of all elements x for which there exists a positive integer n_1, such that $T_{n_1}a\cap c_1\leqslant x$ for some $c_1\in\nabla$. Let ∇_2 be the $\neg\Gamma$-filter generated by b and ∇, i.e. ∇_2 is the set of all elements x for which there exists positive integer n_2 such that $T_{n_2}b\cap c_2\leqslant x$ for some $c_2\in\nabla$. We shall prove that $\nabla=\nabla_1\cap\nabla_2$. Indeed, $\nabla\subset\nabla_1\cap\nabla_2$. If $y\in\nabla_1$ and $y\in\nabla_2$ then there exist a positive integers n_1, n_2 such that $T_{n_1}a\cap c_1\leqslant y$ and $T_{n_2}b\cap c_2\leqslant y$ for any $c_1, c_2\in\nabla$. Hence $T_{n_1}a\cap c_1\cup T_{n_2}b\cap c_2\leqslant y$. This implies that

$$(T_m a\cup T_m b)\cap c_1\cap c_2\leqslant y\quad\text{ where }\quad m=\max(n_1, n_2)\,.$$

Thus $y\in\nabla$ since $T_m a\cup T_m b\in\nabla$ and $c_1\cap c_2\in\nabla$. In consequence, ∇ is not an irreducible $\neg\Gamma$-filter. ■

Let ∇ be a $\neg\Gamma$-filter in a semi-Boolean algebra $\mathfrak{A}=(A,\cup,\cap,\Rightarrow,\dot{-},\neg,\Gamma)$. For any elements $a, b\in A$ we shall write

(1) $\quad a\leqslant_\nabla b\quad$ if and only if $\quad a\Rightarrow b\in\nabla$,

and

(2) $\quad a\sim b(\mathrm{mod}\nabla)\quad$ if and only if $\quad a\Rightarrow b\in\nabla$ and $b\Rightarrow a\in\nabla$.

Relation (1) is a quasi-ordering in A, i.e. $\leqslant_\nabla$ is reflexive and transitive. Relation (2) is an equivalence relation. We consider the set A/∇ of all equivalence classes of the relation $\sim$. Elements in A/∇ will be denoted by $|a|$ ($a\in A$). Relation (1) in A determines an ordering relation [1] $\leqslant$ in A/∇, namely

(3) $\quad |a|\leqslant|b|\quad$ if and only if $\quad a\Rightarrow b\in\nabla$.

We recall that every semi-Boolean algebra is a pseudo-Boolean algebra and every $\neg\Gamma$-filter is a filter, thus

2.7. *The algebra $(A/\nabla,\cup,\cap,\Rightarrow,\neg)$ is a pseudo-Boolean algebra. The algebra $(A/\nabla,\cup,\cap,\Rightarrow,\neg)$ is non-degenerate if and only if ∇ is a proper $\neg\Gamma$-filter.* ■

2.8. *For every $\neg\Gamma$-filter ∇ in a semi-Boolean algebra $\mathfrak{A}=(A,\cup,\cap,\Rightarrow,\dot{-},\neg,\Gamma)$ the equivalence relation $\sim$ defined by (2) is a congruence in this algebra. The algebra $\mathfrak{A}'=(A/\nabla,\cup,\cap,\Rightarrow,\dot{-},\neg,\Gamma)$ where*

(4) $$|a|\cup|b|=|a\cup b|\,,$$

(5) $$|a|\cap|b|=|a\cap b|\,,$$

(6) $$|a|\Rightarrow|b|=|a\Rightarrow b|\,,$$

(7) $$|a|\dot{-}|b|=|a\dot{-}b|\,,$$

(8) $$\neg|a|=|\neg a|\,,$$

(9) $$\Gamma|a|=|\Gamma a|\,.$$

is a semi-Boolean algebra. The algebra $\mathfrak{A}'$ is non-degenerate if and only if ∇ is a proper $\neg\Gamma$-filter.

On account of 2.7 it is sufficient to show that conditions (7) and (9) are fulfilled.

To prove (7) it suffices to show that for any $a, b, x\in A$

$$|a\dot{-}b|\leqslant|x|\quad\text{ if and only if }\quad|a|\leqslant|x|\cup|b|\,.$$

Suppose that $|a|\leqslant|x|\cup|b|=|x\cup b|$, then $a\Rightarrow(x\cup b)=\vee$. By the assumption ∇ is a $\neg\Gamma$-filter, thus $\neg\Gamma(a\Rightarrow(x\cup b))\in\nabla$. By (29), (26), (17) and (30) we infer that

$$\neg\Gamma(a\Rightarrow(x\cup b))\leqslant\neg(a\dot{-}(b\cup x))=\neg((a\dot{-}b)\dot{-}x)\leqslant(a\dot{-}b)\leqslant x\,.$$

[1] A binary relation $\leqslant$ defined in a set A is said to be an *ordering* if it is reflexive, antisymmetric and transitive.

Hence $(a \dot{-} b) \leqslant x \in \nabla$ i.e. $|a \dot{-} b| \leqslant |x|$. On the other hand, if $|a \dot{-} b| \leqslant x$, then by (4) and ($1_4$)

$$|a| \leqslant |a \cup b| = |(a \dot{-} b) \cup b| = |a \dot{-} b| \cup |b| \leqslant |x| \cup |b| .$$

This proves (7). Condition (9) follows immediately from the definition of $\ulcorner$ and (7). ■

It is easy to see that

2.9. *A $\neg\ulcorner$-filter ∇ in a semi-Boolean algebra $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ is maximal if and only if A/∇ has exactly two elements.* ■

An ideal Δ is said to be a *semi-Boolean ideal* ($\neg\ulcorner$-*ideal*) provided that for any $a \in A$ if $a \in \Delta$ then $\ulcorner\neg a \in \Delta$. The notion of $\ulcorner\neg$-ideal is dual to that of a $\neg\ulcorner$-filter. The theory of semi-Boolean algebras can be considered by using the notion of a $\ulcorner\neg$-ideal.

§ 3. Bi-topological Boolean algebras. We shall say that an abstract algebra $\mathfrak{B} = (B, \cup, \cap, \rightarrow, -, \mathfrak{I}, C)$ is a *bi-topological Boolean algebra* if $(B, \cup, \cap, \rightarrow, -)$ is a Boolean algebra, and $\mathfrak{I}$ and C are an interior and a closure operations respectively, such that the following condition is satisfied

$$(*) \qquad \mathfrak{I}a = C\mathfrak{I}a , \quad Ca = \mathfrak{I}Ca \quad \text{for every } a \in B .$$

The operations $\mathfrak{I}$ and C will be called the *conjugate operations over B* when they satisfy $(*)$. An element $a \in B$ is said to be $\mathfrak{I}$-*open* (C-*closed*) when $\mathfrak{I}a = a$ ($Ca = a$). We have seen that in every bi-topological Boolean algebra an element a is $\mathfrak{I}$-open if and only if it is C-closed.

By a topological space we shall understand a system $\langle X, \mathfrak{I}\rangle$ ($\langle X, C\rangle$) where the set X is non-empty and $\mathfrak{I}$ is an interior operation (C is a closure operation). If the systems $\langle X, \mathfrak{I}\rangle$ and $\langle X, C\rangle$ are any topological spaces, then the system $\langle X, \mathfrak{I}, C\rangle$ will be called a *bi-topological space*. Let $\mathfrak{B}_0(X)$ denote the class of all subset of X. If for every $Y \in \mathfrak{B}(X)$, $\mathfrak{B}(X) \subset \mathfrak{B}_0(X)$

$$(**) \qquad \mathfrak{I}Y = C\mathfrak{I}Y, \quad CY = \mathfrak{I}CY,$$

then we shall say that the *operations $\mathfrak{I}$ and C are conjugate over* $\mathfrak{B}(X)$. If $\langle X, \mathfrak{I}, C\rangle$ is a bi-topological space, $\mathfrak{B}(X)$ is a field of subset of X and the operations $\mathfrak{I}$ and C are conjugate over $\mathfrak{B}(X)$, then the algebra $\mathfrak{P} = (\mathfrak{B}(X), \cup, \cap, -, \mathfrak{I}, C)$ as well as each of its subalgebras will be called a *bi-topological field of sets* (more exactly: a *bi-topological field of subset of X*).

An example of a bi-topological field of sets can be constructed in the following way. Let $(X, \leqslant)$ be an arbitrary ordered set, i.e. let the relation $\leqslant$ be an ordering in the set X. Let $\mathfrak{U}(x)$ for any $x \in X$ denote the set of all $y \in X$ such that $y \leqslant x$, i.e.

$$(1) \qquad \mathfrak{U}(x) = \{y \in X\colon\ y \leqslant x\} .$$

Let $\mathfrak{B}(X)$ be the class of all subsets of X. We define an interior operation $\mathfrak{I}$ and a closure operation C in the following way:

$$(2) \qquad \mathfrak{I}Y = \bigcup_{\mathfrak{U}(x) \subset Y} \mathfrak{U}(x) ,$$

$$(3) \qquad CY = \bigcup_{Y \subset \mathfrak{U}(x)} \mathfrak{U}(x) , \qquad \text{for any } Y \in \mathfrak{B}(X).$$

The systems $\langle X, \mathfrak{I}\rangle$ and $\langle X, C\rangle$ are topological spaces. Thus the system $\langle X, \mathfrak{I}, C\rangle$ is a bi-topological space. It is easy to prove that the operations $\mathfrak{I}$ and C defined above are conjugate over $\mathfrak{B}(X)$. The algebra $(\mathfrak{B}(X), \cup, \cap, -, \mathfrak{I}, C)$ — where $\mathfrak{B}(X)$ is the class of all subsets of the ordered set $(X, \leqslant)$, the operations $\cup$, $\cap$ and $-$ are set-theoretical union, intersection and complementation respectively, $\mathfrak{I}$ is the interior defined by (2) and C is the closure operation defined by (3) — is an example of a bi-topological field of sets.

3.1. *For every bi-topological Boolean algebra $\mathfrak{B}$ there exists a bi-topological field of sets $\mathfrak{P}$ and an isomorphism of $\mathfrak{B}$ onto $\mathfrak{P}$.*

Let $\mathfrak{B} = (B, \cup, \cap, \rightarrow, -, \mathfrak{I}^*, C^*)$ be a bi-topological Boolean algebra. Let us denote by X the set of all prime filters ∇ of a Boolean algebra $\mathfrak{B}_0 = (B, \cup, \cap, \rightarrow, -)$, and for every $a \in B$ let $h(a)$ denote the set of all $\nabla \in X$ such that $a \in \nabla$. It follows from [4] that the Stone space $\langle X, I\rangle$ — where the interior operation I is determined by assuming the class $\{h(a)\}_{a \in B}$ as a subbasis — is a compact (²) totally disconnected Hausdorff space. Moreover, the class $\{h(a)\}_{a \in B}$ is the field of all both open and closed subsets of the topological space $\langle X, I\rangle$ and h is an isomorphism of $\mathfrak{B}_0$ onto $\mathfrak{P}' = (\mathfrak{B}(X), \cup, \cap, -)$ where $\mathfrak{B}(X) = \{h(a)\}_{a \in B}$. Now, a new interior operation and a new closure operation in X will be defined in the following way:

$$(4) \qquad \mathfrak{I}Y = \bigcup_{\substack{h(a) \subset Y \\ a = \mathfrak{I}^* a}} h(a) ,$$

$$(5) \qquad CY = \bigcap_{\substack{Y \subset h(b) \\ b = C^* b}} h(b) , \qquad \text{for every } Y \subset X .$$

It will be shown that the operations defined above are conjugate over $\mathfrak{B}(X)$, i.e. that the condition $(**)$ is satisfied. Let $Y \in \mathfrak{B}(X)$, i.e. $Y = h(x)$ for some $x \in B$. By the definition of the operation $\mathfrak{I}$ we have

$$\mathfrak{I}Y = \mathfrak{I}h(x) = \bigcup_{\substack{h(a) \subset h(x) \\ a = \mathfrak{I}^* a}} h(a) .$$

(²) A topological space $\langle X, I\rangle$ is said to be *compact* if for every indexed set $\{A_t\}_{t \in T}$ of open subsets the equation $X = \bigcup_{t \in T} A_t$ implies the existence of a finite set $T_0 \subset T$ such that $X = \bigcup_{t \in T_0} A_t$.

