$\tau\colon P\to R^T$ направленна и истинна. Действительно, пусть A и B — подмножества диагонали в $P\times P$, идентифицируемые с множествами пространства P, и $(\xi\times\eta)(A, B)=\gamma\neq 0$. Мы утверждаем, что существуют $\alpha\in R_+^{\Delta}$ и $\beta\in R_+^{\Sigma}$, оба не равные 0, с $(x, y)\in A\times B\Rightarrow \xi(x, y)>\alpha\vee\eta(x, y)>\beta$: в противном случае для любого $q=(q', q'')\in\Delta\times\Sigma$ с $\gamma_q\neq 0$ можно было б подобрать $(x, y)\in A\times B$ с $\xi_{q'}(x, y)<\gamma>\eta_{q''}(x, y)$, то есть получилось бы

$$(\xi\times\eta)_q\times(x, y)<\gamma_q.$$

Ввиду истинности τ справедливо $\tau>\xi$ и $\tau>\eta$ и, следовательно, существуют $\varepsilon_1, \varepsilon_2\in R_+^T$, для которых $(x, y)\in A\times B\Rightarrow\tau(x, y)>\varepsilon_1\vee\tau(x, y)>\varepsilon_2$. Пусть $(\varepsilon_1)_{q_1}=a\neq 0$, $(\varepsilon_2)_{q_2}=b\neq 0$; пользуясь направленностью τ, выбираем $q\in\tau$ с $q>q_1, q_2$. Итак,

$$(x, y)\in A\times B\Rightarrow\tau_q(x, y)\geqslant(\varepsilon_1)_{q_1}\vee\tau_q(x, y)\geqslant(\varepsilon_2)_{q_2},$$

то есть $\tau_q(A, B)\geqslant\min\{a, b\}>0$ и, таким образом, $\tau(A, B)\neq 0$.

Теперь мы можем заключить, что в суперпозиции

$$P=P^{\delta(\tau)}\to(P\times P)^{\delta(\xi\times\eta)}\to(G\times H)^{\delta(\varrho\times\sigma)}$$

все составляющие отображения эквинепрерывны и, таким образом, $P\to\to(G\times H)^{\delta(\varrho\times\sigma)}$ эквинепрерывно, каковы бы ни были метрики ϱ и σ в пространствах близости G и H. Предполагая эти метрики истинными, получаем эквинепрерывность $P\to G\times H$ и, следовательно, правильность пространства P.

Автор благодарен проф. М. Я. Антоновскому, консультации которого были весьма полезными, за постоянное внимание к работе.

ЛИТЕРАТУРА

[1] М. Я. Антоновский, В. Г. Болтянский и Т. А. Сарымсаков, *Очерк теории топологических полуполей* 21 (4) (1966).

[2] — — *Тихоновские полуполя и некоторые проблемы общей топологии* 25 (3) (1970).

[3] — *Гомоморфизмы тихоновских полуполей и обобщенные полуметрики* 69 (2) (1970).

[4] A. Weil, *Sur les éspaces à structure uniforme et sur la topologie générale*, Paris 1937.

[5] S. Mrówka, *A necessary and sufficient condition for m-almost-metrizability*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 6 (1957), pp. 627–629.

[6] В. З. Поляков, *Правильность, произведение и спектры пространств близости* 154 (1) (1964).

[7] — *Правильность и произведение пространств близости*, Матем. сб. 67 (3) (1965).

[8] — *Равномерные свойства пространств близости*, Москва 1968.

Reçu par la Rédaction le 22. 5. 1972

Discrete ordered rings

by

G. A. Heuer (*) (Moorhead, Minn.)

Abstract. A fully ordered ring is called *discrete* if the positive class has a least element; otherwise it is dense. The paper studies the embedding of a discrete ordered ring in a discrete ordered ring with unity; conditions for the existence of a discrete ring in a discrete ordered ring; discrete orders in direct sums; nonisomorphic discrete orders for the same full order; embedding discrete ordered rings in dense ones, and vice versa; discrete subrings of ordered rings; rings with well ordered positive class, and rings with Archimedean order; order in quadratic extensions of ordered rings. Special attention is given to integral domains.

1. Introduction. The integers have been somewhat popularly characterized as constituting the unique ordered (commutative) integral domain (with unity) with well-ordered positive class. They constitute, in fact, the unique ordered ring with unity having well-ordered positive class, as we show in Corollary 10.4. However, ordered rings (with or without unity) in which the positive class is not necessarily well-ordered, but has a least element, abound, and these are the subject of the present study.

If the positive class has a least element e, then each element r of the ring has an immediate successor $r+e$ and an immediate predecessor $r-e$. If the positive class has no least element, then between any two distinct elements of the ring lie infinitely many ring elements. Thus the order type of an ordered ring (indeed, any ordered group) is either discrete throughout or dense throughout. We shall call ordered rings *discrete* or *dense* according as the positive class has or has not a least element.

Except in Section 4, where we consider the extension of discrete partial orders to discrete full orders, only fully ordered rings are considered, and hereafter "ordered ring" is understood to mean "fully-ordered ring".

In Section 3 we show that whenever a discrete ordered ring is order-embeddable in an ordered ring with unity, it is order-embeddable in a discrete ordered ring with unity. In Section 4 we obtain a criterion for the existence of a discrete full order, analogous to that of Fuchs for the existence of a full order. Section 5 gives necessary conditions, and suf-

(*) Research supported by NSF Grant GY7675. Roger Bjorgan served as a very valuable student assistant in this project.

ficient conditions, for a direct sum of ordered rings to be discrete orderable. It turns out that none of the summands need be discrete orderable. Section 6 provides examples of rings which admit infinitely many non-isomorphic discrete orders. In Section 7 we show that every discrete ordered ring may be order-embedded in a dense ordered ring, and in Section 8, that every dense ordered ring may be order-embedded in a discrete one. A necessary and sufficient condition for embeddability of an ordered integral domain in a discrete ordered integral domain is given. Section 9 deals with discrete subrings of ordered rings. In Section 10 we describe all rings with well-ordered positive class, and in Section 11 consider the extension of an order in R to $R[x]/(x^2+bx+c)$.

