Normality in function spaces

by

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Abstract. In this note we characterize metrizable spaces for which the function space I^X is normal, Lindelöf, or has the k-property. We give also some related examples.

This paper concerns the normality, the Lindelöf property and the k-property of the space I^X of continuous mappings of a metrizable space X into a segment I. In part one we shall formulate two theorems characterizing metrizable spaces X for which I^X is a normal space or a Lindelöf space (Theorem 1) and spaces for which I^X is a k-space (Theorem 2). Part two is devoted to the proof of Theorem 1, part three contains a certain generalization of that theorem, and part four gives examples related to the above facts. In part five we shall prove Theorem 2.

We adopt the terminology and notation of [3] and [4]. In particular, the word "mapping" and the symbol $f\colon X \to Y$ always denote a continuous function from X to Y. For topological spaces X and Y the symbol Y^X will be used to denote the space of continuous mappings of space X into Y considered with a compact-open topology. If Y is a metrizable space, the base of the space Y^X is formed by the sets

$$M(f, Z, \varepsilon) = \{f' \in Y^X|\ \varrho(f(z), f'(z)) < \varepsilon \text{ for } z \in Z\}$$

where $f \in Y^X$, $Z \subset X$ is compact, $\varepsilon > 0$, and ϱ is a fixed metric on Y ([3], T. 8.2.3). The symbol σY^X will denote the set $\{f | f\colon X \to Y\}$ with a topology of pointwise convergence. The symbol $Y^{|X|}$ will denote the product of $\overline{\overline{X}}$ copies of the space Y indexed by the set X; the mapping $f \in Y^X$ will also be regarded as an element of $Y^{|X|}$. By $D(\mathfrak{m})$ we shall denote a discrete space of power $\mathfrak{m}$, by N — natural numbers, by Q — rational numbers, by I — the segment $[0, 1]$, by T — the unit complex circle, by R — real numbers.

1. Theorem 1. *For a metrizable space X the following statements are equivalent*:

(i) I^X *is normal*,

(i)′ σI^X *is normal*,

(ii) X^d *is separable*,

(iii) *for any compact metrizable space K, the space K^X is Lindelöf.*

As can be seen from the examples in part four, the assumption of metrizability in the above theorem is essential.

THEOREM 2. *For a paracompact space X satisfying the first axiom of countability the following statements are equivalent*:

(i) I^X *is a k-space,*

(ii) $X = D(\mathfrak{m}) \oplus X_0$, *where X_0 is a locally compact Lindelöf space,*

(iii) *for any compact metrizable space K, the space K^X is paracompact and complete in the sense of Čech.*

The example quoted in part five shows that the assumptions on the space X are essential.

Remark. Theorem 2 does not hold for σI^X. Consider the rational numbers Q: the space σI^Q is metrizable but Q does not satisfy (ii) and σI^Q is not complete.

2. We begin the proof of Theorem 1 with a simple lemma.

LEMMA 1. *If the derivative of a paracompact space X satisfying the first axiom of countability is not Lindelöf, then I^X and σI^X contain the space $N^{\aleph_1}$ as a closed set.*

Proof. We shall restrict ourselves to the space I^X; the proof for σI^X is identical. Suppose that X^d is not Lindelöf. Then there exists a family, discrete in the space X, of closed sets $\{F_s|\ s \in S\}$ such that $\overline{\overline{S}} = \aleph_1$ and $\operatorname{Int} F_s \cap X^d \ni x_s$. Let

$$A_s = \{f \in I^X|\ f|(X \setminus F_s) = 0\} \quad \text{and} \quad A = \{f \in I^X|\ f|(X \setminus \bigcup_{s \in S} F_s) = 0\}.$$

Then $A = \mathbf{P}_{s \in S} A_s$, and since A is a closed subset of I^X, it is sufficient to prove that every A_s contains a discrete countable closed subset (cf. [3], Ex. 3.3.E). Let us fix s and let $x_n \in \operatorname{Int} F_s$, $x_n \neq x_m \neq x_s$ for $m \neq n$ and $x_n \to x_s$. Let us choose a function $f_n \in A_s$ such that

$$f_n(x_m) = \begin{cases} 0, & m > n, \\ 1, & m \leqslant n, \end{cases} \qquad f_n(x_s) = 0 .$$

The set $\{f_n|\ n = 1, 2, \ldots\}$ is discrete and closed in the space A_s.