Since h is an isomorphism of the Boolean algebra $\mathfrak{B}_0$ onto $\mathfrak{P}$ the condition $h(a) \subset h(x)$ is equivalent to $a \leqslant x$ for any $a, x \in B$. Since we have $a = \mathfrak{I}^*a$, the last inequality is equivalent to the following one $a \leqslant \mathfrak{I}^*x$, i.e. to $h(a) \subset h(\mathfrak{I}^*x)$. Thus

$$\mathfrak{I}h(x) = \bigcup_{\substack{h(a)\subset h(x)\\ a=\mathfrak{I}^*a}} h(a) = \bigcup_{\substack{h(a)\subset h(\mathfrak{I}^*x)\\ a=\mathfrak{I}^*a}} h(a) = h(\mathfrak{I}^*x) .$$

In the same way it could be shown that

$$Ch(x) = h(C^*x) .$$

Since the operation $\mathfrak{I}^*$ and C^* are conjugate over B, it is true that

$$\mathfrak{I}Y = \mathfrak{I}h(x) = h(\mathfrak{I}^*x) = h(C^*\mathfrak{I}^*x) = C\mathfrak{I}h(x) = C\mathfrak{I}Y$$

and

$$CY = Ch(x) = h(C^*y) = h(\mathfrak{I}^*C^*x) = \mathfrak{I}Ch(x) = \mathfrak{I}CY .$$

This proves that condition (**) is satisfied. The algebra $\mathfrak{P} = (\mathfrak{B}(X), \cup, \cap, -, \mathfrak{I}, C)$ is the required bi-topological field of sets and h is the isomorphism of the bi-topological Boolean algebra $\mathfrak{B}$ onto $\mathfrak{P}$. ■

§ 4. The connection between semi-Boolean algebras and bi-topological Boolean algebras. For any bi-topological Boolean algebra $\mathfrak{B} = (B, \cup, \cap, \rightarrow, -, \mathfrak{I}, C)$ we shall denote by $G_{\mathfrak{I}}(B)$ the set of all $\mathfrak{I}$-open elements in $\mathfrak{B}$. By condition (*) § 3 the elements of $G_{\mathfrak{I}}(B)$ are simultaneously $\mathfrak{I}$-open and C-closed.

4.1. *The algebra* $(G_{\mathfrak{I}}(B), \cup, \cap, \Rightarrow, \dot{-})$ — *where* $G_{\mathfrak{I}}(B)$ *is the set of all* $\mathfrak{I}$-*open elements of a bi-topological Boolean algebra* $\mathfrak{B} = (B, \cup, \cap, \rightarrow, -, \mathfrak{I}, C)$ — *is a semi-Boolean algebra. For all* $a, b \in G_{\mathfrak{I}}(B)$

(1) $$a \Rightarrow b = \mathfrak{I}(-a \cup b) ,$$

(2) $$a \dot{-} b = C(a \cap -b) .$$

To prove (1) it suffices to show that $a \cap x \leqslant b$ if and only if $x \leqslant \mathfrak{I}(-a \cup b)$ for any $a, b, x \in G_{\mathfrak{I}}(B)$. In fact, the condition $a \cap x \leqslant b$ is equivalent to $x \leqslant -a \cup b$, where sign $-$ denotes the complement in the bi-topological Boolean algebra $\mathfrak{B}$. Since x is $\mathfrak{I}$-open the last inequality is equivalent to $x \leqslant \mathfrak{I}(-a \cup b)$, which proves (1). Condition (2) we prove similarly. On account of the definition of $\dot{-}$ we have to prove that, for any $a, b, x \in G_{\mathfrak{I}}(B)$, $a \leqslant b \cup x$ if and only if $C(a \cap -b) \leqslant x$. The condition $a \leqslant b \cup x$ is equivalent to $a \cap -b \leqslant x$. Since $x \in G_{\mathfrak{I}}(B)$, we infer that x is C-closed. Thus, the last inequality is equivalent to $C(a \cap -b) \leqslant x$. This proves (2). ■

Theorem 4.1 yields an important example of semi-Boolean algebras. We shall prove that every semi-Boolean algebra $\mathfrak{A}$ is of the form $(G_{\mathfrak{I}}(B), \cup, \cap, \Rightarrow, \dot{-})$ where B is the set of elements of a bi-topologiaal Boolean algebra $\mathfrak{B} = (B, \cup, \cap, \Rightarrow, -, \mathfrak{I}, C)$; more precisely.

4.2 *For every semi-Boolean algebra* $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ *there exists a bi-topological Boolean algebra* $\mathfrak{B} = (B, \cup, \cap, \rightarrow, -, \mathfrak{I}, C)$ *such that* $A = G_{\mathfrak{I}}(B)$.

Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ be an arbitrary semi-Boolean algebra. It is well known that there exists a Boolean algebra $(B, \cup, \cap, \rightarrow, -)$ such that

a) $(A, \cup, \cap)$ is a sublatice $(B, \cup, \cap)$ the zero (unit) element of $(A, \cup, \cap)$ is the zero (unit) element of $(B, \cup, \cap)$,

b) every element $b \in B$ is of the form

(1)′ $$b = (a_1 \rightarrow a_1') \cap \ldots \cap (a_n \rightarrow a_n')$$

where $a_1, a_1', \ldots, a_n, a_n' \in A$. The symbol $\rightarrow$ denotes the Boolean relative complement,

c) every element $b \in B$ is of the form

(2)′ $$b = (a_1 - a_1') \cup \ldots \cup (a_n - a_n')$$

where $a_1, a_1', \ldots, a_n, a_n' \in A$, the symbol $-$ denotes the Boolean difference, i.e. $a_j - a_j' = a_j \cap -a_j'$ for every $j = 1, \ldots, n$.

We observe that for arbitrary $a, a' \in A$

$$a \Rightarrow a' \leqslant a \rightarrow a' \quad \text{and} \quad a - a' \leqslant a \dot{-} a'.$$

It is easy to prove that for any $a_1, a_1', \ldots, a_n, a_n' \in A$ the inequality

$$(a_1 \rightarrow a_1') \cap \ldots \cap (a_n \rightarrow a_n') \leqslant a \rightarrow a'$$

implies

$$(a_1 \Rightarrow a_1') \cap \ldots \cap (a_n \Rightarrow a_n') \leqslant a \Rightarrow a'$$

and the inequality

$$a - a' \leqslant (a_1 - a_1') \cup \ldots \cup (a_n - a_n')$$

implies

$$a \dot{-} a' \leqslant (a_1 \dot{-} a_1') \cup \ldots \cup (a_n \dot{-} a_n') .$$

We define an interior operation $\mathfrak{I}$ and the closure operation C in a Boolean algebra $(B, \cup, \cap, \rightarrow, -)$ as follows: for every $b \in B$

(3) $$\mathfrak{I}b = (a_1 \Rightarrow a_1') \cap \ldots \cap (a_n \Rightarrow a_n') ,$$

(4) $$Cb = (a_1 \dot{-} a_1') \cup \ldots \cup (a_n \dot{-} a_n') .$$

It follows from the above notions that $\mathfrak{I}$ and C do not depend on the representation of the element b in the forms (1′) and (2′). Obviously, the operations $\mathfrak{I}$ and C defined by (3) and (4) are the interior operation and the closure operation, respectively. In particular,

$$\mathfrak{I}b \in A \quad \text{and} \quad Cb \in A \quad \text{for} \quad b \in B, \tag{5}$$

$$\mathfrak{I}b = b \quad \text{and} \quad Cb = b \quad \text{for} \quad b \in A. \tag{6}$$

We prove that the operations $\mathfrak{I}$ and C are conjugate over B, i.e. that condition (*) is satisfied. Suppose that $b \in B$, then by (5) $\mathfrak{I}b \in A$ and $Cb \in A$. Thus, by (6) we have that $C\mathfrak{I}b = \mathfrak{I}b$ and $\mathfrak{I}Cb = Cb$. Conditions (5) and (6) imply immediately the equation $A = G_{\mathfrak{I}}(B)$. This proves that the algebra $(B, \cup, \cap, \rightarrow, -, \mathfrak{I}, C)$ is the required bi-topological Boolean algebra. ■

§ 5. **Semi-fields of sets.** We recall that every semi-Boolean algebra is of the form $(G_{\mathfrak{I}}(B), \cup, \cap, \Rightarrow, \dot{-})$, where $(G_{\mathfrak{I}}(B), \cup, \cap)$ is the lattice of all $\mathfrak{I}$-open elements of a bi-topological Boolean algebra $(B, \cup, \cap, \rightarrow, -, \mathfrak{I}, C)$. The operations $\mathfrak{I}, C$ are defined by (1) and (2) § 4. If $B = \mathfrak{B}(X)$, i.e. if B is a bi-topological field of sets of the bi-topological space $\langle X, \mathfrak{I}, C\rangle$ such that the operations $\mathfrak{I}$ and C are conjugate over B then the algebra $\mathfrak{C} = (G_{\mathfrak{I}}(\mathfrak{B}(X)), \cup, \cap, \Rightarrow, \dot{-})$ and each of its subalgebra will be called a *semi-field of sets* (more exactly: a *semi-field of subsets of* X). The following theorem explains the connection between semi-Boolean algebras and a semi-field of sets.

5.1. *For every semi-Boolean algebra* $\mathfrak{A}$ *there exists a semi-field of sets* $\mathfrak{C}$ *and an isomorphism of* $\mathfrak{A}$ *onto* $\mathfrak{C}$.

Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ be an arbitrary semi-Boolean algebra. Let us denote by X the set of all prime filtres V of the lattice $(A, \cup, \cap)$ and for every $a \in A$ let $h(a)$ denote the set of all $V \in X$ such that $a \in A$. Let $\mathfrak{R}$ be the class of all $h(a)$, $a \in A$, i.e. $\mathfrak{R} = \{h(a)\}_{a \in A}$.