2. Conventions, some preliminary observations, and examples. Except as otherwise noted, *ordered ring* will always mean fully ordered associative ring. *Integral domain* or *domain* will mean associative ring without divisors of zero. A *discrete* ordered ring is an ordered ring with least positive element. This element we also call an *atom*, or *atomic element*, and we usually denote it e. A ring which admits a total order (= full order) is called an $\mathfrak{O}$*-ring*; one which admits a full discrete order we shall call a $\mathfrak{D}$*-ring*. To indicate that $na<|b|$ for all n in Z (the integers), we write $a\ll b$. If neither $a\ll b$ nor $b\ll a$, then a and b are in the same *Archimedean class*.

The additive group of an ordered ring is always torsion-free. Thus if the ring has a unity, 1, the subring which it generates is isomorphic to the ring Z of integers. Whether or not the identity is atomic (when the order is discrete) depends upon whether the ring has divisors of zero. The following facts are immediately verified for discrete rings:

(2.1) If $rs=0$, $0\leqslant u\leqslant r$ and $0\leqslant v\leqslant s$, then $uv=0$.

(2.2) If e is an atomic element in R and $e^2\neq 0$, then R is an integral domain.

Discrete ordered rings may be classified broadly according to the size of e^2: (1) $e^2=0$; (2) $e^2\in Ze$; (3) $e^2>Ze$. Only in the second case is e necessarily central.

(2.3) If $u^2=mu$ for some $m\in Z$ and u is not a divisor of 0 in the ordered ring R, then $ur=ru=mr$ for all $r\in R$.

For, $u^2r=mur=u(mr)\Rightarrow ur=mr$. From (2.2) and (2.3) we have immediately

(2.4) If $e^2=me$, then $er=re=mr$ for all $r\in R$.

(2.5) A discrete ordered Archimedean domain is isomorphic to mZ for some $m\in Z$.

Examples (2.13) and (2.16) show that e need not be central when $e^2=0$ or $e^2>Ze$. In Section 5 we describe all Archimedean discrete ordered rings.

(2.6) If $ex=x$ for some $x\neq 0$, then R is an integral domain and $e=1$.

For, $0\neq x=ex=e^2x$, so $e^2\neq 0$.

(2.7) If R has a 1, then $e=1$ or $e^2=0$,

since $0<e\leqslant 1\Rightarrow 0\leqslant e^2\leqslant e$, and hence e^2 is 0 or e.

(2.8) If $e^2=me$ and $r^2=kr$ for some nonzero $m,k\in Z$ and $r\ (\neq 0)\in R$, then $r=ne$ and $k=nm$ for some $n\in Z$.

For $ke=re=mr$ by (2.3), so r is in the Archimedean class of e, and hence $r=ne$ for some n. Since $kne=kr=r^2=n^2e^2=n^2me$, $k=nm$.

(2.9) If $e^2>me$ for all $m\in Z$, and r,s are any positive elements of R, then $rs>mr+ns$ for all $m,n\in Z$.

(2.10) If $e^2\neq 0$ then $er\geqslant r$ for all $r>0$ in R.

(2.11) If $e^2\neq 0$ then the convex ideal I generated by e is R.

Since every semi-simple Artinian $\mathfrak{O}$-ring is an $\mathfrak{O}$-field [1; Prop. 5, p. 115], and obviously no field admits a discrete order, a $\mathfrak{D}$-ring cannot be semi-simple Artinian. In fact,

(2.12) No $\mathfrak{D}$-ring is Artinian.

For if $m,n\in Z$, with $m>1$ and $n\neq 0$, then $ne\notin(mne)$, so that (ne) properly contains (mne).

(For terms not defined here the reader is referred to [1].)

The following examples illustrate a variety of behavior, and will be referred to at appropriate points later.

(2.13) Example. Let $R=Z^2$, $u=(1,0)$, $e=(0,1)$. For given positive integers m and k, define $u^2=mu$, $ue=ke$, and $eu=e^2=0$. R may be ordered lexicographically, and e is atomic.

(2.14) Example. $R=Z^2$, $u=(1,0)$, $e=(0,1)$; $u^2=u+e$, $ue=eu=e$, $e^2=0$. If R is ordered lexicographically, e is atomic. In this example R is commutative; in fact R is generated by u, since $e=u^2-u$.

(2.15) Example. R is the semigroup ring of the additive semigroup of nonnegative rational numbers over the ring of integers. Equivalently, R is the set of "polynomials" in x with integer coefficients and nonnegative rational exponents, and an element of R is positive if the coefficient of the highest power of x is positive. The ordinary ring of polynomials over the integers, ordered in the usual way, is a subring of R.

(2.16) Example. R is the ring of polynomials in two noncommuting indeterminates, x and y, with integer coefficients and no constant terms. (Equivalently, the semigroup ring of the free semigroup on two generators over the ring of integers.) If u and v are two monomials and the degree of u (in x and y) is smaller than that of v, put $u \ll v$. If $u = a_1 a_2 \dots a_n \neq v = b_1 b_2 \dots b_n$ where each a_i and each b_i is x or y, put $u \ll v$ if in the first place where $a_i \neq b_i$ we have $a_i = x$. This determines an order in R in which x is atomic.

3. Embedding in a ring with unity. Every ordered ring having an element which is not a left divisor of zero and one which is not a right divisor of zero may be order-embedded in a ring with unity, and under certain conditions the embedding is possible when all elements are divisors of zero. We shall show that whenever a discrete ordered ring is order-embeddable in an ordered ring with unity, it is order-embeddable in a discrete ordered ring with unity. There are three cases, described in the three theorems following. (Not every ordered extension with unity is discrete; see Theorem 7.2.)