The implications (i), (i)$' \Rightarrow$ (ii) follow from Lemma 1 and from the fact that $N^{\aleph_1}$ is not normal ([3] P. 2. T).

For proving Theorem 1 it is sufficient to prove the implication (ii) $\Rightarrow$ (iii) (the remaining ones follow *a fortiori*); this implication results from Proposition 1 given below.

To begin with, define a class $\mathfrak{A}$ of all regular topological spaces X such that there exists a *COSMIC* space E ([7], Def. 10.1) and upper semi-continuous set-valued function Φ ([4], § 18) from E into the family of compact, non-empty subsets of the space X and

$$\bigcup \{\Phi(x)|\ x \in E\} = X . \tag{1}$$

This class has the following properties:

(2) $\mathfrak{A}$ is $\aleph_0$-multiplicative, hereditary with respect to closed subspaces and closed with respect to the continuous images,

(3) *COSMIC* spaces belong to $\mathfrak{A}$,

(4) if $X \in \mathfrak{A}$, then X is Lindelöf.

PROPOSITION 1. *Let X be a metrizable space with a separable derivative. Then for any compact metrizable space K the space K^X and σK^X belong to $\mathfrak{A}$. Thus a countable product of the space K^X or σK^X and a product of this space by any separable, metrizable space is a Lindelöf space.*

Let X be a metrizable space with a separable derivative. From (2) and (4) it follows that it is sufficient to show $K^X \in \mathfrak{A}$. This fact, as will be shown later, can easily be reduced to the following lemma.

LEMMA 2. *Let $I_0(X) = \{f \in I^X|\ f|X^d = 0\}$. Then $I_0(X) \in \mathfrak{A}$.*

Proof. Adopt the notation $X^d = X_0$, $I_k = [0, 1/k]$, $k = 1, 2, \ldots$ Establish a metric ϱ on the space X for which the diameter of X is less than 1, and a retraction $r\colon X \xrightarrow[\text{onto}]{} X_0$ ([5], T. O.). For $\varphi \in I^{X_0}$ such that $\varphi(x) > 0$ for $x \in X_0$ define a set

$$W(\varphi) = \{x \in X|\ \varrho(x, r(x)) < \varphi(r(x))\} . \tag{5}$$

Then set $W(\varphi)$ is a open-and-closed neighbourhood of X_0, and if $\varphi_1 \geqslant \varphi_2 > 0$, then $W(\varphi_1) \supset W(\varphi_2)$. As E let us take a subspace of the product $I_1^{X_0} \times I_2^{X_0} \ldots$ consisting of sequences $a = (a_i)$ such that $a_1 = 1 \geqslant a_2 \geqslant \ldots > 0$. For $a \in E$ the following sequence of open-and-closed sets is defined:

$$W(a_1) = X \supset W(a_2) \supset \ldots \supset X_0 . \tag{6}$$

Assume $X_i(a) = W(a_i) \setminus W(a_{i+1})$, for $i = 1, 2, \ldots$ Then

$$\bigcup_{i=1}^{\infty} X_i(a) = X \setminus X_0, \quad X_i(a) \cap X_j(a) = \emptyset, \quad \text{for} \quad i \neq j \tag{7}$$

and it follows that we can define the function $F(a)$ as

$$F(a)(x) = \begin{cases} 1/k & \text{for} \quad x \in X_k(a) \\ 0 & \text{for} \quad x \in X_0 . \end{cases} \tag{8}$$

Assume:

$$\Phi(a) = \{u \in I^{|X|}|\ u(x) \leqslant F(a)(x), \text{ for } x \in X\} . \tag{9}$$

The openness of $W(a_i)$ and the definition of $F(a)$ imply that

$$\Phi(a) \subset I_0(X) . \tag{10}$$

We shall prove that

(11) $\bigcup \{\Phi(a) |\ a \in E\} = I_0(X)$,

(12) $\Phi(a)$ is compact for $a \in E$,

(13) Φ is upper semi-continuous.