Let us define an interior operation $\mathfrak{I}$ and a closure operation C in X in the following way: for every $Y \subset X$

$$\mathfrak{I}Y = \bigcup_{\substack{h(a) \in \mathfrak{R} \\ h(a) \subset Y}} h(a), \qquad CY = \bigcap_{\substack{h(b) \in \mathfrak{R} \\ Y \subset h(b)}} h(b).$$

The system $\langle X, \mathfrak{I}, C\rangle$ is a bi-topological space. Let $\mathfrak{B}(X)$ be the field of subsets of X generated by $\mathfrak{R}$ and such that the following condition is satisfied

$$\text{if} \quad Y \in \mathfrak{B}(X) \quad \text{then} \quad \mathfrak{I}Y \in \mathfrak{R} \text{ and } CY \in \mathfrak{R}.$$

We observe that the operations $\mathfrak{I}$ and C defined above are conjugate over $\mathfrak{B}(X)$, i.e. that condition (**) is fulfilled. Indeed, suppose that $Y \in \mathfrak{B}(X)$. Then $\mathfrak{I}Y \in \mathfrak{R}$ and $CY \in \mathfrak{R}$. On account of the definition of the operations $\mathfrak{I}$ and C we have $C\mathfrak{I}Y = \mathfrak{I}Y$ and $\mathfrak{I}CY = CY$. Thus the algebra $(\mathfrak{B}(X), \cup, \cap, -, \mathfrak{I}, C)$ is a bi-topological field of subsets of the bi-topological space $\langle X, \mathfrak{I}, C\rangle$. The class of all $\mathfrak{I}$-open elements in $\mathfrak{B}(X)$

coincides with $\mathfrak{R}$, i.e. $G_{\mathfrak{I}}(\mathfrak{B}(X)) = \mathfrak{R}$. In consequence, the algebra $\mathfrak{C} = (\mathfrak{R}, \cup, \cap, \Rightarrow, \dot{-})$, where the operations $\cup, \cap$ are the set-theoretical union and intersection, respectively, and the operations $\Rightarrow$ and $\dot{-}$ are defined as follows: for every $Y, Z \in G_{\mathfrak{I}}(\mathfrak{B}(X))$

$$Y \Rightarrow Z = \mathfrak{I}((X - Y) \cup Z), \tag{1}$$

$$Y \dot{-} Z = C(Y \cap (X - Z)), \tag{2}$$

is a semi-field of sets.

It will be proved that the mapping h is the required isomorphism of the semi-Boolean algebra $\mathfrak{A}$ onto the semi-field of sets $\mathfrak{C}$. It is well known that the mapping h is one-to-one and

$$h(a \cup b) = h(a) \cup h(b), \tag{3}$$

$$h(a \cap b) = h(a) \cap h(b). \tag{4}$$

It is sufficient to prove that the following conditions are satisfied:

$$h(a \Rightarrow b) = h(a) \Rightarrow h(b), \tag{5}$$

for every $a, b \in$

$$h(a \dot{-} b) = h(a) \dot{-} h(b), \tag{6}$$

On account of (1) and (2) we have to show that

$$h(a \Rightarrow b) = \mathfrak{I}((X - h(a)) \cup h(b)), \tag{7}$$

$$h(a \dot{-} b) = C(h(a) \cap (X - h(b))). \tag{8}$$

Condition (7) follows from [5].

Clearly condition (8) is equivalent to the following two conditions:

$$h(a) \cap X - h(b) \subset h(a \dot{-} b), \tag{9}$$

$$\text{if} \quad h(a) \cap X - h(b) \subset h(c) \quad \text{then} \quad h(a \dot{-} b) \subset h(c). \tag{10}$$

By (1_4) § 1 we have $a \cup b = b \cup (a \dot{-} b)$. Hence $h(a) \cup h(b) = h(b) \cup h(a \dot{-} b)$. This implies that $h(a) \cap X - h(b) \subset h(a \dot{-} b)$ which proves (9).

Let us suppose that, for some $c \in A$, $h(a) \cap X - h(b) \subset h(c)$. Thus $h(a) \subset h(c) \cup h(b) = h(c \cup b)$. It is easy to show that

$$h(x) \subset h(y) \quad \text{if and only if} \quad h(x \dot{-} y) = \emptyset.$$

Thus and by (12) § 1 we obtain $h(a \dot{-} b) \subset h(c)$ which proves condition (10). We infer from (3), (4), (5) and (6) that h is the required isomorphism of $\mathfrak{A}$ onto $\mathfrak{C}$, which completes the proof of 5.1. ■

§ 6. **$(X, \mathfrak{R})$-topological semi-Boolean algebras.** The aim of this section is to give a method of the construction of semi-Boolean algebras and a certain representation theorem of these algebras.

Let $\langle X, I\rangle$ be an arbitrary compact topological space and let $\mathfrak{B}(X)$ be a field of all both open and closed subsets of the topological space $\langle X, I\rangle$. Let $\mathfrak{R}$ be an arbitrary ring of sets such that the field $\mathfrak{B}(X)$ is generated by $\mathfrak{R}$ and the following conditions are satisfied:

(i) the empty set $\emptyset \in \mathfrak{R}$,

(ii) $X \in \mathfrak{R}$,

(iii) if $Z \in \mathfrak{B}(X)$ then $\bigcup\limits_{\substack{A \in \mathfrak{R} \\ A \subset Z}} A$ and $\bigcup\limits_{\substack{B \in \mathfrak{R} \\ Z \subset B}} B$ belong to $\mathfrak{B}(X)$.

Let $\mathfrak{I}$ be an interior operation in X defined as follows

$$\mathfrak{I}Y = \bigcup_{\substack{A \in \mathfrak{R} \\ A \subset Y}} A \quad \text{for every } Y \subset X . \tag{1}$$

The system $\langle X, \mathfrak{I}\rangle$ is a topological space.

Let C be a closure operation in the set X defined as follows

$$CY = \bigcap_{\substack{B \in \mathfrak{R} \\ Y \subset B}} B \quad \text{for every } Y \subset X . \tag{2}$$

The system $\langle X, C\rangle$ is a topological space. Thus the system $\langle X, \mathfrak{I}, C\rangle$ is a bi-topological space. We observe that if $A \in \mathfrak{R}$ then $\mathfrak{I}A = A$ and $CA = A$ i.e. the elements of the ring $\mathfrak{R}$ are both $\mathfrak{I}$-open and C-closed.

6.1. *If* $A \in \mathfrak{B}(X)$ *then* $\mathfrak{I}A \in \mathfrak{R}$ *and* $CA \in \mathfrak{R}$.

In fact, if $A \in \mathfrak{B}(X)$ then by (iii) and the definition of the interior operation $\mathfrak{I}$ it is true that $\mathfrak{I}A$ is simultaneously an open and a closed subset of a compact space $\langle X, I\rangle$. Hence $\mathfrak{I}A$ is a finite union of the elements of the ring $\mathfrak{R}$, i.e. $\mathfrak{I}A \in \mathfrak{R}$. In the same way it could be shown that $CA \in \mathfrak{R}$. ■

Thus the following theorem holds:

6.2. *The field* $\mathfrak{B}(X)$ *is a bi-topological field of sets.* ■

Thus by an easy verification we obtain the next theorem:

6.3. *The algebra* $\mathfrak{R} = (\mathfrak{R}, \cup, \cap, \Rightarrow, \dot{-})$ *where* $\mathfrak{R}$ *is the ring defined above, the operations* $\cup, \cap$ *are the set-theoretical union and intersection respectively, and the operations* $\Rightarrow, \dot{-}$ *are defined by* (1) *and* (2), § 5, *respectively, is a semi-Boolean algebra.* ■

Every semi-Boolean algebra of this kind is said to be an $(X, \mathfrak{R})$-*topological semi-Boolean algebra.* To illustrate the notion of a $(X, \mathfrak{R})$-topological semi-Boolean algebra let us consider the case in which X is the Cantor discontinuum [3], i.e. X is the Cartesian product U^E, where U is the set consisting of the integers 0 and 1 only, and E is a non-empty set. By definition, X is the set of all mappings $u = \{u_a\}_{a \in E}$ such that $u_a = 0$ or $u_a = 1$, $a \in E$. Let A^a $(a \in E)$ be the set of all $u \in X$ such that $u_a = 1$. Denote by $\mathfrak{D}$ the class of all sets A^a and their set-theoretical complements.

Let $\mathfrak{B}(X)$ be the field of subsets of X generated by $\mathfrak{D}$. It is known that $\mathfrak{B}(X)$ is the field of all both open and closed subsets of the topological space $\langle X, I\rangle$, where I is the interior operation in X determined by the class $\mathfrak{D}$ assumed as a subbasis. Now, let $\mathfrak{R}$ be a ring of the sets which belong to the class $\{A^a\}_{a \in E}$ and be such that conditions (i) and (ii) are satisfied. It is easy to see that the field $\mathfrak{B}(X)$ is generated by the ring $\mathfrak{R}$, i.e. if $Y \subset \mathfrak{B}(X)$ then

$$Y = \bigcap_{i=1}^{k} (A^{a_{i_1}} \cup \ldots \cup A^{a_{i_n}} \cup B^{\beta_{i_1}} \cup \ldots \cup B^{\beta_{i_m}}) , \tag{3}$$

where for every i, j: $A^{a_{ij}} \in \mathfrak{R}$, $B^{\beta_{ij}}$ is a set-theoretical complement of some $A^a \in \mathfrak{R}$ (i.e. $X - B^{\beta_{ij}} \in \mathfrak{R}$) and $a_{ij} \neq \beta_{ij}$.

Let $\mathfrak{I}$ be an interior operation defined by (1), and let C be a closure operation defined by (2). It will be shown that if $Y \in \mathfrak{B}(X)$ i.e. if Y is of form (3), then

$$\mathfrak{I}Y = \bigcap_{i=1}^{k} (A^{a_{i_1}} \cup \ldots \cup A^{a_{i_n}}) , \tag{4}$$

$$CY = \bigcup_{i=1}^{k} \big((X - B^{\beta_{i_1}}) \cap \ldots \cap (X - B^{\beta_{i_m}})\big) . \tag{5}$$

We prove condition (4). The proof of (5) is similar. Obviously, if $Y = \emptyset$ or $Y = X$, condition (4) is satisfied. Let $Y \neq \emptyset$ and $Y \neq X$. On account of the definition of the interior operation it is sufficient to show that if Y is of the form (3) then the following equation is fulfilled:

$$\bigcup_{\substack{A \in \mathfrak{R} \\ A \subset Y}} A = \bigcap_{i=1}^{k} (A^{a_{i_1}} \cup \ldots \cup A^{a_{i_n}}) . \tag{6}$$

It easy to see that

$$\bigcap_{i=1}^{k} (A^{a_{i_1}} \cup \ldots \cup A^{a_{i_n}}) \subset Y \quad \text{and} \quad \bigcap_{i=1}^{k} (A^{a_{i_1}} \cup \ldots \cup A^{a_{i_n}}) \in R .$$

Thus it is sufficient to show that

$$\text{if} \quad Z \in \mathfrak{R} \text{ and } Z \subset Y \quad \text{then} \quad Z \subset \bigcap_{i=1}^{k} (A^{a_{i_1}} \cup \ldots \cup A^{a_{i_n}}) . \tag{7}$$

Let us suppose that $Z \in \mathfrak{R}$ and $Z \subset Y$. Hence $Z = \bigcup\limits_{p=1}^{l} (A^{\gamma_{p_1}} \cap \ldots \cap A^{\gamma_{p_s}})$. Obviously, for every i and p we have the conclusion

$$A^{\gamma_{p_1}} \cap \ldots \cap A^{\gamma_{p_s}} \subset A^{a_{i_1}} \cup \ldots \cup A^{a_{i_n}} \cup B^{\beta_{i_1}} \cup \ldots B^{\beta_{i_m}} . \tag{8}$$

We observe that for every i and p there exists a $j = 1, \ldots, s$ such that $\gamma_{p_j} \in \{a_{i_1}, \ldots, a_{i_n}\}$, i.e. there exist $j = 1, \ldots, s$ and $t = 1, \ldots, n$ such that

$A^{\gamma_{p_j}} = A^{\alpha_{i_t}}$. Suppose the contrary, i.e. that for all $j = 1, \ldots, s$, $\gamma_{p_j} \notin \{\alpha_{i_1}, \ldots, \alpha_{i_n}\}$. Let $u = \{u_a\}_{a \in E}$ be a mapping such that for fixed i and p

$$u_{\gamma_{p_1}} = \ldots = u_{\gamma_{p_s}} = u_{\beta_{i_1}} = \ldots = u_{\beta_{i_m}} = 1$$

and

$$u_{\alpha_{i_1}} = \ldots = u_{\alpha_{i_n}} = 0 .$$

Thus u belongs to $A^{\gamma_{p_1}} \cap \ldots \cap A^{\gamma_{p_s}}$ but $u \notin A^{\alpha_{i_1}} \cup \ldots \cup A^{\alpha_{i_n}} \cup B^{\beta_{i_1}} \cup \ldots \cup B^{\beta_{i_m}}$. This is impossible on account of (8). Hence for every i and p there exist j and t such that

$$A^{\gamma_{p_1}} \cap \ldots \cap A^{\gamma_{p_s}} \subset A^{\gamma_{p_j}} = A^{\alpha_{i_t}} \subset A^{\alpha_{i_1}} \cap \ldots \cap A^{\alpha_{i_m}} .$$

Consequently

$$\bigcup_{p=1}^{l} (A^{\gamma_{p_1}} \cap \ldots \cap A^{\gamma_{p_s}}) \subset \bigcap_{i=1}^{k} (A^{\alpha_{i_1}} \cup \ldots \cup A^{\alpha_{i_n}}) .$$

This proves that condition (7) is fulfilled, and that $\mathcal{R}$ is a ring satisfying conditions (i)-(iii). By Theorem 6.2 we infer that $(\mathcal{R}, \cup, \cap, \Rightarrow, \dot{-})$ — where $\mathcal{R}$ is the ring defined above and the operations $\cup, \cap, \Rightarrow \dot{-}$, are defined as usual — is an $(X, \mathcal{R})$-topological semi-Boolean algebra.