(3.1) Theorem. *Let R be an ordered ring with least positive element e, and $e^2 \neq 0$. Then R may be embedded in an order-preserving manner in a discrete ordered integral domain S with unity (and 1 is atomic in S).*

Proof. (i) First suppose that $e^2 > Ze$. Then the standard extension with unity, $S = R \times Z$ with vector addition and $(a, m)(b, n) = (ab + mb + na, mn)$, ordered lexicographically (i.e. $(a, m) > 0$ iff $a > 0$, or $a = 0$ and $m > 0$), is discrete, with the unity $(0, 1)$ as atom. ((2.9) implies that $(a, m)(b, n) > 0$ when $(a, m) > 0$ and $(b, n) > 0$.)

(ii) Now suppose that $e^2 = me$ for some integer $m > 0$. If $m = 1$, e itself is an identity, by (2.4). If $m > 1$, let $\bar{S} = R \times Z$ as in (i). Now $(e, -m)(r, 0) = (er - mr, 0) = (0, 0)$ by (2.4), so $\bar{S}$ is not orderable with the identity $(0, 1)$ as least positive element, in view of (2.2). Let J be the ideal generated by $(e, -m)$. Since $(r, j)(e, -m) = (e, -m)(r, j) = (je, -jm)$, $J = \{k(e, -m)\colon k \in Z\}$. We have $R \xrightarrow{\delta} \bar{S} \xrightarrow{\varphi} S = \bar{S}/J$, where $r\delta = (r, 0)$ and φ is the natural map. One notes that $\ker(\delta \circ \varphi) = (0)$ so R is embedded in S, and S has unity $(0, 1) + J$.

It is easy to see that the elements of S are uniquely representable in the form $[r, j] = (r, j) + J$ where $0 \leqslant j < m$. With elements written in this form, define $[r, j] > 0$ if $r > 0$, or $r = 0$ and $j > 0$. It is routine to check that S is fully ordered, with unity $[0, 1]$ as atom (and therefore S is an integral domain), and that the embedding $\delta \circ \varphi$ preserves order. ∎

(3.2) Theorem. *Let R be an ordered ring with atomic element e, $e^2 = 0$. Assume that R has an element which is not a left divisor of 0 and one which is not a right divisor of 0. Then R may be order-embedded in a discrete ordered ring S with unity.*

Proof. That R may be order-embedded as an ideal in a ring S with unity under the given conditions is known [1, Cor. 7, p. 111]. By replacing S if necessary by the subring generated by 1 and R, we may assume that each element of S is expressible in the form $r - k$, where $r \in R$ and $k = k \cdot 1 \in Z$. (Whether these expressions are unique is immaterial to our argument.) We wish to show there is a smallest positive element of this type.

If $(r-k)^2 > 0$, then $|r-k| > e$, since $e^2 = 0$, so we need be concerned only with elements $r - k$ where $(r-k)^2 = 0$. We note that in this case $k^2 = 2kr - r^2 \in R$.

Let $J = \{j \in Z\colon (r-j)^2 = 0 \text{ in } S \text{ for some } r \in R\}$ and $K = Z \cap R$. Then J and K are ideals in Z, say $J = (j_0)$ and $K = (k_0)$. Then we have $(j_0^2) \subseteq (k_0) \subseteq (j_0)$. Write $k_0 = mj_0$ and $j_0^2 = nk_0$. Then $j_0 = nm$. If, now, $(r - hj_0)^2 = 0$ in S, then $mr - mhj_0 \in R$, since $mj_0 = k_0 \in R$, so $m(r - hj_0) \geqslant e$. Thus no positive element of S is smaller than e/m. ∎

If R does not satisfy the conditions of either of the two foregoing theorems, all elements of R must be divisors of 0. For this case we have

(3.3) Theorem. *Let R be an ordered ring in which every element is a divisor of 0. Then R is order-embeddable in a ring with unity if and only if*

$$uv \leqslant \min\{|u|, |v|\} \quad \text{for all } u, v \in R. \tag{3.3.1}$$

If the order in R is discrete and (3.3.1) holds, then R is order-embeddable in a discrete ordered ring with unity (and with the same atomic element).

Proof. We show first that the condition (3.3.1) is necessary. Suppose that S is an ordered ring with unity having R as an ordered subring, and that $u, v \in R$. Since u is a divisor of 0, $|u| < 1$ in S, so $uv \leqslant |u||v| \leqslant |v|$. Similarly we see that $uv \leqslant |u|$.

Suppose, conversely, that (3.3.1) holds. Let $S = R \times Z$ be the standard extension with unity, now ordered antilexicographically. It is easy to check that S is ordered, the subring $\bar{R} = R \times (0)$ is order-isomorphic to R under the map $(r, 0) \mapsto r$, and that if R is discrete with atomic element e, then $(e, 0)$ is atomic in S. ∎

4. Conditions for existence of a discrete order. For use in this section only we shall make some additional definitions.

(4.1) Definition. A *discrete subsemiring* of the ring R is a pair (S, e) where S is a semiring and $S \subset R$, e a nonzero element of S, and $s \neq 0$ in $S \Rightarrow s - e \in S$. The element e will be called the *atomic element*, or *atom*, of (S, e).

(4.2) Definition. A *discrete partial order* (d.p.o.) in a ring R is a discrete semiring (P, e), where $0 \in P \subseteq R$ and P is conic. Thus (i) $P \cap -P = (0)$, (ii) $P + P \subseteq P$, (iii) $P \cdot P \subseteq P$, and (iv) $r \neq 0$ in P implies $r - e \in P$.

A discrete full order (d.f.o.) is then a d.p.o. (P, e) with $P \cup -P = R$. One notes immediately:

(4.3) The atomic element in a d.p.o. is unique,

for if e_1 and e_2 were atomic then by (iv) $e_1 - e_2$ and $e_2 - e_1$ would be in P, and hence by (i), $e_1 = e_2$.