Let $f \in I_0(X)$ and assume that $L_i = f^{-1}(I_i)$, $i = 1, 2, \ldots$ We shall define by induction a sequence $a_1, a_2, \ldots$ such that $a = (a_i) \in E$ and $W(a_i) \subset L_i$. For $a_1 = 1$ we have $W(a_1) = X = L_1$. Assume that $a_1, \ldots, a_k$ have already been defined and let $\beta(x) = \varrho(x, X \setminus L_{k+1})$, for $x \in X_0$ and $a_{k+1} = (1/k+1)\min(\beta, a_k)$. Then $a_{k+1} \in I_{k+1}^{X_0}$, $a_k \geqslant a_{k+1} > 0$, and $W(a_{k+1}) \subset W(\beta)$. It is sufficient to show that $W(\beta) \subset L_{k+1}$. Take $x \in W(\beta)$; then $\varrho(x, r(x)) < \varrho(r(x), X \setminus L_{k+1})$, i.e. $x \in L_{k+1}$. Now for the a defined above and for $x \in X_k(a)$ we have $x \in W_k(a) \subset L_k$; hence $f(x) \leqslant 1/k = F(a)(x)$, and thus $f \in \Phi(a)$, which concludes the proof of (11).

Let us now fix $a = (a_i) \in E$ and, for every $a = (a_i) \in I_1^{X_1(a)} \times I_2^{X_2(a)} \times \ldots = A$, put

$$D(a)(x) = \begin{cases} 0, & x \in X_0, \\ a_i(x), & x \in X_i(a). \end{cases}$$

We shall show that D is a continuous mapping of the compact space A onto $\Phi(a)$, which proves (12). Let $f = D(a)$, let $M(f, Z, \varepsilon)$ be a neighbourhood of f and let $2/k < \varepsilon$. Assume that $Z_i = Z \cap X_i(a)$,

$$U = M(a_1, Z_1, \varepsilon) \times \ldots \times M(a_{k-1}, Z_{k-1}, \varepsilon) \times I_k^{X_k(a)} \times \ldots$$

Let $a' = (a_i') \in U$ and $z \in Z$. If $z \in W(a_k)$, then

$$|D(a')(z) - f(z)| \leqslant 2/k < \varepsilon ,$$

and if $z \in X_i(a)$ for $i < k$, then

$$|D(a')(z) - f(z)| = |a_i'(z) - a_i(z)| < \varepsilon .$$

Thus $D(U) \subset M(f, Z, \varepsilon)$.

We shall now prove (13). Let $U \subset I^X$ be an open set and, for a certain $a = (a_i) \in E$, let $\Phi(a) \subset U$ hold. Using (12), let us choose $f_1, \ldots, f_p \in \Phi(a)$ and their neighbourhoods $M(f_i, Z_i, \varepsilon_i)$, $i = 1, \ldots, p$ so as to have

$$\Phi(a) \subset \bigcup_{i=1}^{p} M(f_i, Z_i, \varepsilon_i) \quad \text{and} \quad M(f_i, Z_i, 2\varepsilon_i) \subset U, \quad i = 1, \ldots, p. \tag{14}$$

Let $Z = \bigcup_{i=1}^{p} Z_i$, $\varepsilon = \min\{\varepsilon_i |\ i \leqslant p\}$. The set $C_k = Z \cap W(a_k)$ is compact and for $z \in C_k$ we have $a_k(r(z)) - \varrho(z, r(z)) > 0$; hence

(15) there exists a $\delta_k > 0$ such that $a_k(r(z)) - \varrho(z, r(z)) > \delta_k$ for $z \in C_k$.

Let $2/k_0 < \varepsilon$, $\delta = \min\{\delta_i |\ i \leqslant k_0\}$, $Z' = r(Z)$. Let us take a neighbourhood of a of the form

$$V = \left(M(a_1, Z', \delta) \times \ldots \times M(a_{k_0}, Z', \delta) \times I_{k_0+1}^{X_0} \times \ldots\right) \cap E . \tag{16}$$

We shall first show that

(17) if $a' \in V$, $z \in Z$, and $F(a')(z) > F(a)(z)$, then $F(a')(z) - F(a)(z) < \varepsilon$.