6.4. *For every semi-Boolean algebra $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ there exists an $(X, \mathcal{R})$-topological semi-Boolean algebra $\mathfrak{R} = (\mathcal{R}, \cup, \cap, \Rightarrow, \dot{-})$ and an isomorphism h of $\mathfrak{A}$ onto $\mathfrak{R}$.*

By Theorem 4.2 we can assume that $A = G_{\mathfrak{I}^*}(B)$ where B is the set of all elements of a bi-topological Boolean algebra $\mathfrak{B} = (B, \cup, \cap, \rightarrow, -, \mathfrak{I}^*, C^*)$. Let $\langle X, I\rangle$ be the Stone space of the Boolean algebra $\mathfrak{B}_0 = (B, \cup, \cap, -)$. Let $\mathfrak{I}$ and C be new interior and closure operations in the set X which are defined by (1) and (2). It follows from 3.1 that these operations are conjugate over the Stone field $\mathfrak{B}(X) = \{h(a)\}_{a \in B}$. Let $\mathcal{R}$ be the class of all $h(a)$ such that $a \in V$ for $a \in G_{\mathfrak{I}^*}(B)$. Suppose that $Y \in \mathcal{R}$, then $Y = h(x)$ for some $x = \mathfrak{I}^* x = C^* x \in G_{\mathfrak{I}^*}(B)$. Thus $\mathfrak{I} Y = Y$ and $CY = Y$ i.e. the elements of $\mathcal{R}$ are simultaneously $\mathfrak{I}$-open and C-closed. It is easy to see that if $Y \in \mathfrak{B}(X)$ then $\mathfrak{I} Y \in \mathcal{R}$ and $CY \in \mathcal{R}$. From 6.3 it follows that the algebra $\mathfrak{R} = (\mathcal{R}, \cup, \cap, \Rightarrow, \dot{-})$, where $\cup, \cap$ are the set-theoretical union and intersection, respectively, and $\Rightarrow$, $\dot{-}$ are defined by (1) and (2) § 5, is an $(X, \mathcal{R})$-topological semi-Boolean algebra. We shall now prove that the mapping h is the required isomorphism of $\mathfrak{A}$ onto $\mathfrak{R}$. It is sufficient to prove that

$$\left.\begin{aligned} h(a \Rightarrow b) &= h(a) \Rightarrow h(b) , \\ h(a \dot{-} b) &= h(a) \dot{-} h(b) , \end{aligned}\right\} \quad \text{for} \quad a, b \in A .$$

By the definition of the operation $\Rightarrow$ in $\mathfrak{R}$ we have $h(a) \Rightarrow h(b) = \mathfrak{I}\big((X - h(a)) \cup h(b)\big)$. On the other hand, $h(a \Rightarrow b) = h\big(\mathfrak{I}^*(-a \cup b)\big)$,

where the signs $-$, $\cup$, $\mathfrak{I}^*$ denote a complement, a join and an interior operation in the bi-topological Boolean algebra $\mathfrak{B}$, respectivly. Thus

$$\begin{aligned} h(a \Rightarrow b) &= h\big(\mathfrak{I}^*(-a \cup b)\big) = \mathfrak{I} h(-a \cup b) \\ &= \mathfrak{I}\big((X - h(a)) \cup h(b)\big) = h(a) \Rightarrow h(b) . \end{aligned}$$

The proof of the equation $h(a \dot{-} b) = h(a) \dot{-} h(b)$ is similar. This completes the proof of 6.4. ■

§ 7. Infinite joins and meets in semi-Boolean algebras. In this section we shall consider the semi-Boolean algebras with infinite joins and meets. Our aim is to give a representation theorem analogous to the Rasiowa-Sikorski lemma.

First, let us prove that the infinite distributive laws are satisfied in this algebras.

7.1. *Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ be a semi-Boolean algebra. If an infinite join $\bigcup_{t \in T} b_t$ exists in $\mathfrak{A}$, then for every $a \in A$ the join $\bigcup_{t \in T} a \cap b_t$ also exists in $\mathfrak{A}$ and*

(1) $$a \cap \bigcup_{t \in T} b_t = \bigcup_{t \in T} a \cap b_t .$$

If an infinite meet $\bigcap_{t \in T} b_t$ exists in $\mathfrak{A}$ then for every $a \in A$ the meet $\bigcap_{t \in T} (a \cup b_t)$ also exists in $\mathfrak{A}$ and

(2) $$a \cup \bigcap_{t \in T} b_t = \bigcap_{t \in T} (a \cup b_t) .$$

The first part of 7.1 is fulfilled in every pseudo-Boolean algebra [2]. Thus it is satisfied in semi-Boolean algebra $\mathfrak{A}$. We prove the second part of (2). Let us assume that $b = \bigcap_{t \in T} b_t$ exists. We have $b \leqslant b_t$, for every $t \in T$. Thus $a \cup b \leqslant a \cup b_t$, $t \in T$. Suppose that there exists c such that $c \leqslant a \cup b_t$, $t \in T$. The last inequality is equivalent to $c \dot{-} a \leqslant b_t$ and this implies $c \dot{-} a \leqslant b$. Thus $c \leqslant a \cup b$, which completes the proof of (2). ■

7.2. *In every semi-Boolean algebra $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$*

(3) $$\ulcorner \bigcup_{t \in T} a_t \leqslant \bigcap_{t \in T} \ulcorner a_t ,$$

(4) $$\ulcorner \bigcap_{t \in T} a_t = \bigcup_{t \in T} \ulcorner a_t ,$$

(5) $$\urcorner \bigcap_{t \in T} a_t \leqslant \bigcup_{t \in T} \ulcorner a_t ,$$

(6) $$\bigcap_{t \in T} \urcorner a_t \leqslant \ulcorner \bigcup_{t \in T} a_t ,$$

(7) $$\bigcap_{t \in T} (b \dot{-} a_t) \leqslant b \dot{-} (\bigcap_{t \in T} a_t) ,$$

$$(8)\qquad b \dot{-} \Big(\bigcup_{t\in T} a_t\Big) \leqslant \bigcap_{t\in T} (b \dot{-} a_t)\,,$$

$$(9)\qquad \bigcap_{t\in T} b_t \dot{-} a \leqslant \bigcap_{t\in T} (b_t \dot{-} a)\,,$$

$$(10)\qquad \Big(\bigcup_{t\in T} b_t\Big) \dot{-} a = \bigcup_{t\in T} (b_t \dot{-} a)\,,$$

$$(11)\qquad \Big(\bigcup_{t\in T} a_t\Big) \dot{-} \Big(\bigcup b_t\Big) \leqslant \bigcup_{t\in T} (a_t \dot{-} b_t)\,,$$

$$(12)\qquad \Big(\bigcap_{t\in T} a_t\Big) \dot{-} \Big(\bigcap_{t\in T} b_t\Big) \leqslant \bigcup_{t\in T} (a_t \dot{-} b_t)\,,$$

$$(13)\qquad b \dot{-} \Big(\bigcup_{t\in T} a_t\Big) \leqslant \Big(\bigcap_{t\in T} a_t\Big) \Rightarrow b\,,$$

$$(14)\qquad b \dot{-} \Big(\bigcap_{t\in T} a_t\Big) = \bigcup_{t\in T} (b \dot{-} a_t)$$

provided all the joins and meets exist in $\mathfrak{A}$.

The simple prove of 7.2 is omitted. ■

7.3. *Let* $\mathfrak{B} = (B, \cup, \cap, \rightarrow, -, \mathfrak{J}, C)$ *be a bi-topological Boolean algebra. Denote by* $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ *a semi-Boolean algebra such that* $A = G_{\mathfrak{J}}(B)$. *For every* $t \in T$ *let* $a_t \in A$. *Then the meet* $\bigcap^{\mathfrak{A}}_{t\in T} a_t$ *exists if and only if the meet* $\bigcap^{\mathfrak{B}}_{t\in T} a_t$ *exists and*

$$(15)\qquad \bigcap^{\mathfrak{A}}_{t\in T} a_t = \bigcap^{\mathfrak{B}}_{t\in T} a_t\,.$$

Similarly, the join $\bigcup^{\mathfrak{A}}_{t\in T} a_t$ *exists if and only if the join* $\bigcup^{\mathfrak{B}}_{t\in T} a_t$ *exists*

$$(16)\qquad \bigcup^{\mathfrak{A}}_{t\in T} a_t = \bigcup^{\mathfrak{B}}_{t\in T} a_t\,.$$

Suppose that $a = \bigcap^{\mathfrak{A}}_{t\in T} a_t$ exists and that $d \leqslant a_t$ for every $t \in T$, $d \in B$. Then $Cd \leqslant Ca_t = a$ for every $t \in T$. Hence $Cd \leqslant a$ and consequently $d \leqslant a$, which proves that $a = \bigcap^{\mathfrak{B}}_{t\in T} a_t$. Conversely, suppose that $a = \bigcap^{\mathfrak{B}}_{t\in T} a_t$ exists. The meet of any number of C-closed elements being C-closed we have $a \in A$, which implies $a = \bigcup^{\mathfrak{A}}_{t\in T} a_t$.