(4.4) If $\{(P\alpha, e)\}$ is a collection of discrete subsemirings of a ring R, then $(\bigcap P\alpha, e)$ is a discrete semiring.

If $A \subset R$, let $K(A; e)$ be the intersection of all discrete semirings in R containing A and with atom e. We shall write

$$K(A, a_1, \ldots, a_n; e) \quad \text{for} \quad K(A \cup \{a_1, \ldots, a_n\}; e),$$
$$K(A, B; e) \quad \text{for} \quad K(A \cup B; e), \quad \text{etc.}$$

The criterion for existence of a d.f.o. is analagous to that for existence of a full order. Cf. [1, pp. 113–114].

(4.5) THEOREM. *A necessary and sufficient condition that a d.p.o. (P, e) in a ring R extend to a d.f.o. in R is that*

(4.5.1) *For every finite set $\{a_1, \ldots, a_n\}$ in R, there are signs $\varepsilon_1, \ldots, \varepsilon_n$ (each $\varepsilon_i = +$ or $-$) such that $K(P, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n; e)$ is conic.*

The proof will be immediate from two lemmas.

(4.6) LEMMA. *If (P, e) is a d.p.o. satisfying (4.5.1), then for each x in R, one of $K(P, x; e)$ and $K(P, -x; e)$ defines a d.p.o. P' satisfying (4.5.1).*

Proof. Note first that $K(K(A; e), B; e) = K(A, B; e)$ for any sets A, B. Let $P_1 = K(P, x; e)$ and $P_2 = K(P, -x; e)$. If both (P_1, e) and (P_2, e) fail to satisfy (4.5.1), then there are elements $a_1, \ldots, a_n, b_1, \ldots, b_m$ in R such that for all choices of signs $\varepsilon_1, \ldots, \varepsilon_n$ and $\delta_1, \ldots, \delta_m$, both $K(P, x, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n; e)$ and $K(P, -x, \delta_1 b_1, \ldots, \delta_m b_m; e)$ fail to be conic. But then $K(P, \varepsilon_0 x, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, \delta_1 b_1, \ldots, \delta_m b_m; e)$ is conic for no choice of signs $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n, \delta_1, \ldots, \delta_m$, contrary to the assumption that P satisfies (4.5.1). ■

(4.7) LEMMA. *Let $\mathfrak{F}$ be the set of all d.p.o.'s in R containing a given d.p.o. (P, e) and satisfying (4.5.1). Then $\mathfrak{F}$, partially ordered by inclusion, has a maximal element.*

Proof. It is straightforward to verify that the union of any chain in $\mathfrak{F}$ is again an element of P, so that Zorn's Lemma applies. ■

Proof of (4.5). The condition is obviously necessary. For the sufficiency, let $\overline{P}$ be a maximal element of $\mathfrak{F}$. If neither x nor $-x$ were in $\overline{P}$ for some x in R, then by (4.6) one of $K(\overline{P}, x; e)$ and $K(\overline{P}, -x; e)$ would be a d.p.o. satisfying (4.5.1), contrary to the maximality of $\overline{P}$. ■

(4.8) COROLLARY. *R is a $\mathfrak{D}$-ring with e atomic if and only if $P = K(e; e)$ satisfies (4.5.1).*

(4.9) COROLLARY. *If every finitely generated subring of R containing e is a $\mathfrak{D}$-ring with e atomic, then so is R.*

As an application of Theorem 4.5 we may obtain a condition that a discrete partial order be the intersection of discrete full orders.

(4.10) THEOREM. *Let (P, e) be a d.p.o. in a ring R. A necessary and sufficient condition that P be the intersection of a collection $\{Q_\lambda\}$ of d.f.o.'s with e atomic is*

(4.10.1) *If there exist $a_1, \ldots, a_n \in R$ such that $K(P, -a, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n; e)$ is not conic for any choice of signs ε_i, then $a \in P$.*

Proof. To show the condition necessary, suppose $P = \bigcap_\lambda Q_\lambda$, each Q_λ a d.f.o. with atom e. If $a \notin P$ then $a \notin Q_\lambda$ for some λ, so $-a \in Q_\lambda$; i.e. $P' = K(P, -a; e)$ extends to a d.f.o. Q_λ with e atomic. By (4.5), $K(P, -a, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n; e) = K(P', \varepsilon_1 a_1, \ldots, \varepsilon_n a_n; e)$ is conic for some choice of signs ε_i.

Now assume that (4.10.1) holds, and let a be any element of $R \setminus P$. From (4.10.1) we know that for every $a_1, \ldots, a_n \in R$, $K(P, -a, \varepsilon_1 a_1, \ldots$ $\ldots, \varepsilon_n a_n; e)$ is conic for some choice of signs ε_i. Thus $P' = K(P, -a; e)$ satisfies (4.5.1), so P' extends to a d.f.o. Q_a with atom e. Then $P = \bigcap_{a \in R \setminus P} Q_a$. ■

As an application of (4.8), let R be the ring of Example 2.15, and R_0 the subring of "polynomials" with zero constant term. In the given order, R_0 is a dense ring. We show that no discrete order exists for R_0. If $e = \sum m_q x^q$, choose a positive rational s smaller than the minimum q occurring in e, and let $a_1 = x^{s/2}$, $a_2 = \sum m_q x^{q-s}$. Then $(\varepsilon_1 a_1)^2 = x^s$ and $(\varepsilon_1 a_1)^2 a_2 = e$ are in $K(\varepsilon_1 a_1, \varepsilon_2 a_2; e)$ for any choice of signs $\varepsilon_1, \varepsilon_2$. Since $-e \in K(\varepsilon_1 a_1, -a_2; e)$, this set is not conic, and $K(\varepsilon_1 a_1, a_2; e)$ contains $(a_2 - e)(\varepsilon_1 a_1)^2 = e - e(\varepsilon_1 a_1)^2$ and its negative, so is not conic.