Let $a' \in V$ and $i \leqslant k_0$; then $W(a_i) \cap Z \subset W(a_i')$. Indeed, if $z \in W(a_i) \cap Z = C_i$, we have, by (15), $a_i(r(z)) - \delta > \varrho(z, r(z))$, and since $r(z) \in Z'$, we have, by (16), $a_i(r(z)) - \delta < a_i'(r(z))$, i.e. $a_i'(r(z)) > \varrho(z, r(z))$ and thus $z \in W(a_i')$. Now let $z \in Z$. If $z \in W(a_{k_0})$, then $z \in W(a_{k_0}')$, and thus

$$|F(a)(z) - F(a')(z)| \leqslant F(a)(z) + F(a')(z) \leqslant 2/k_0 < \varepsilon ;$$

on the other hand, if $z \notin W(a_{k_0})$, then $z \in X_k(a)$ for $k < k_0$, and thus $z \in W(a_k) \cap Z \subset W(a_k')$ whence $F(a')(z) \leqslant 1/k = F(a)(z)$.

We shall now prove that for $a' \in V$ we have $\Phi(a') \subset U$, which will complete the proof of (13). Let $f' \in \Phi(a')$, $f'' = \min\{f', F(a)\}$. Thus $f'' \in \Phi(a)$, i.e. $f'' \in M(f_{i_0}, Z_{i_0}, \varepsilon_{i_0})$ for $i_0 \leqslant p$ (from (14)). Let $z \in Z$. If $f'(z) > F(a)(z)$, then $F(a')(z) \geqslant f'(z) > F(a)(z)$, whence

$$|f'(z) - f''(z)| = |f'(z) - F(a)(z)| \leqslant F(a')(z) - F(a)(z) < \varepsilon ,$$

by (17). On the other hand, if $f'(z) \leqslant F(a)(z)$, then $|f'(z) - f''(z)| = 0$. Thus we always have $|f'(z) - f''(z)| < \varepsilon$, whence, for $z \in Z_{i_0}$,

$$|f'(z) - f_{i_0}(z)| \leqslant |f'(z) - f''(z)| + |f''(z) - f_{i_0}(z)| < 2\varepsilon_{i_0} ,$$

which, by (14), gives $f' \in U$.

The lemma now follows from (11), (12), (13) and from the fact that the space E is *COSMIC* ([7], Proposition 10.3).

We shall now derive Proposition 1 from the lemma. Let $T_0(X) = \{f \in T^X |\ f|X_0 = 1\}$, and let $h\colon T \to I^2$ be a homeomorphism of T onto the perimeter of a square such that $h(1) = (0, 0)$. Then $\varphi(f) = h \circ f$ is a homeomorphic embedding of the space $T_0(X)$ onto the closed subset of $I_0(X) \times I_0(X)$, whence, by (2) and Lemma 2, we have $T_0(X) \in \mathfrak{A}$. For $(f_1, f_2) \in T_0(X) \times T^{X_0}$ put $\varphi'(f_1, f_2) = f_1 \cdot \dfrac{1}{f_2 \circ r}$; then $\varphi'\colon T_0(X) \times T^{X_0} \xrightarrow[\text{onto}]{} T^X$. By (2), (4) and the fact that T^{X_0} is *COSMIC*, we have $T^X \in \mathfrak{A}$. From the exponential law ([8], Theorem 2) we have $(T^{\aleph_0})^X = (T^X)^{\aleph_0} \in \mathfrak{A}$, and since K is embedded in $T^{\aleph_0}$ as a closed subspace, K^X is homeomorphic with a closed subspace of $(T^{\aleph_0})^X$; hence it is an element of $\mathfrak{A}$.

3. Let X, Y, Z be topological spaces and $p\colon X \xrightarrow[\text{onto}]{} Y$. For $f \in Z^Y$ let us assume $p^*(f) = f \circ p$. Then: p^* embeds σZ^Y homeomorphically in σZ^X; if p is a compact-covering mapping ([7], § 7), then p^* embeds Z^Y homeomorphically in Z^X; if p is a quotient mapping, then $p^*(Z^Y)$ is a closed set in σZ^X, and thus also in Z^X.