The proof of (16) is similar. ■

Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ be a semi-Boolean algebra and let (Q) be a set of infinite joins and meets in $\mathfrak{A}$:

$$(\mathrm{Q})\qquad \begin{aligned} a_s &= \bigcup^{\mathfrak{A}}_{t\in T_s} a_{s_t} \qquad (s \in S)\,,\\ b_s &= \bigcap^{\mathfrak{A}}_{t\in T_s} b_{s_t} \qquad (s \in S')\,. \end{aligned}$$

We shall say that an isomorphism h from a semi-Boolean algebra $\mathfrak{A}$ into a semi-Boolean algebra $\mathfrak{B} = (B, \cup, \cap, \Rightarrow, \dot{-})$ is a *Q-isomorphism* provided it preserves all the infinite joins and meets in (Q), i.e. if

$$(17)\qquad h(a_s) = \bigcup^{\mathfrak{B}}_{t\in T_s} h(a_{s_t}) \qquad (s \in S)\,,$$

$$(18)\qquad h(b_s) = \bigcap^{\mathfrak{B}}_{t\in T_s} h(b_{s_t}) \qquad (s \in S')\,.$$

7.4. *For every semi-Boolean algebra* $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$, *if the set* (Q) *is a most enumerable, i.e. if the sets* S *and* S' *are at most enumerable, then there exist a semi-field of sets* $\mathfrak{C}$ *and a* Q*-isomorphism from* $\mathfrak{A}$ *onto* $\mathfrak{C}$ *such that the infinite joins and meets on the right-hand sides of the equations* (17) *and* (18) *coincide with the set-theoretical unions and intersections' respectively.*

Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$, be a semi-Boolean algebra. On account of 3.1 we can assume that $A = G_{\mathfrak{J}^*}(B)$ where B is the set of all elements of a bi-topological Boolean algebra $\mathfrak{B} = (B, \cup, \cap, \rightarrow, -, \mathfrak{J}^*, C^*)$. Denote by $\mathfrak{B}_0 = (B, \cup, \cap, \rightarrow, -)$ the Boolean algebra of $\mathfrak{B}$. It is known [3] that there exists a Boolean Q-isomorphism h of the Boolean algebra $\mathfrak{B}_0$ into the field $\mathcal{B}(X)$ of all subsets of space X. We define an interior operation $\mathfrak{J}$ in X and a closure operation C in X as follows:

$$\mathfrak{J}Y = \bigcup_{\substack{h(a)\subset Y\\ a=\mathfrak{J}^*a}} h(a)\,, \qquad CY = \bigcap_{\substack{Y\subset h(b)\\ b=C^*b}} h(b)\,.$$

These operations are conjugate over $h(B)$. Thus the algebra $(h(B), \cup, \cap, -, \mathfrak{J}, C)$ is a bi-topological field of subsets of X, and if $a, b \in A$ then

$$(19)\qquad h(a \cup b) = h(a) \cup h(b)\,,$$

$$(20)\qquad h(a \cap b) = h(a) \cap h(b)\,,$$

$$(21)\qquad h(-a) = X - h(a)\,,$$

$$(22)\qquad h(\mathfrak{J}^*a) = \mathfrak{J}h(a)\,,$$

$$(23)\qquad h(C^*a) = Ch(a)\,.$$

Let $G_{\mathfrak{J}}(h(B))$ denote the class of all $\mathfrak{J}$-open elements of $h(B)$. The algebra $\mathfrak{C} = \big(G_{\mathfrak{J}}(h(B)), \cup, \cap, \Rightarrow, \dot{-}\big)$ where the operations $\Rightarrow, \dot{-}$ are defined as follows for every $a, b \in A$

$$h(a) \Rightarrow h(b) = \mathfrak{J}\big((X - h(a)) \cup h(B)\big)\,,$$
$$h(a) \dot{-} h(b) = C\big(h(a) \cap X - h(b)\big)$$

is a semi-field of sets of the bi-topological space $\langle X, \mathfrak{J}, C\rangle$. We shall prove that the mapping h is the required Q-isomorphism of the semi-

Boolean algebra $\mathfrak{A}$ onto the semi-field of sets $\mathfrak{C}$. Suppose that $a, b \in A$. This implies that $a = \mathfrak{I}^* a$, $b = \mathfrak{I}^* b$. Thus from (19) and (22) we infer that $h(a) \cup h(b) \in G_{\mathfrak{I}}(h(B))$. Similarly, we prove that the right-hand side of (20) belongs to $G_{\mathfrak{I}}(h(B))$. Now we show that if $a, b \in A$ then

$$h(a \Rightarrow b) = h(a) \Rightarrow h(b), \tag{24}$$

$$h(a \dot{-} b) = h(a) \dot{-} h(b), \tag{25}$$

where the signs $\Rightarrow$, $\dot{-}$ on the right-hand sides of (24) and (25) denote the relative pseudo-complement and pseudo-difference in $\mathfrak{C}$. On account of the definition of $\Rightarrow$ and $\dot{-}$ in $\mathfrak{C}$, (22), (23) and from 4.1 we have

$$h(a \Rightarrow b) = h(\mathfrak{I}^*(-a \cup b)) = \mathfrak{I}\big((X - h(a)) \cup h(b)\big) = h(a) \Rightarrow h(b)$$

and

$$h(a \dot{-} b) = h(C^*(a \cap -b)) = C\big(h(a) \cap (X - h(b))\big) = h(a) \dot{-} h(b).$$

Now, it is sufficient to prove that the equations corresponding to (17) and (18) — where the sets S and S' are at most enumerable and the infinite joins $\bigcup_{t \in T}^{\mathfrak{B}} a_t$ and meets $\bigcap_{t \in T}^{\mathfrak{B}} a_t$ on the right-hand sides of these equations coincide with the set-theoretical unions and intersections, respectively — are satisfied. Suppose that $a_s = \bigcup_{t \in T_s}^{\mathfrak{A}} a_{s,t}$ and for every $t \in T_s$, $a_{s,t} \in A$. By 7.3 we can write that $a_s = \bigcup_{t \in T_s}^{\mathfrak{B}} a_{s,t}$. Since h is a Boolean Q-isomorphism of the Boolean algebra $\mathfrak{B}_0 = (B, \cup, \cap, \rightarrow, -)$ into the field $\mathfrak{B}(X)$ of all subsets of X, we infer that $h(a_s) = \bigcup_{t \in T_s} h(a_{s,t})$ where the sign $\bigcup$ denotes the set-theoretical union. It remains to show that $\bigcup_{t \in T_s} h(a_{s,t})$ is an $\mathfrak{I}$-open element in $h(B)$, i.e. that $\bigcup_{t \in T_s} h(a_{s,t}) \in G_{\mathfrak{I}}(h(B))$. This follows immediately from the fact, that for every $s \in S$ and $t \in T_s$, $h(a_{s,t})$ is an $\mathfrak{I}$-open set. Thus the condition (17) is satisfied. Let $b_s = \bigcap_{t \in T_s}^{\mathfrak{A}} b_{s,t}$ and for every $t \in T_s$, $b_{s,t} \in A$. By 7.3 we obtain $b_s = \bigcap_{t \in T_s}^{\mathfrak{B}} b_{s,t}$. Thus $h(b_s) = \bigcap_{t \in T_s} h(b_{s,t})$, where the sign $\bigcap$ denotes the set-theoretical intersection and h is a Boolean Q-isomorphism of $\mathfrak{B}_0$ into $\mathfrak{B}(X)$. The intersection of any number of C-closed elements is C-closed. Thus we have $\bigcap_{t \in T} h(b_{s,t}) \in G_{\mathfrak{I}}(h(B))$, which completes the proof of 7.4. ■

§ 8. Finite semi-Boolean algebras.

8.1. *Let $\mathfrak{B} = (B, \cup, \cap, \rightarrow, -, \mathfrak{I}^*, C^*)$ be a bi-topological Boolean algebra, let A_0 be a finite subset of B and let $\mathfrak{B}' = (B', \cup, \cap, \rightarrow, -)$ be the Boolean algebra generated by A_0. Then there exist an interior operation $\mathfrak{I}$ in B' and a closure operation C in B' such that condition $(*)$ is satisfied, and for every $a \in B$*

$$\text{if} \quad \mathfrak{I}^* a \in A_0 \quad \text{then} \quad \mathfrak{I}^* a = \mathfrak{I} a, \tag{1}$$

$$\text{if} \quad C^* a \in A_0 \quad \text{then} \quad C^* a = C a. \tag{2}$$

The algebra $\mathfrak{B}'$ contains at most 2^{2^r} elements, where r is the number of elements in A_0.

Let $G_{\mathfrak{I}^*}(A_0)$ denote the class of all $\mathfrak{I}$-open elements in $\mathfrak{B}$ which belong to A_0. Obviously the elements of $G_{\mathfrak{I}^*}(A_0)$ are C-closed. It is well known that there exists an interior operation $\mathfrak{I}$ in $\mathfrak{B}'$ such that the algebra $\mathfrak{B}'$ may be considered as a topological Boolean algebra with the interior operation $\mathfrak{I}$, an property (1) is satisfied. Namely, the operation $\mathfrak{I}$ is defined by the formula

$$\mathfrak{I} a = \bigcup_{\substack{b \in G_{\mathfrak{I}^*}(A_0) \\ b \leqslant a}} b \quad (a \in B').$$

Now let the closure operation C be defined in $\mathfrak{B}'$ in the following way:

$$C a = \bigcap_{\substack{d \in G_{\mathfrak{I}^*}(A_0) \\ b \leqslant d}} d \quad (a \in B').$$

It is easy to see that the operations $\mathfrak{I}$ and C are conjugate over B' and (2) is fulfilled. The proof the second part of 8.1 is known. ■

8.2. *Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ be a semi-Boolean algebra, and let $A_0 \subset A$ be a finite set containing r elements. Then there exists a finite semi-Boolean algebra $\mathfrak{A}' = (A', \cup, \cap, \Rightarrow, \dot{-})$ containing at most 2^{2^r} elements such that $V \in A'$, $\Lambda \in A'$, $A_0 \subset A'$ and for every a, b, c $A_0' = A_0 \cup (\Lambda, V)$*

(a) *if c is the join of a, b in $\mathfrak{A}$, then c is the join of a, b in $\mathfrak{A}'$,*

(b) *if c is the meet of a, b in $\mathfrak{A}$, then c is the meet of a, b in $\mathfrak{A}'$,*

(c) *if c is the pseudo-complement of a relative to b in $\mathfrak{A}$, then c is the pseudo-complement of a relative to b in $\mathfrak{A}'$,*

(d) *if c is the pseudo-difference a and b in $\mathfrak{A}$, then c is the pseudo-difference a and b in $\mathfrak{A}'$.*

This follows immediately from 4.2 and 8.1. ■

8.3. *Every finite topological Boolean algebra can be extended to bi-topological Boolean algebra.*

Let $\mathfrak{B} = (B, \cup, \cap, \rightarrow, -, \mathfrak{I})$ be a finite topological Boolean algebra. A closure operation C in B is defined as follows: for every $a \in B$

$$C a = \bigcap_{\substack{a \leqslant c \\ c = \mathfrak{I} a}} c.$$

It is easy to see that $C\mathfrak{I}a = \mathfrak{I}a$ and $\mathfrak{I}Ca = Ca$, i.e. the operations $\mathfrak{I}$ and C are conjugate over B. ■

8.4. *Every finite pseudo-Boolean algebra can be extended to a semi-Boolean algebra.*

This statement follows immediately from 8.3, 4.2 and from the fact [1] that every pseudo-Boolean algebra is of the form $(G_{\mathfrak{I}}(B), \cup, \cap, \Rightarrow, \neg)$, where $G_{\mathfrak{I}}(B)$ is the set of $\mathfrak{I}$-open elements of a topological Boolean algebra $(B, \cup, \cap, \rightarrow, -, \mathfrak{I})$ and the operations $\Rightarrow$ and $\neg$ are defined as follows:

$$a \Rightarrow b = \mathfrak{I}(-a \cup b), \qquad \neg a = \mathfrak{I}(-a)$$

for any $a, b \in G_{\mathfrak{I}}(B)$. ■

§ 9. Definition and some properties of the propositional calculi of the H-B logic. By the alphabet of a propositional calculus we shall understand any ordered system

$$A = \{V, L_1, L_2, U\}$$

where V, L_1, L_2, U are disjoint sets, the set V is infinite and elements in V are called *propositional variables* and denote by $a, b, c, \ldots$; L_1 contains two elements denoted by $\neg$ and $\ulcorner$ and called the *Heyting negation sign* and the *Brouwerian negation sign*, respectively; L_2 contains four elements denoted by $\cup, \cap, \Rightarrow, \dot{-}$ and called the *disjunction sign*, the *conjunction sign*, the *implication sign*, and the *explication sign*, respectively; the set U contains only two elements denoted by (,) and called *parentheses in A*.