5. Direct sums. We recall that a direct sum of $\mathfrak{O}$-rings is an $\mathfrak{O}$-ring if and only if at most one of the summands has nontrivial multiplication [1, Prop. 4, p. 114]. The question of when a direct sum of rings is a $\mathfrak{D}$-ring is a bit more involved. For example, it is not necessary that any summands be $\mathfrak{D}$-rings (although they must obviously be $\mathfrak{O}$-rings). On the other hand, it is necessary that at most one summand have nontrivial multiplication (since a $\mathfrak{D}$-ring is an $\mathfrak{O}$-ring), so we are led first to some investigation of discrete ordered abelian groups. An is well-known, an abelian group admits a full order if and only if it is torsion-free. For the discrete case the condition (4.5.1) may be immediately adapted, of course, but some other characterizations will be more useful.

(5.1) THEOREM. *Let G be a torsion-free (additive) abelian group, and $e \in G$. The following are equivalent:*

(A) *G admits a discrete full order with atom e.*

(B) *Given elements $a_1, \ldots, a_n$ in G, if $\exists$ signs $\varepsilon_1, \ldots, \varepsilon_n$ such that $\sum_1^n k_i(\varepsilon_i a_i) + ke = 0$ with $k_i \geqslant 0$ in Z and k in Z, then $k = 0$ and each $k_i a_i = 0$.*

(C) *Every finitely generated subgroup of G containing e admits a d.f.o. with atom e.*

(D) *For every finitely generated subgroup of G containing e, there is a minimal generating set containing e.*

(E) *e is not divisible in G by an integer >1.*

Proof. The equivalence of (A), (B) and (C) is immediate from Theorem 4.5 and Corollary 4.9, since (4.5.1) reduces to (B) when R has trivial multiplication. That (A) implies (E) is obvious, and that (D) implies (C) is easy (extend e to a basis and order lexicographically). We shall complete the proof of equivalence by showing that (E) implies (D). Let $\bar{G}$ be a subgroup of G having dimension n. We may suppose $\bar{G} = Z^n$ and that the assertion is valid for dimension $n-1$. Let G_1 be an $(n-1)$-dimensional subgroup of $\bar{G}$ containing e, and H the $(n-1)$-dimensional vector space (over the reals) determined by G_1. Then $G_2 = \bar{G} \cap H$ is an $(n-1)$-dimensional subgroup of $\bar{G}$ containing G_1. Extend e to a basis $\{e, b_2, \ldots, b_{n-1}\}$ of G_2. Each coset of G_2 in $\bar{G}$ is the part of $\bar{G}$ in some hyperplane parallel to H, and there is one of these whose positive distance from H (measured in $\boldsymbol{R}^n$) is minimal. If b_n is any element of $\bar{G}$ in this nearest hyperplane, then $\bar{G} = G_2 + Zb_n$; i.e. $\{e, b_2, \ldots, b_{n-1}, b_n\}$ is a basis for $\bar{G}$. ∎

(5.2) COROLLARY. *A ring with trivial multiplication is a $\mathfrak{D}$-ring if, and only if, its additive group is torsion free and it has an element not divisible by an integer >1.*

This corollary allows us incidentally to describe all discrete Archimedean ordered rings. Hion [2] has shown that every Archimedean ordered ring is (up to order-isomorphism) either a subring of the real numbers $\boldsymbol{R}$ with usual order, or a ring with trivial multiplication and with additive group a subgroup of $\boldsymbol{R}$. The former case we have already mentioned in (2.5); in the latter case Corollary 5.2 applies.

(5.3) COROLLARY. *Every discrete ordered Archimedean ring R is order-isomorphic to mZ for some $m \in Z$ or to a trivial ring whose additive group is a subgroup of the real numbers and contains an element not divisible in R by an integer >1. Conversely, all such rings are discrete Archimedean ordered. (Cf.* (10.2) *and* (10.3).)

To turn now to the question of direct sums, we show first that the summands need not be $\mathfrak{D}$-rings.

(5.4) EXAMPLE. *Let $Q_2 = \{m \cdot 2^{-n}\colon m, n \in Z\}$ and $Q_3 = \{m \cdot 3^{-n}\colon m, n \in Z\}$ with the usual addition but trivial multiplication, and $R = Q_2 \oplus Q_3$. Then R is a $\mathfrak{D}$-ring, but neither Q_2 nor Q_3 is. For, neither summand has an indivisible element, but $(1, 1)$ is indivisible in R.*

We have not succeeded in finding a satisfactory necessary and sufficient condition that a direct sum of $\mathfrak{O}$-rings be a $\mathfrak{D}$-ring. (For a direct sum of $\mathfrak{D}$-rings see Corollary 5.7.) Certain conditions are obviously necessary: the atomic element must be indivisible and not a sum of two or more even products (products in which each expressed factor occurs an even number of times). We have also:

(5.5) THEOREM. *Let R be the direct sum of the $\mathfrak{O}$-rings $\{R_\gamma\colon \gamma \in \Gamma\}$. (Assume that Γ and each R_γ has at least two elements.) If R is a $\mathfrak{D}$-ring with e as an atomic element, then*

(A) *at most one R_γ has nontrivial multiplication, and*

(B) *$eR = Re = 0$. (In particular, the component of e in the nontrivial R_γ annihilates R_γ.)*

Proof. (A) is necessary in order that R be an $\mathfrak{O}$-ring, as we have already remarked. (B) Let R be ordered, with $e > 0$ atomic. Since at least one R_γ has trivial multiplication and has a nonzero element, $\exists r > 0$ in R_γ such that $rR = Rr = 0$. Then $eR = Re = 0$, since $0 < e \leqslant r$. ∎

We have not found a counterexample to show that the conditions of (5.5), along with the necessary conditions of the preceding paragraph, are not sufficient; however, our sufficient conditions are not sharply different from the necessary ones:

(5.6) THEOREM. *Let R be the direct sum of the $\mathfrak{O}$-rings $\{R_\gamma\}$, and write $R = R_1 \oplus R_2$, where R_1 is the sum of all R_γ with trivial multiplication and R_2 the sum of the remaining ones. If*

(A) *R_2 is* (0) *or consists of a single R_γ, and*

(B) *R_1 is a $\mathfrak{D}$-ring (i.e., has an indivisible element) or R_2 is a $\mathfrak{D}$-ring $\neq$* (0) *with annihilating atomic element, then R is a $\mathfrak{D}$-ring.*

Proof. If $R_2 = (0)$ we have $R = R_1$ is a $\mathfrak{D}$-ring by (B), so we suppose $R_2 \neq (0)$. If R_1 is a $\mathfrak{D}$-ring, let R_1 be given a discrete order and R_2 any order. Then it is easy to check that antilexicographic order in $R = R_1 \oplus R_2$ is a discrete full order. ($r = r_1 + r_2$, $r_i \in R_i$, is positive if $r_2 > 0$ in R_2, or $r_2 = 0$ and $r_1 > 0$ in R_1.) If R_2 has an annihilating atomic element e, let R be ordered as follows: For $r_i \in R_i$, $r_1 + r_2 > 0$ if (i) $r_2 > 0$ and $r_2 \gg e$ in R_2, or (ii) $r_2 \in Ze$ and $r_1 > 0$ in R_1, or (iii) $r_1 = 0$ and $r_2 = ke$ for some integer $k > 0$. Then $(r_1 + r_2)(s_1 + s_2) = r_2 s_2 = 0$ unless both $r_2 \gg e$ and $s_2 \gg e$; all the required order properties easily follow, and e is the least positive element in R. ∎

(5.7) COROLLARY. *A direct sum of $\mathfrak{D}$-rings is a $\mathfrak{D}$-ring if, and only if, at most one summand has nontrivial multiplication.*

That condition (B) in (5.6) is not necessary is seen from

(5.8) EXAMPLE. Let $R = R_1 \oplus R_2$, where $R_1 = \{p\cdot 3^{-q}e_1 \colon\ p,\, q \in Z\}$ with all products 0, and $R_2 = \{ka + m\cdot 2^{-n}e_2 \colon\ k, m, n \in Z\}$ with $a^2 = e_2$, $e_2 a = a e_2 = e_2^2 = 0$. Then neither R_1 nor R_2 is a $\mathfrak{D}$-ring, and R_2 has non-trivial multiplication (and R_2 is not itself a direct sum), but R is a $\mathfrak{D}$-ring. For: each element of R_1 is divisible by 3, so R_1 is not a $\mathfrak{D}$-ring by (5.2). In any order in R_2 we must have $|a| \gg |e_2|$ since e_2 is an annihilator; since there is no smallest multiple of e_2, R_2 is not a $\mathfrak{D}$-ring. In R, let $p\cdot 3^{-q}e_1 + ka + m\cdot 2^{-n}e_2 > 0$ if (i) $k > 0$, or (ii) $k = 0$, and $p\cdot 3^{-q} + m\cdot 2^{-n} > 0$, or (iii) $k = 0$, $p\cdot 3^{-q} + m\cdot 2^{-n} = 0$ and $p > 0$. One verifies readily that this defines a full order in R, with $e_1 - e_2$ the least positive element.

6. D-rings with many possible atomic elements. This section is primarily concerned with some examples. In some $\mathfrak{D}$-rings, all discrete orders have the same atomic element. This is the case, for example, in an integral domain with 1 (where the atom must be 1). If R is a discrete ordered ring and α an automorphism of R, then α induces another discrete order in R; if e is atomic in the original order, $e\alpha$ is atomic in the new order. In some cases discrete orders with different atomic elements exist, but for any two, one may be obtained from the other by applying an automorphism. For example, this is the case when R is a finitely generated ring with trivial multiplication, as one sees from Theorem 5.1. However, there are $\mathfrak{D}$-rings which admit infinitely many discrete orders and no automorphism mapping the atomic element of one onto that of a different one. We give an example in each of the three cases, (i) $eR = Re = (0)$, (ii) $e^2 = 0$ but $eR \neq 0$, and (iii) $e^2 \neq 0$. (Any two atomic elements must, of course, have the same left and right annihilators.)

(6.1) EXAMPLE. Let R be the ring of Example 5.8. Choose relatively prime integers m, n, neither divisible by 2 or 3, and define $P_{m,n}$ in R by $u_1e_1 + u_2e_2 + ka \in P_{m,n}$ if $k > 0$, or $k = 0$ and $mu_2 + nu_1 > 0$, or $k = 0$ and $u_1 = u_2 = 0$. Then (A) $P_{m,n}$ is a discrete full order in R, with $e = me_1 - ne_2$ atomic, and (B) if α is any automorphism of R, then $(me_1 - ne_2)\alpha = bme_1 - cne_2$ for some $b, c \in Z$. Thus there are infinitely many choices of (m, n) such that for no two of them is there an automorphism mapping the one atomic element onto the other. The proofs of (A) and (B) are left to the reader. For (B), note that if α is an automorphism of R, then $e_1\alpha$ and $e_2\alpha$ must annihilate R, so have the form $u_1e_1 + u_2e_2$.

(6.2) EXAMPLE. With R as in (6.1), let $S = R \times Z$, with the order described in the proof of (3.3). For each atomic element e in R, $(e, 0)$ is atomic in S, and again there are infinitely many, with no automorphism of S mapping one onto another. Here $(e, 0)^2 = (0, 0)$ but $(e, 0)S = S(e, 0) \neq (0, 0)$.