LEMMA 3. *A regular space with a point-countable base and a separable derivative is a compact-covering image of a metrizable space with a separable derivative.*

Proof. Let X be a regular space with a point-countable base $\mathfrak{B}$. Put $X^d = X_0$. We shall begin by constructing a pseudometric d continuous on X such that $d(x, y) < 1$ for $x, y \in X$ and

(18) if $d(x_n, x_0) \to 0$ and $x_0 \in X_0$, then $x_n \to x_0$.

We shall use the classical construction of Urysohn [10]. Let $\mathfrak{U} = \{(V, W) |\ V, W \in \mathfrak{B},\ \overline{V} \subset W,\ X_0 \cap V \neq \emptyset\}$. Then $\overline{\overline{\mathfrak{U}}} \leqslant \aleph_0$ and let $(V_1, W_1), (V_2, W_2), \ldots$ be a sequence of elements from $\mathfrak{U}$. Since X_0 satisfies the second axiom of countability, it can easily be seen that X is paracompact, whence there exist functions $f_i\colon X \to I$ such that $f_i|\overline{V}_i = 0$, $f_i|(X \setminus W_i) = 1$. The pseudometric $d(x, y) = \frac{1}{2}\sum_{i=1}^{\infty} 2^{-i}|f_i(x) - f_i(y)|$ satisfies (18). Let $\mathfrak{m} = \overline{\overline{X}}$ and let $J^*(\mathfrak{m})$ be a subspace of the hedgehog $J(\mathfrak{m})$ (the definition and the notations used in the sequel are taken from ([3], E. 4.1.3)) consisting of points $q_{s,n} = (s, 1/n)$ $n = 1, 2, \ldots$, $s \in S$, and the point $q_0 = (s, 0)$. Let ϱ^* be a standard metric on $J(\mathfrak{m})$. In the product $E = X_0 \times J^*(\mathfrak{m})$ we shall introduce a metric by a formula ([3], E. 4.1.4.):

$$\varrho\big((x, q), (x', q')\big) = \begin{cases} \varrho^*(q, q') & \text{for } x = x', \\ \varrho^*(q, q_0) + \varrho^*(q', q_0) + d(x, x') & \text{for } x \neq x'. \end{cases}$$

The derivative of E is homeomorphic with X_0, and thus it is separable. Let us introduce the following notation: for $A \subset X$ let

$$K(A, \varepsilon) = \{x \in X |\ d(x, A) < \varepsilon\};$$
$$A_n(x) = K(x, 1/n) \setminus K(x, 1/n+1) \setminus X_0,$$
$$B_n(x) = \{(x, q_{s,n}) |\ s \in S\}, \quad x \in X_0,\ n = 1, 2, \ldots$$

For $x \in X_0$ and $n \in N$ we shall choose a function $p_{x,n}$ in the following way:

$$p_{x,n}\colon B_n(x) \xrightarrow[\text{onto}]{} A_n(x) \quad \text{if} \quad A_n(x) \neq \emptyset$$

and

$$p_{x,n}\colon B_n(x) \to \{x\} \quad \text{if} \quad A_n(x) = \emptyset.$$

Define the function $p\colon E \xrightarrow[\text{onto}]{} X$ by the formula

$$p\big((x, q)\big) = \begin{cases} p_{x,n}(q) & \text{for} \quad q \in B_n(x), \\ x & \text{for} \quad q = q_0. \end{cases}$$

We shall verify that p is continuous and compact-covering. Continuity follows from the fact that for $u_0 = (x, q_0) \in E^d$ and $u = (x', q)$ we have $d\big(p(u_0), p(u)\big) \leqslant \varrho(u_0, u)$; hence if $u_n \in E$ and $u_n \to u_0$ then $d\big(p(u_0), p(u_n)\big) \to 0$ and by (18) also $p(u_n) \to p(u_0)$. Now let $Z \subset X$ be a compact set. Put $Z_0 = Z \cap X_0$, $Z_n = Z \cap \big(K(Z_0, 1/n) \setminus K(Z_0, 1/n+1)\big)$. The set Z_n is finite. For every $z \in Z_n$ choose $z' \in Z_0$ such that $z \in K(z', 1/n)$ and $\tilde{z} \in B_n(z')$ such that $p(\tilde{z}) = z$. Let $\tilde{Z} = \{\tilde{z} |\ z \in Z_n,\ n = 1, 2, \ldots\} \cup (Z_0 \times \{q_0\})$. The set $\tilde{Z} \setminus \{u \in E |\ \varrho(u, Z_0 \times \{q_0\}) < 1/n\}$ is finite and the set $Z_0 \times \{q_0\}$ is compact; hence $\tilde{Z}$ is compact and $p(\tilde{Z}) = Z$.