The class F of formulas is the smallest class of expressions of this system, which contains all the variables and satisfies the following conditions:

(i) if α is a formula, then so are $\neg\alpha$ and $\ulcorner\alpha$,

(ii) if α and β are arbitrary formulas, then so are

$$(\alpha \cup \beta),\ (\alpha \cap \beta),\ \alpha(\Rightarrow \beta),\ (\alpha \dot{-} \beta).$$

By the formalized language of the propositional calculus or the formalized language of zero order we shall understand the pair

$$\mathfrak{L} = \{A, F\}.$$

Let the set $\mathcal{A}_1$ of axioms consists of all formulas of the form:

(A$_1$) $\big((\alpha \Rightarrow \beta) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma))\big)$,

(A$_2$) $(\alpha \Rightarrow (\alpha \cup \beta))$,

(A$_3$) $(\beta \Rightarrow (\alpha \cup \beta))$,

(A$_4$) $\big((\alpha \Rightarrow \gamma) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow ((\alpha \cup \beta) \Rightarrow \gamma))\big)$,

(A$_5$) $((\alpha \cap \beta) \Rightarrow \alpha)$,

(A$_6$) $((\alpha \cap \beta) \Rightarrow \beta)$,

(A$_7$) $\big((\gamma \Rightarrow \alpha) \Rightarrow ((\gamma \Rightarrow \beta) \Rightarrow (\gamma \Rightarrow (\alpha \cap \beta)))\big)$,

(A$_8$) $\big((\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \cap \beta) \Rightarrow \gamma)\big)$,

(A$_9$) $\big(((\alpha \cap \beta) \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow (\beta \Rightarrow \gamma))\big)$,

(A$_{10}$) $(\alpha \Rightarrow (\beta \cup (\alpha \dot{-} \beta)))$,

(A$_{11}$) $((\alpha \Rightarrow \beta) \Rightarrow (\neg\beta \Rightarrow \neg\alpha))$,

(A$_{12}$) $((\alpha \dot{-} \beta) \Rightarrow \ulcorner(\alpha \Rightarrow \beta))$,

(A$_{13}$) $\big(((\alpha \dot{-} \beta) \dot{-} \gamma) \Rightarrow (\alpha \dot{-} (\beta \cup \gamma))\big)$,

(A$_{14}$) $(\neg(\alpha \dot{-} \beta) \Rightarrow (\alpha \Rightarrow \beta))$,

(A$_{15}$) $((\alpha \Rightarrow (\gamma \dot{-} \gamma)) \Rightarrow \neg\alpha)$,

(A$_{16}$) $(\neg\alpha \Rightarrow (\alpha \Rightarrow (\gamma \dot{-} \gamma)))$,

(A$_{17}$) $(((\gamma \Rightarrow \gamma) \dot{-} \alpha) \Rightarrow \ulcorner\alpha)$,

(A$_{18}$) $(\ulcorner\alpha \Rightarrow ((\gamma \Rightarrow \gamma) \dot{-} \alpha))$,

where α, β, γ are any formulas in $\mathfrak{L}$.

By a consequence operation in $\mathfrak{L}$ we will understand a mapping $\mathfrak{C}$ of 2^F to 2^F such that for any $F_1 \subset F$, $\mathfrak{C}(F_1)$ is the smallest set containing $\mathcal{A}_1$ and F_1 and closed under of the rules of inference: modus ponens and

$$\text{(r)} \quad \frac{\alpha}{\neg\ulcorner\alpha},$$

where α is any formula in $\mathfrak{L}$.

The deductive system

$$\mathcal{S} = \{\mathfrak{L}, \mathfrak{C}\}$$

will be called the *propositional calculus of the Heyting-Brouwer logic based on the language* $\mathfrak{L}$ or *briefly the propositional calculus of the* H-B *logic*. A formula α is said to be *derivable in* $\mathcal{S}$ provided it is in $\mathfrak{C}(\emptyset)$.

By the formalized theories of zero order based on the language (for brevity theories) we shall understand the system

$$\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\},$$

where $\mathcal{A}$ is a set of formulas. Formulas in $\mathfrak{C}(\mathcal{A})$ are called *theorems of the theory* $\mathfrak{T}$. The theory $\mathfrak{T}$ is consistent if there exists a formula which is not a theorem of $\mathfrak{T}$.

Let $(F, \cup, \cap, \Rightarrow, \dot{-}, \neg, \ulcorner)$ be the algebra of formulas of the language $\mathfrak{L}$. It is known that the relation $\approx$ defined as

(1) $\alpha \approx \beta$ if and only if formulas $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are both theorems in $\mathfrak{T}$ is a congruence relation in the algebra $(F, \cup, \cap, \Rightarrow, \dot{-}, \neg, \ulcorner)$.

We shall denote the quotient algebra by $\mathfrak{A}(\mathfrak{T})$, i.e. $\mathfrak{A}(\mathfrak{T}) = (F/\approx, \cup, \cap,$

$\Rightarrow, \dot{-}, \neg, \ulcorner$). Elements in $\mathfrak{A}(\mathfrak{T})$ will be denoted by $\|\alpha\|$ for $\alpha \in F$. The relation $\leqslant$ defined as follows: for any formulas α, β in F

(2) $\|\alpha\| \leqslant \|\beta\|$ if and only if $(\alpha \Rightarrow \beta)$ is a theorem in $\mathfrak{T}$ is an ordering relation in $F/\approx$.

9.1. *The algebra $\mathfrak{A}(\mathfrak{T})$ is a semi-Boolean algebra. Moreover, for any formulas α, β*

$$\|\alpha\| \cup \|\beta\| = \|(\alpha \cup \beta)\|, \tag{3}$$

$$\|\alpha\| \cap \|\beta\| = \|(\alpha \cap \beta)\|, \tag{4}$$

$$\|\alpha\| \Rightarrow \|\beta\| = \|(\alpha \Rightarrow \beta)\|, \tag{5}$$

$$\|\alpha\| \dot{-} \|\beta\| = \|(\alpha \dot{-} \beta)\|, \tag{6}$$

$$\neg\|\alpha\| = \|\neg\alpha\|, \tag{7}$$

$$\ulcorner\|\alpha\| = \|\ulcorner\alpha\|. \tag{8}$$

A formula α is a theorem in $\mathfrak{T}$ if and only if the element $\|\alpha\|$ is the unit V of $\mathfrak{A}(\mathfrak{T})$. The theory $\mathfrak{T}$ is consistent if and only if the semi-Boolean algebra $\mathfrak{A}(\mathfrak{T})$ is non-degenerate.

The proofs of (3), (4) and (5) are known [2]. We prove (6). Let $\|\alpha\|$ and $\|\beta\|$ be any elements in $\mathfrak{A}(\mathfrak{T})$. The formula $(\alpha \Rightarrow ((\alpha \dot{-} \beta) \cup \beta))$ is a theorem because it is of the form (A_{10}). By (2) and (3) we can write $\|\alpha\| \leqslant \|(\alpha \dot{-} \beta) \cup \beta\|$.

Now we shall prove that, for every element $\|\gamma\|$ in $\mathfrak{A}(\mathfrak{T})$,

$$\text{if} \quad \|\alpha\| \leqslant \|\beta\| \cup \|\gamma\| \quad \text{then} \quad \|(\alpha \dot{-} \beta)\| \leqslant \|\gamma\|.$$

Suppose that $\|\alpha\| \leqslant \|\beta\| \cup \|\gamma\|$; then by (2) and (3) the formula $(\alpha \Rightarrow (\beta \cup \gamma))$ is a theorem. By the rule (r) the formula $\neg\ulcorner(\alpha \Rightarrow (\beta \cup \gamma))$ is a theorem. The formula $(((\alpha \dot{-} (\beta \cup \gamma)) \Rightarrow \ulcorner(\alpha \Rightarrow (\beta \cup \gamma))) \Rightarrow (\neg\ulcorner(\alpha \Rightarrow (\beta \cup \gamma)) \Rightarrow \neg(\alpha \dot{-} (\beta \cup \gamma))))$, is also a theorem since it is of the form (A_{11}). Thus by (A_{12}) and by applying modus ponens twice the formula $\neg(\alpha \dot{-} (\beta \cup \gamma))$ is a theorem in $\mathfrak{T}$. The formula $((((\alpha \dot{-} \beta) \dot{-} \gamma) \Rightarrow (\alpha \dot{-} (\beta \cup \gamma))) \Rightarrow (\neg(\alpha \dot{-} (\beta \cup \gamma)) \Rightarrow \neg((\alpha \dot{-} \beta) \dot{-} \gamma)))$ is a theorem because it is of the form (A_{11}). Using (A_{13}) and applying modus ponens twice we find that the formula $\neg((\alpha \dot{-} \beta) \dot{-} \gamma)$ is a theorem. On account of (A_{14}) we infer that the formula $(\neg((\alpha \dot{-} \beta) \dot{-} \gamma) \Rightarrow ((\alpha \dot{-} \beta) \Rightarrow \gamma))$ is a theorem. Thus by modus ponens we have that the formula $((\alpha \dot{-} \beta) \Rightarrow \gamma)$ is a theorem in $\mathfrak{T}$, i.e. that $\|(\alpha \dot{-} \beta)\| \leqslant \|\gamma\|$, which completes the proof of condition (6). Identity (7) and (8) follows directly from (A_{15}), (A_{16}), (A_{17}) and (A_{18}) and the definition of the operations $\neg$ and $\ulcorner$ in a semi-Boolean algebra.

The simple proof of the last part of 9.1 is omitted. ■

Let $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ be a non-degenerate semi-Boolean algebra, and, as usual, let F denote the set of all formulas. Every formula α in F uniquely determines a mapping $\alpha_{\mathfrak{A}}\colon A^V \to A$, where V is the set of all propositional variables. Every element $v = \{v_a\}_{a \in V}$ of the Cartesian product A^V, i.e. every mapping $v\colon V \to A$, is called a *valuation in* $\mathfrak{A}$. Sometimes we shall take as $\mathfrak{A}$ the algebra $\mathfrak{A}(\mathfrak{T})$ of a formalized theory $\mathfrak{T} = (\mathfrak{L}, \mathfrak{C}, \mathcal{A})$. Then the valuation

$$v^0 = \{\|a\|\}_{a \in V}$$

in $\mathfrak{A}(\mathfrak{T})$ is called the *canonical valuation for* $\mathfrak{T}$. In particular, for every formula,

$$\alpha_{\mathfrak{A}(\mathfrak{T})}(v^0) = \|\alpha\|,$$

where v^0 is a canonical valuation.

A valuation $v \in A^V$ is said to be a *model for a set* $\mathfrak{A}$ *of formulas* if

$$\alpha_{\mathfrak{A}}(v) = V \quad \text{for every formula } \alpha \text{ in } \mathfrak{A}.$$

If $\mathcal{A}$ consists of one formula only, then the valuation v is said to be a *model for a formula* α if $\alpha_{\mathfrak{A}}(v) = V$. A valuation v is said to be a model for a theory $\mathfrak{T} = (\mathfrak{L}, \mathfrak{C}, \mathcal{A})$ provided v is the model for the set $\mathcal{A}$ of axioms of $\mathfrak{T}$. Every model in the two-element Boolean algebra is said to be *semantic*. A formula α is said to be *valid in a semi-Boolean algebra* $\mathfrak{A}$ if every valuation $v \in A^V$ is a model for α, i.e. if the mapping $\alpha_{\mathfrak{A}}$ is identically equal to the unit V of $\mathfrak{A}$. We then write $\alpha_{\mathfrak{A}} = V$. A formula is said to be a H-B *logic propositional tautology* if it is valid in every semi-Boolean algebra. We shall write that a formula α is a tautology instead of writing that it is a H-B logic propositional tautology if this does not lead to confusion.