(6.3) EXAMPLE. Let R be the ring of polynomials in a denumerably infinite set of commuting indeterminates $\{x_1, x_2, \ldots\}$, with integral coefficients, and with the constant terms and the coefficients of x_i^j equal to zero for $1 \leqslant j < i$. Then for each i there is a discrete order in R for which x_i^i is atomic, and if $i \neq j$ no automorphism of R maps x_i^i to x_j^j. This time R is an integral domain. To obtain the desired order, order the indeterminates with x_i first, and decree that if $x_m < x_n$ then every polynomial in x_m is $< x_n$ (and, of course, $x_n^{k+1} \gg x_n^k$ for all n and k).

7. Embedding a discrete ring in a dense ring. If D is a discrete ordered commutative integral domain, the order in D extends in a unique way to the field F of quotients, and the order in F is necessarily dense. We show in this section how to embed an arbitrary discrete ordered ring in a dense ordered ring. The following lemma is probably well-known, but we have not encountered it before.

(7.1) LEMMA. *Let R be a ring whose additive group is torsion-free. Then R may be embedded in a ring S such that if $x \in S$ and $n \neq 0$ in Z then $x = ny$ for some $y \in S$. (Every torsion-free Z-module may be embedded in a Q-module, where Q is the field of rational numbers.)*

Proof. One roughly imitates the formation of quotient field, as follows: In $Q \times R$, define $(m/n, r) \sim (m'/n', r')$ to mean $mn'r = m'nr'$. This is an equivalence relation; let S be the set of equivalence classes, and denote the class of (q, r) by $[q, r]$. In S, define the sum and product of $[m/n, r]$ and $[m'/n', r']$ by $[1/nn', mn'r + m'nr']$ and $[mm'/nn', rr']$, respectively. Then one routinely checks that the operations are well-defined, S is a ring under these operations, and the map $\varphi\colon R \to S$ defined by $r\varphi = [1, r]$ is an isomorphism onto a subring R' of S. ∎

(7.2) THEOREM. *Every discrete ordered ring may be embedded in an order-preserving manner in a dense ordered ring.*

Proof. If R is a discrete ordered ring, let S be the extension constructed in the lemma. Call $[m/n, r] > 0$ in S if $mnr > 0$ in R. This order is well-defined in S, is dense, and extends the order induced in the subring R' by the isomorphism φ in the proof of (7.1). ∎

8. Embedding an ordered ring in a discrete ring. Not every subring of a discrete ordered ring is discrete in the induced order. For instance, in the ring of Example (2.15), the subring of polynomials having zero constant term is not discrete. In Example (5.4) we have a direct sum of two rings, with a discrete order in the sum, while the induced order in each summand is dense (indeed, the summands are not even $\mathfrak{D}$-rings). These examples, it turns out, are not exceptional. On the contrary, we shall show that every dense ordered ring may be found as an ordered subring of some discrete ordered ring.

(8.1) Lemma. *If $nr < r^2 < (n+1)r$ for some integer n and some $r > 0$ in an ordered integral domain D, then there exists $x > 0$ in D such that $0 < xr < r$.*

Proof. Either $2nr < 2r^2 \leqslant (2n+1)r$ or $(2n+1)r < 2r^2 < (2n+2)r$. We assume the former, the proof being similar in the latter case. Thus, $0 < 2r^2 - 2nr \leqslant r$. Choose k so that $2^k > n+1$. Then

$$\begin{aligned}(n+1)(r-n)^k r^2 &< (2r-2n)^k r^2 = (2r-2n)^{k-1}(2r^2-2nr)r\\ &\leqslant (2r-2n)^{k-1}r^2 \leqslant \ldots \leqslant (2r-2n)r^2 = (2r^2-2nr)r\\ &\leqslant r^2 < (n+1)r,\end{aligned}$$

so we have $0 < [(r-n)^k r]r < r$. ∎

(8.2) Theorem. (A) *Every ordered ring R is order-embeddable in a discrete ordered ring S.* (B) *If R is an ordered integral domain, it may be order-embedded in a discrete ordered integral domain if, and only if,*

$$(8.2.1) \qquad |xy| \geqslant \max\{x, y\} \quad \text{for all } x, y \text{ in } R.$$

Proof. (A) If R is not itself discrete, let $\bar{R}$ be any discrete ordered ring with trivial multiplication. Then $S = R \oplus \bar{R}$, with lexicographic order, has the required properties. In particular, the atomic element is $(0, \bar{e})$, where $\bar{e}$ is atomic in $\bar{R}$. (B) If S is a discrete ordered domain containing R, with atomic element e, and $x, y \in R$, then

$$|xy| \geqslant \max\{|x|e, e|y|\} \geqslant \max\{|x|, |y|\},$$

by (2.10). Thus (8.2.1) is necessary.

Assume, conversely, that (8.2.1) holds. In view of Lemma (8.1) and (8.2.1), if $r > 0$ in R, either $r^2 \gg r$ or $r^2 = mr$ for some $m \in Z$. In the latter case $r = m$ (see (2.3)), and the smallest positive integer in R is atomic; i.e., R itself is discrete. In the former case, $S = R \times Z$, the standard extension with unity, ordered lexicographically (see proof (3.1), (i)) is a discrete ordered extension of R, with no divisors of zero.

9. Discrete subrings of ordered rings. Given a dense ordered ring, it seemed natural to inquire about the discrete subrings it might contain. We considered primarily the extreme cases of singly generated subrings and maximal subrings.