Lemma 3, Proposition 1, the fact that the compact-covering mapping onto a first countable Hausdorff space is quotient ([1], Lemma 11.2) and the initial remarks in this part imply

PROPOSITION 2. *If X is a regular space with a point-countable base and a separable derivative and K is a compact metrizable space, then K^X is a Lindelöf space (more precisely: $K^X \in \mathfrak{A}$).*

Remark. Proposition 2 and Lemma 1 imply that the assumptions of Theorem 1 can be weakened to paracompactness and the existence of a point-countable base.

4. We shall now give three examples related to our situations.

EXAMPLE 1. Let K be a segment $[0, 1)$ with the topology of the right-hand arrow ([3], E. 1.2.1). We shall show that the spaces I^K and σI^K are not normal.

For $x \in [0, 1]$ let us define a function $f_x \in I^K$ by the formula

$$f_x(t) = \begin{cases} 1 & \text{for} \quad t \geqslant x, \\ 0 & \text{for} \quad t < x. \end{cases}$$

It can easily be seen that the mapping $\varphi(x) = f_{(1-x)}$ is a homeomorphism of the space K onto the closed subspace of the space $I^K(\sigma I^K)$. Since $K \underset{\text{top}}{=} K \oplus K$, we have $I^K \underset{\text{top}}{=} I^K \times I^K$ $(\sigma I^K \underset{\text{top}}{=} \sigma I^K \times \sigma I^K)$, and so both these spaces contain, as a closed subspace, the space $K \times K$, which is not normal ([3], *E.* 2.3.2.).

EXAMPLE 2. ([2], Proposition 5). Let $\mathfrak{F}$ be a filter on a space $D(\aleph_1)$ consisting of sets with denumerable complements. Let $A(\mathfrak{F}) = D(\aleph_1) \cup$

$\cup\{\mathfrak{F}\}$ be a topological space connected with $\mathfrak{F}$. Then $T^{A(\mathfrak{F})} = \sigma T^{A(\mathfrak{F})}$ is homeomorphic with the Σ-product of $\aleph_1$-copies of the space T, and thus it is normal but not Lindelöf.

EXAMPLE 3. For $t \in R$ let

$$V(t, n) = \{x \in R\times R \mid |x-(t\pm 1/n, 0)| < 1/n\} \cup \{t, 0\}.$$

In $R\times R$, let us take, as the base of neighbourhoods of points $(t, 0)$ the sets $V(t, n)$, $n = 1, 2, \ldots$, and as the base of neighbourhoods at the remaining points, Euclidean balls. We obtain the well-known *COSMIC* space (cf. [6]). Let us denote by A the subspace $I\times I$ of that space. Let $B = ((Q\times Q)\cup I)\cap A$ be a subspace A, and let B_I ([3], E. 5.1.2) be a space in which the neighbourhoods of points belonging to I are such as in B and the points from $B\setminus I$ are isolated.

We shall prove that

(19) the spaces I^A and $I^{(B_I)}$ are not normal,

(20) the space σI^A is Lindelöf.

To begin with, let us observe that the spaces I^A and I^{B_I} are both separable, because A and B_I can be mapped in a one-to-one way into a plane ([11], Theorem 5). For the proof of (19) it is sufficient to show that both of them contain a closed discrete set of power $2^{\aleph_0}$ (see [3], E. 1.5.2).

Let $F(t) = A\setminus V(t, 1)$ for $t \in I$. Define a function $f_t \in I^A$ so as to have

$$f_t(x) = \begin{cases} 1 & \text{for} \quad x \in F(t), \\ 0 & \text{for} \quad x \in I. \end{cases}$$

Since $F(t)$ and I are closed and disjoint in the Lindelöf space A, such functions exist.