Since a pseudo-Boolean algebra is a semi-Boolean algebra, the last definition immediately implies that

9.2. *Every intuitionistic propositional tautology* (³) *is a tautology.* ■

The next theorem is an immediate consequence of the above definitions and of some properties of semi-Boolean algebras (see 1.2 and 1.3).

9.3. *Every derivable formula is a tautology.* ■

The theorem converse to 9.3 is also true and it will be proved in § 10. Now we prove the following theorem

9.4. *The* H-B *propositional calculus is consistent if and only if the formulas α and $\ulcorner\alpha$ are not both tautologies for any formula α.*

(³) A formula α is said to be an *intuitionistic propositional tautology* if it is valid in every pseudo-Boolean algebra.

First we observe that the formulas of the form

(9) $$\big((\alpha \cap \neg\alpha) \Rightarrow \beta\big),$$

(10) $$(\neg\alpha \Rightarrow \ulcorner\alpha)$$

are tautologies in $\mathcal{S}$. Indeed, by 9.2 a formula of form (9) is a tautology in $\mathcal{S}$. Let v be a valuation in a semi-Boolean algebra $\mathfrak{A}$. Using 1.3 (26) we have $(\neg\alpha \Rightarrow \ulcorner\alpha)_{\mathfrak{A}}(v) = \neg\alpha_{\mathfrak{A}}(v) \Rightarrow \ulcorner\alpha_{\mathfrak{A}}(v) = V$, i.e. that the a formula of form (10) is a tautology in $\mathcal{S}$.

Now, we suppose that $\mathcal{S}$ is consistent, i.e. that there exists a formula which is not a tautology in $\mathcal{S}$. Let the formulas α and $\ulcorner\alpha$ be tautologies in $\mathcal{S}$. By the rule (r) $\neg\ulcorner\alpha$ and $\neg\ulcorner\ulcorner\alpha$ are tautologies in $\mathcal{S}$. Replacing $\ulcorner\alpha$ by α in (10) we infer that a formula of the form $(\neg\ulcorner\alpha \Rightarrow \ulcorner\ulcorner\alpha)$ is a tautology. By (A_{11}) the formula $\big((\neg\ulcorner\alpha \Rightarrow \ulcorner\ulcorner\alpha) \Rightarrow (\neg\ulcorner\ulcorner\alpha \Rightarrow \neg\neg\ulcorner\alpha)\big)$ is a tautology in $\mathcal{S}$. Thus, by applying modus ponens twice we infer that the formula $\neg\neg\ulcorner\alpha$ is a tautology. Replacing $\neg\ulcorner\alpha$ by α in (9) and by modus ponens and (A_9), we find that every formula of the form $(\neg\ulcorner\alpha \Rightarrow (\neg\neg\ulcorner\alpha \Rightarrow \beta))$ is a tautology in $\mathcal{S}$. Thus we find that every formula β in $\mathfrak{L}$ is a tautology in $\mathcal{S}$ in contradiction to our assumption. On the other hand, the proof is obvious. ■

On account of this theorem we infer that for no formula α are the formulas α and $\neg\alpha$ both tautologies in $\mathcal{S}$. Indeed, we suppose that exists a formula α, such that α and $\neg\alpha$ are tautologies in $\mathcal{S}$. By (10) we infer that $\ulcorner\alpha$ is a tautology and this contradicts 9.4.

§ 10. The completeness theorem. Theorem 9.3 states that every derivable formula is a tautology. The converse statement is also true, namely

10.1. *A formula α in $\mathfrak{L}$ is a tautology if and only if it is derivable in the* H-B *logic.*

This theorem is called the *completeness theorem for* H-B *propositional calculi* and it is part of the following theorem:

10.2. *For every formula α in $\mathfrak{L}$, the following conditions are equivalent:*

(i) *α is derivable in $\mathcal{S}$,*

(ii) *α is a tautology,*

(iii) *α is valid in every semi-Boolean algebra of all $\mathfrak{J}$-open sets of a bi-topological field of subsets of a bi-topological space $\langle X, \mathfrak{J}, C\rangle$,*

(iv) *$\alpha_{\mathfrak{A}(\mathcal{S})}(v^0) = V$ where v^0 is a canonical valuation for $\mathcal{S}$,*

(v) *α is valid in every finite semi-Boolean algebra,*

(vi) *α is valid in every semi-Boolean algebra with at most 2^{2^r} elements, where r is the number of all subformulas of the formula α.*

Condition (i) implies (ii) by 9.3. Clearly (ii) implies (iii) because every semi-Boolean algebra is isomorphic with a semi-field of subsets of a bi-topological space $\langle X, \mathfrak{J}, C\rangle$ (see 5.1). In particular the algebra $\mathfrak{A}(S)$ is isomorphic with a semi-field of sets. Thus the condition (iii) implies (iv).

Now, we shall prove that the condition (iv) implies (i). Suppose that a formula α is not derivable in $\mathcal{S}$. Hence, by § 9 the element $\|\alpha\|$ of $\mathfrak{A}(\mathcal{S})$ is not the unit element, i.e. $\alpha_{\mathfrak{A}(\mathcal{S})}(v^0) \neq V$. We have proved that the conditions (i)-(iv) are equivalent. Clearly (ii) implies (v) and (v) implies (vi). To complete the proof of 10.2 it suffices to show that (vi) implies (iv). Suppose that (iv) does not hold for a formula α, i.e. that $\alpha_{\mathfrak{A}(\mathcal{S})}(v^0) = \|\alpha\| \neq V$. Let the formula α contain r subformulas and let A_0 be the set consisting of all $\|\beta\|$ such that β is a subformula of α. It follows from 8.2 that there exists a finite semi-Boolean algebra $\mathfrak{A} = (A, \cup, \cap, \Rightarrow, \dot{-})$ containing at most 2^{2^r} elements and such that $A_0 \subset A$ and the operations in $\mathfrak{A}$ are extensions of operations in $A_0' = A_0 \cup (\Lambda, V)$.

Let v be a valuation of $\mathcal{S}$ in $\mathfrak{A}$ defined as follows:

$$v_a = \begin{cases} V & \text{for any propositional variable which does not occur in } \alpha, \\ v_a^0 = \|a\| & \text{for any propositional variable } a \text{ in } \alpha. \end{cases}$$

We can prove by induction on the length of subformula β of the formula α that $\beta_{\mathfrak{A}}(v) = \|\beta\| \in A_0'$. The simple proof is omitted. In particular we infer that $\alpha_{\mathfrak{A}}(v) = \|\alpha\| \neq V$, i.e. that condition (vi) does not hold for α. Thus condition (vi) implies (iv). In consequence, condition (i)-(vi) are equivalent. ■

10.3. *For any formulas α, β, γ the following formulas are tautologies:*

(1) $(\ulcorner\alpha \cup \alpha)$,

(2) $\big(\ulcorner(\alpha \cap \ulcorner\alpha)\big)$,

(3) $(\neg\ulcorner\alpha \Rightarrow \ulcorner\neg\alpha)$,

(4) $(\ulcorner\ulcorner\alpha \Rightarrow \neg\neg\alpha)$,

(5) $\big((\alpha \dot{-} \beta) \Rightarrow (\alpha \cap \ulcorner\beta)\big)$,

(6) $(\ulcorner\ulcorner\ulcorner\alpha \Rightarrow \ulcorner\alpha)$,

(7) $(\ulcorner\alpha \Rightarrow \ulcorner\ulcorner\ulcorner\alpha)$,

(8) $\Big(\big((\alpha - \gamma) \dot{-} (\alpha \dot{-} \beta)\big) \Rightarrow \ulcorner(\beta \Rightarrow \gamma)\Big)$,

(9) $\big((\alpha \dot{-} \beta) \Rightarrow (\neg\ulcorner\beta \Rightarrow \alpha)\big)$,

(10) $\big((\neg\alpha \dot{-} \beta) \Rightarrow (\alpha \Rightarrow \ulcorner\beta)\big)$,

(11) $\big((\alpha \dot{-} \beta) \Rightarrow (\neg\alpha \Rightarrow \ulcorner\beta)\big)$,

(12) $\big(\ulcorner(\alpha \Rightarrow \beta) \Rightarrow (\ulcorner\alpha \Rightarrow \ulcorner\beta)\big)$,

(13) $\big((\neg\alpha \dot{-} \neg\beta) \Rightarrow \neg(\alpha \dot{-} \beta)\big)$,

(14) $\Big(\big(\neg\ulcorner\alpha \cup \neg\ulcorner\beta\big) \Rightarrow \big(\neg\ulcorner(\alpha \cup \beta)\big)\Big)$,

(15) $\Big(\big(\neg\ulcorner\alpha \cap \neg\ulcorner\beta\big) \Rightarrow \big(\neg\ulcorner(\alpha \cap \beta)\big)\Big)$,

(16) $\big(\neg\ulcorner(\alpha \cap \beta) \Rightarrow (\neg\ulcorner\alpha \cap \neg\ulcorner\beta)\big)$,

(17) $\big((\ulcorner\alpha \cup \ulcorner\beta) \Rightarrow \ulcorner(\alpha \cap \beta)\big)$,

(18) $(\ulcorner(\alpha\cap\beta)\Rightarrow(\ulcorner\alpha\cup\ulcorner\beta))$,

(19) $(\ulcorner(\alpha\cup\beta)\Rightarrow(\ulcorner\alpha\cap\ulcorner\beta))$,

(20) $(\neg\ulcorner(\alpha\Rightarrow\beta)\Rightarrow(\neg\ulcorner\alpha\Rightarrow\neg\ulcorner\beta))$,

(21) $((\alpha\dot{-}\beta)\Rightarrow(\ulcorner\alpha\Rightarrow\ulcorner\beta))$,

(22) $((\gamma\dot{-}(\alpha\cup\beta))\Rightarrow((\gamma\dot{-}\beta)\dot{-}\alpha))$,

(23) $((\alpha\dot{-}\beta)\Rightarrow(\beta\Rightarrow\alpha))$.

This theorem follows from 1.2, 1.3 and from the definition of tautology in $\mathcal{S}$. ■

10.4. *The following formulas are not tautologies*

(24) $(a\Rightarrow\ulcorner\ulcorner a)$,

(25) $(a\Rightarrow\neg\ulcorner a)$,

(26) $((\ulcorner b\cap a)\Rightarrow(a\dot{-}b))$,

(27) $((a\Rightarrow b)\Rightarrow(b\dot{-}a))$,

(28) $((\ulcorner b\Rightarrow\ulcorner a)\Rightarrow(a\Rightarrow b))$,

(29) $((\ulcorner a\Rightarrow b)\Rightarrow(\ulcorner b\Rightarrow a))$,

(30) $((\ulcorner a\cap\ulcorner b)\Rightarrow\ulcorner(a\cup b))$,

(31) $((a\Rightarrow b)\Rightarrow(\ulcorner b\Rightarrow\ulcorner a))$,

where a, b are different propositional variables.