Let $r\colon I^A \to I^{(B_I)}$ be a restriction $r(f) = f|B$ to the set B. Since B is dense in A, the mapping r is one-to-one. Let $f'_t = r(f_t)$, $F = \{f'_t \mid t \in I\} \subset r(I^A)$. It suffices to show that F is closed and discrete in the space $I^{(B_I)}$. Let $f \in I^{(B_I)}$ and $f \in \overline{F}$. Then $f(I) = 0$, and thus $V = f^{-1}[0, \frac{1}{2})$ is a neighbourhood of I. There exist $t_1, \ldots, t_p \in I$ and $n_1, \ldots, n_p \in N$ such that $I \subset \big(\bigcup_{i=1}^{p} V(t_i, n_i)\big)\cap B \subset V$. For every $i \leqslant p$ choose a compact set $Z_i \subset B \cap V(t_i, n_i)$ such that if $t \in V(t_i, n_i)\cap I$ and $t \notin \{t_i, t_i - 2/n_i, t_i + 2/n_i\} = C_i$, then $F(t)\cap Z_i \neq \emptyset$ (we can take as Z_i the suitable sequences tending to the points of C_i).

Let $Z = \bigcup_{i=1}^{\infty} Z_i$, $U = M(f, Z, \frac{1}{2})$. If $f_t \in U$, for a certain $i \leqslant p$ the point t belongs to the set $V(t_i, n_i)\cap I$, and since for $z \in Z_i$ we have $f_t(z) \leqslant |f(z) - f_t(z)| + f(z) < 1$, we obtain $Z_i \cap F(t) = \emptyset$ and it follows that $t \in C_i$. Hence $U\cap F \subset \{f_t \mid t \in \bigcup_{i=1}^{p} C_i\}$ is a finite set, i.e. F is closed and discrete. Property (20) follows from the fact that there exist $\aleph_0$-space E and a quotient mapping E onto A ([7], Example 12.7), and thus, in accordance with the remark at the beginning of section 3, the space σI^A is contained topologically as a closed subspace in the Lindelöf space σI^E ([7], Theorem 9.3).

5. We shall begin the proof of Theorem 2 by showing the implication (i) $\Rightarrow$ (ii). First we shall prove that

(21) X is locally compact.

Assume that it is not and let $x_0 \in X$ be a point which has no compact neighbourhood. Let $V'_1 \supset \overline{V}'_2 \supset V'_2 \ldots$ be a base in x_0 and let $P'_i = \overline{V}'_i \setminus V'_{i+1}$. Since none of the sets V'_i is compact, there exists a sequence $k_1 < k_2 < \ldots$ such that P'_{k_i} is not compact and $k_{i+1} > k_i + 1$. Put $P_i = P'_{k_i}$, $V_i = V'_{k_i}$. Then $\{P_i\}_{i=1}^{\infty}$ is a sequence of closed, non-compact subsets of X, $\{V_i\}_{i=1}^{\infty}$ is a descending base in x_0 and

(22) $$P_i \subset \overline{V}_i \quad \text{and} \quad P_i \cap \overline{V}_{i+1} = \emptyset.$$

The paracompactness of X implies that P_i contains a discrete closed subspace of power $\aleph_0$ whose elements can be arranged in a sequence $x_{j,i}$, $j = 1, 2, \ldots$ From (22) it follows that there exists a family $\{V_{j,i}\}_{j=1}^{\infty}$ open in X and such that

(23) $$x_{j,i} \in V_{j,i}, \quad V_{j,i} \cap \overline{V}_{i+1} = \emptyset, \quad \{V_{j,i}\}_{j=1}^{\infty} \text{ is discrete in } X.$$

For $p, q \in N$ such that $q > p \geqslant 1$ let us choose an $f_{q,p} \in I^X$ so as to have

(24) $$f_{q,p}(x_{q,p}) = 1, \quad f_{q,p}(x_{q,q}) = 0, \quad f_{q,p}(x_0) = 1/p,$$
$$f_{q,p}(x) \leqslant 1/p \quad \text{for} \quad x \notin V_{q,p}.$$

We shall show that for $A = \{f_{q,p} \mid q > p \geqslant 1\}$ the following condition is satisfied:

(25) if $K \subset I^X$ is compact, then $K\cap A$ is finite.