In order to prove that a formula from among those mentioned in 10.4 is not a tautology it suffices to find a semi-Boolean algebra $\mathfrak{A}$ and a valuation v in $\mathfrak{A}$ such that $\alpha_{\mathfrak{A}}(v)\neq V$. As an algebra $\mathfrak{A}$ we shall take here the $(X,\mathfrak{K})$-topological semi-Boolean algebra which was considered page 232. As the example we show that the formula α of the form (28) is not a tautology. Indeed the formula α contains two propositional variables a and b. Thus $\alpha_{\mathfrak{A}}(v)$ depends only on the values v_a and v_b. Let $v_a=A^1$ and $v_b=X-A^1$. On account of the definition of the operation $\Rightarrow$ and $\dot{-}$ in an $(X,\mathfrak{K})$-topological semi-Boolean algebra we have

$$\begin{aligned}((\ulcorner b\Rightarrow\ulcorner a)\Rightarrow(a\Rightarrow b))_{\mathfrak{A}}(v) &= \mathfrak{J}\big(-\mathfrak{J}\big(-C(-v_b)\cup C(-v_a)\big)\cup\mathfrak{J}(-v_a\cup v_b)\big)\\ &= \mathfrak{J}\big(-\mathfrak{J}\big(C(A^1)\cup C(-A^1)\big)\cup\mathfrak{J}(-A^1\cup -A^1)\big)\\ &= \mathfrak{J}\big(-\mathfrak{J}(-A^1\cup X)\cup\mathfrak{J}(-A^1)\big)\\ &= \mathfrak{J}(-\mathfrak{J}X\cup\emptyset)=\mathfrak{J}(\emptyset)=\emptyset,\end{aligned}$$

i.e.

$$((\ulcorner b\Rightarrow\ulcorner a)\Rightarrow(a\Rightarrow b))_{\mathfrak{A}}(v)\neq V. \ ■$$

The following theorem explains the connection between tautologies and intuitionistic propositional tautologies.

10.5. *If formula α does not contain the connectives explication and Brouwerian negation, then α is a tautology if and only if it is an intuitionistic propositional tautology.*

Suppose that α does not contain the connections $\dot{-}$ and $\ulcorner$ and it is not an intuitionistic propositional tautology. On account of the intuitionistic analogue of Theorem 10.2 [2] there exists a finite pseudo-Boolean algebra $\mathfrak{A}=(A,\cup,\cap,\Rightarrow,\neg)$ and a valuation v such that $\alpha_{\mathfrak{A}}(v)\neq V$. By 8.4 the algebra $\mathfrak{A}$ can be extended to a finite semi-Boolean algebra $\mathfrak{A}=(A,\cup,\cap,\Rightarrow,\dot{-})$. Thus we can interpret the valuation v as a valuation in $\mathfrak{A}$. Hence $\alpha_{\mathfrak{A}}(v)\neq V$, i.e. α is not valid in the finite semi-Boolean algebra. By 10.2 we infer that α is not a tautology. Thus the necessity is proved. The sufficiency follows from the theory 9.1. ■

§ 11. Consistency and the existence of models. This section contains theorems concerning connections between the consistency of a theory $\mathfrak{T}=(\mathfrak{L},\mathcal{C},\mathcal{A})$ and the existence of models in semi-Boolean algebras.

11.1. *If α is a theorem of a theory $\mathfrak{T}=(\mathfrak{L},\mathcal{C},\mathcal{A})$ then every model for $\mathfrak{T}$ in any semi-Boolean algebra $\mathfrak{A}$ is a model for α.*

The simple proof of this theorem is omitted. ■

A model $v\in A^V$ of a theory $\mathfrak{T}=(\mathfrak{L},\mathcal{C},\mathcal{A})$ is said to be *adequate for* $\mathfrak{T}$ provided, for every formula α in $\mathfrak{L}$, α is a theorem in $\mathfrak{T}$ if and only if v is a model for α.

11.2. *For any theory $\mathfrak{T}=(\mathfrak{L},\mathcal{C},\mathcal{A})$ the following conditions are equivalent:*

(i) *$\mathfrak{T}$ is consistent,*

(ii) *there exists a model for $\mathfrak{T}$,*

(iii) *there exists an adequate model for $\mathfrak{T}$,*

(iv) *there exists an adequate model for $\mathfrak{T}$ in a semi-Boolean algebra of all $\mathfrak{J}$-open sets of a bi-topological field of subsets of a bi-topological space $\langle X,\mathfrak{J},C\rangle$,*

(v) *there exists a semantic model for $\mathfrak{T}$.*

Condition (i) implies (iii). In fact, it $\mathfrak{T}$ is consistent then by 9.1 the algebra $\mathfrak{A}(\mathfrak{T})$ is a non-degenerate semi-Boolean algebra. Let v^0 be the canonical valuation, i.e. let $v_a=\|a\|\in\mathfrak{A}(\mathfrak{T})$ for $a\in V$. Thus $\alpha_{\mathfrak{A}(\mathfrak{T})}(v^0)=\|\alpha\|$ for any α in F. By 9.1 we infer that $\alpha_{\mathfrak{A}(\mathfrak{T})}(v^0)=V$ if and only if $\alpha\in\mathcal{C}(\mathcal{A})$. This proves that v^0 is an adequate model for $\mathfrak{T}$.

Clearly (iii) implies (ii). We prove that (ii) implies (i). Let v be a model for $\mathfrak{T}$ in a semi-Boolean algebra $\mathfrak{A}$. If the formulas α and $\ulcorner\alpha$ are both theorems in $\mathfrak{T}$, then by 11.1 $\alpha_{\mathfrak{A}}(v)=V$ and $(\ulcorner\alpha)_{\mathfrak{A}}(v)=V$. Hence $V=\Lambda$ which does not hold in any non-degenerate semi-Boolean algebra. Thus (i)-(iii) are equivalent. It follows from 5.1 that (iii) is equivalent to (iv). We shall prove that (i) implies (v). Suppose that $\mathfrak{T}$ is consistent. By 9.1 the semi-Boolean algebra $\mathfrak{A}(\mathfrak{T})$ is non-degenerate. It is easy to see that there exists a maximal $\neg\ulcorner$-filter ∇ in $\mathfrak{A}(\mathfrak{T})$. Let h be the natural homomorphism from $\mathfrak{A}(\mathfrak{T})$ onto the two-element Boolean algebra $\mathfrak{A}_0=\mathfrak{A}(\mathfrak{T})/\nabla$. The valuation $v=hv^0$, where v^0 is the canonical valuation in $\mathfrak{A}(\mathfrak{T})$ is a semantic model. In fact, for every formula $\alpha_{\mathfrak{A}_0}(v) = h(\alpha_{\mathfrak{A}(\mathfrak{T})}(v^0)) = h(\|\alpha\|)$. Hence $\alpha_{\mathfrak{A}_0}(v) = h(\|\alpha\|) = h(V) = V$, i.e. v is a semantic model for $\mathfrak{T}$. This proves that (i) implies (v). Clearly (v) implies (ii), which completes the proof of 11.2. ■

Now we formulate conditions which are necessary and sufficient for any formula to be a theorem of a theory $\mathfrak{T}$.

11.3. *For any formula α in a consistent theory $\mathfrak{T}$ of zero order the following conditions are equivalent:*

(i) *α is a theorem of $\mathfrak{T}$,*

(ii) *every model for $\mathfrak{T}$ is a model for α,*

(iii) *every model for $\mathfrak{T}$ is a semi-Boolean algebra of all $\mathfrak{I}$-open sets of a bi-topological field of subsets of a bi-topological space $\langle X, \mathfrak{I}, C\rangle$ is a model for α,*

(iv) *$\alpha_{\mathfrak{A}(\mathfrak{T})}(v^0) = V$ for the canonical valuation v^0.*

The proof of 11.3 is by easy verification. ■

§ 12. Deduction theorems.

12.1. *A formula β is a theorem in a theory $\mathfrak{T}' = (\mathfrak{L}, \mathfrak{C}, \mathcal{A} \cup [\alpha])$ if and only if there exists a positive integer n such that formula $((T_n \alpha) \Rightarrow \beta)$ is a theorem in the theory $\mathfrak{T} = (\mathfrak{L}, \mathfrak{C}, \mathcal{A})$.*

If there exists a positive integer n such that the formula $((T_n \alpha) \Rightarrow \beta)$ is a theorem in $\mathfrak{T}$, then it is also a theorem in $\mathfrak{T}'$. The formula α is an axiom in $\mathfrak{T}'$. Using the rule (r) n-times we find that the formula $(T_n \alpha)$ is a theorem in theory $\mathfrak{T}'$. By modus ponens formula β is a theorem in $\mathfrak{T}'$.

To prove the remaining part of 12.1 we suppose that for every positive integer n the formula $((T_n \alpha) \Rightarrow \beta)$ is not a theorem in $\mathfrak{T}$. By 9.1 we infer that the inequality $T_n\|\alpha\| \Rightarrow \|\beta\| \neq V$ is satisfied in a semi-Boolean algebra $\mathfrak{A}(\mathfrak{T})$. Let ∇ be a $\neg\Gamma$-filter generated by $\|\alpha\|$, i.e. let ∇ be the set of all elements $\|\gamma\|$ in $\mathfrak{A}(\mathfrak{T})$ for which there exists a positive integer n_1 such that $T_{n_1}\|\alpha\| \leqslant \|\gamma\|$. The filter ∇ is proper since $\|\beta\| \notin \nabla$. Indeed, the hypothesis $\|\beta\| \in \nabla$ implies that there exists a positive integer n_1 such that $T_{n_1}\|\alpha\| \leqslant \|\beta\|$, and this proves that the formula $((T_{n_1}\alpha) \Rightarrow \beta)$ is a theorem in $\mathfrak{T}$, in contradiction to our assumption. Thus the quotient algebra $\mathfrak{A}_0 = \mathfrak{A}(\mathfrak{T})/\nabla$, is a semi-Boolean algebra and the mapping h:

$$h(\|\delta\|) = |\|\delta\|| \qquad (\|\delta\| \in \mathfrak{A}(\mathfrak{T}))$$

is an isomorphism from $\mathfrak{A}(\mathfrak{T})$ onto $\mathfrak{A}_0$. Let v^0 be the canonical valuation in $\mathfrak{A}(\mathfrak{T})$. We find that for every formula the identity $\gamma_{\mathfrak{A}_0}(h(v^0)) = h(\gamma_{\mathfrak{A}(\mathfrak{T})}(v^0))$ is satisfied.

Now, if γ is a formula in $\mathcal{A}$, then $\gamma_{\mathfrak{A}(\mathfrak{T})}(v^0) = V$ and consequently $\gamma_{\mathfrak{A}_0}(hv^0) = h(V) = V$, where the last sign V is the unit element in $\mathfrak{A}(\mathfrak{T})$. If γ is the formula α, then the element $\|\alpha\|$ belongs to the $\neg\Gamma$-filter ∇

and $\gamma_{\mathfrak{A}_0}(hv^0) = \alpha_{\mathfrak{A}_0}(hv^0) = h(V) = V$, where the last sign V is the unit element in $\mathfrak{A}_0$. This proves that the valuation hv^0 in $\mathfrak{A}_0$ is a model for the theory $\mathfrak{T}'$ but not for β. Indeed $\beta_{\mathfrak{A}_0}(hv^0) = h(\beta_{\mathfrak{A}(\mathfrak{T})}(v^0)) = h(\|\beta\|) = V$. On account of 11.1 formula β is not a theorem in $\mathfrak{T}'$. ■

12.2. *A formula β is a theorem in a theory $\mathfrak{T} = (\mathfrak{L}, \mathfrak{C}, \mathcal{A})$ (where $\mathcal{A}$ is a non-empty set of formulas) if and only if there exist positive integers $n_1, \ldots, n_m$ such that the implication $(\bigcap_{i=1}^{m} (T_{n_i}\alpha_i) \Rightarrow \beta)$ where α_i for $i = 1, \ldots, m$ are axioms in $\mathcal{A}$, is a tautology.*

The simple proof of this theorem is omitted. ■

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Reçu par la Rédaction le 21. 8. 1972