Let us first observe that

(26) there exists a p_0 such that if $p \geqslant p_0$ and $q > p$, then $f_{q,p} \notin K$.

Otherwise we would be able to choose a sequence $p_1 < q_1 < p_2 < q_2 < \ldots$ such that $f_{q_i,p_i} \in K$. Let

$$Z = \{x_{q_i,p_i}, x_{q_i,q_i} \mid i = 1, 2, \ldots\} \cup \{x_0\}.$$

From (22) it follows that Z is compact, and thus Ascoli's theorem ([3], T. 8.2.5) implies that there exists an $r \in N$ such that if $z', z'' \in Z\cap \overline{V}_r$ and $f \in K$, then $|f(z') - f(z'')| < 1$. But $q_r > p_r \geqslant r$, whence x_{q_r,p_r}, x_{q_r,q_r}

$\in \overline{V}_r \cap Z$, and also $|f_{q_r,p_r}(x_{q_r,p_r}) - f_{q_r,p_r}(x_{q_r,q_r})| = 1$. Since $f_{q_r,p_r} \in K$, we get a contradiction.

We shall now prove that

(27) there exists a q_0 such that if $q \geqslant q_0$, and $q > p$, then $f_{q,p} \notin K$.

Let $Z' = \{x_{j,i} | j \leqslant i\} \cup \{x_0\}$. We infer from (22) that Z' is compact and, again by Ascoli's theorem, it follows that there exists a q_0 such that for $z', z'' \in \overline{V}_{q_0} \cap Z'$ and $f \in K$ we have $|f(z') - f(z'')| < 1/p_0$ where p_0 satisfies (26). The q_0 chosen in this way satisfies (27), since otherwise there would exist a $q \geqslant q_0$ and $p < q$ such that $f_{q,p} \in K$. Then $x_{q,q} \in \overline{V}_{q_0} \cap Z'$, $x_0 \in \overline{V}_{q_0} \cap Z'$, and by (24),

$$|f_{q,p}(x_{q,q}) - f_{q,p}(x_0)| = f_{q,p}(x_0) \geqslant 1/p \geqslant 1/p_0,$$

which gives a contradiction.

From (27) immediately follows (25), because the set $\{f_{q,p} | p < q < q_0\}$ is finite. We shall show that $f_0 = 0$ belongs to $\overline{A} \setminus A$, which by (25) contradicts the assumption that I^X is a k-space.

For a compact $Z \subset X$ and $\varepsilon > 0$ let us choose $1/p_0 < \varepsilon$. From (23) it follows that for $j \geqslant j_0$ holds $Z \cap V_{j,p_0} = \emptyset$. Take $q_0 = p_0 + j_0$. Then $f_{q_0,p_0} \in A$, and for $z \in Z$ we have $z \notin V_{q_0,p_0}$, and thus, in accordance with (24), $f_{q_0,p_0}(z) \leqslant 1/p_0 < \varepsilon$. This concludes the proof of (21).

Lemma 1 implies that X^d is a Lindelöf space; otherwise the space $N^{\aleph_1}$, which is not a k-space ([3a], Problem 7. J), would be embedded in I^X as a closed subspace. It follows from (21) that there exists a set X_0, open-and-closed in X, which is a Lindelöf space, so that $X_0 \supset X^d$. The required decomposition is $X = X_0 \cup (X \setminus X_0)$.

The remaining implications necessary to conclude the proof of Theorem 2 follow from the fact that $K^{(X_0 \oplus D(\mathfrak{m}))} = K^{X_0} \times K^{\mathfrak{m}}$ is a product of a space metrizable in a complete manner by a compact space, and hence it is paracompact and complete in the sense of Čech ([3]), and from the fact that spaces complete in the sense of Čech are k-spaces ([1], III, § 2, Corollary 1).

Example 4. The assumption that X is first countable cannot be omitted in Theorem 2. The space $I^{A(\mathfrak{F})}$ from Example 2 is a k-space ([9], Theorem 2.1) but it is neither complete in the sense of Čech nor paracompact, and $A(\mathfrak{F})$ is not locally compact.

I wish to express my gratitude to Professor R. Engelking for valuable discussions.

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Reçu par la Rédaction le 4. 2. 1973