(9.1) Lemma. *If $n\alpha < \alpha^2 < (n+1)\alpha$ for some $\alpha > 0$ in an ordered ring R and some $n \in Z$, then no subring containing α and having no divisors of zero is discrete ordered.*

Proof. Let S be a subring containing α and having no divisors of zero. If $\beta > 0$ in S, then $0 < \alpha^2 - n\alpha < \alpha$ implies $0 < \alpha^2\beta - n\alpha\beta < \alpha\beta$, and thus $0 < \alpha\beta - n\beta < \beta$. ∎

(9.2) Theorem. *If D is an ordered integral domain and $\alpha \in D$ then the subdomain D_α generated by α is discrete if, and only if, $\alpha^2 \in Z\alpha$ or $\alpha^2 \gg \alpha$.*

Proof. If $\alpha^2 = m\alpha$ for some $m \in Z$, then $D_\alpha = mZ$ by (2.3). If $\alpha^2 \gg \alpha$ then D_α consists of all polynomials in α with integral coefficients and zero constant term, ordered according to the coefficient of the highest power of $|\alpha|$. The least positive element is $|\alpha|$.

The converse is immediate from Lemma 9.1. ∎

One might guess from Lemma 9.1 that if α is not a zero divisor and $n\alpha < \alpha^2 < (n+1)\alpha$ for some integer n, then the subring generated by α fails to be discrete. That this is not the case may by seen from Example 2.14, where $u < u^2 < 2u$, and u is not a divisor of 0, but u generates all of R, and R is discrete.

A dense ordered ring need not, in general, have a maximal discrete ordered subring. If R is the ring of Example (2.15), R_0 the subring of polynomials with zero constant terms, and S any discrete subring of R_0, there is an element x^q in R_0 which is smaller than the atomic element of S, and the subring generated by S and x^q is again discrete. Thus R_0 has no maximal discrete subring.

We do, however, obtain some results about maximal discrete subrings of a special kind.

(9.3) Definition. A commutative ordered ring R is *strongly discrete* if there is a set $T \subset R$ satisfying

(i) T generates R,

(ii) $x < y$ in T implies every polynomial in x with integer coefficients is $< y$, and

(iii) T has a least element t_0, and $y \leqslant \min\{t_0 y, y t_0\}$ for all $y > 0$ in R. (We abbreviate in (iii) by writing $t_0 \geqslant 1$, even if R does not have a 1.)

(9.4) Lemma. *If R is a strongly discrete commutative integral domain, then R is discrete.*

Proof. Let T be the generating set of (9.3) and t_0 its least element. We show first that representation of elements of R as polynomials in T with integral coefficients is unique. If it is not, then 0 may be represented as a polynomial $P(t_1, \ldots, t_n)$ in a nontrivial way, where $t_1, \ldots, t_n \in T$. We assume that each of $t_1, \ldots, t_n$ actually occurs with a nonzero coefficient, that t_n is the largest of them, and that the number of t_i occurring is minimal for all such nontrivial representations of 0. If we write

$$P(t_1, \ldots, t_n) = \sum_{i=0}^{m} P_i(t_1, \ldots, t_{n-1}) t_n^i,$$

then $P_m(t_1, \ldots, t_{n-1}) \neq 0$ so $|P_m(t_1, \ldots, t_{n-1})| \geqslant 1$ (it is $\geqslant$ any t_i which actually occurs). Thus

$$|P_m(t_1, \ldots, t_{n-1})t_n^m| \geqslant t_n^m > \left|\sum_{i=0}^{m-1} P_i(t_1, \ldots, t_{n-1})t_n^i\right|,$$

contrary to $P(t_1, \ldots, t_n) = 0$.

It is clear now that when $t_1 < t_2 < \ldots < t_n$ in T, $\sum_{i=0}^{m} P_i(t_1, \ldots, t_{n-1})t_n^i > 0$ precisely when $P_m(t_1, \ldots, t_{n-1}) > 0$, and therefore t_0 is the least positive element of R. ■

Notation. We write $x \succsim y$ to mean that x is greater than every integral polynomial in y.

(9.5) Theorem. *A necessary condition for a commutative ordered integral domain D to have a maximal strongly discrete subdomain is that*

(9.5.1) *Every infinite subset of D which is totally ordered by $\succsim$ and whose elements are >1 has a lower bound $\geqslant 1$ in D.*

Proof. Let M be a maximal strongly discrete subdomain, T the generating set for M called for in (9.3), $t_0 = \min T$, and $X = \{x_\gamma\colon \gamma \in \Gamma\}$ an infinite subset of D totally ordered by $\succsim$. We assume X has no least element, since otherwise there is nothing to prove. If $t_0 \geqslant x_\gamma$ for some $\gamma \in \Gamma$, then $t_0 \succsim x_\mu$ for some μ, and the subdomain generated by $T \cup \{x_\mu\}$ would violate the maximality of M. Thus $t_0 < x_\gamma$ for each $\gamma \in \Gamma$. Also, $1 \leqslant t_0$ by (2.10) and the fact that M is discrete. ■

Let us call a set T satisfying (ii) and (iii) in (9.3) *qualified*. Following is a partial converse to Theorem 9.5.

(9.6) Theorem. *If D is a commutative ordered integral domain satisfying* (9.5.1), *then every qualified set in D is contained in a maximal qualified set.* (*Thus D contains strongly discrete subdomains for which the generating set is maximal.*)

Proof. Let T_0 be a qualified set and $\mathfrak{C}$ the collection of all qualified sets containing T_0. $\mathfrak{C}$ is partially ordered by inclusion. If $\{T_\gamma\colon \gamma \in \Gamma\}$ is a totally ordered subset of $\mathfrak{C}$, let $T = \bigcup_{\Gamma} T_\gamma$. If T has a least element, then T is a qualified set. If T has no least element, there is by (9.5.1) an element $t_0 \geqslant 1$ in D which is a lower bound for T. Clearly we must have $t \precsim t_0$ for each $t \in T$, so $T \cup \{t_0\}$ is qualified. Zorn's Lemma now gives the desired result. ■

It does not follow, however, that a commutative ordered domain satisfying (9.5.1) has a maximal strongly discrete subdomain. In the domain R_0 described just before (9.3), no two elements satisfy $r \precsim s$, so (9.5.1) holds, but R_0 has no maximal discrete subdomain.

10. Rings with well-ordered positive class. These may be disposed of quite briefly.

(10.1) Lemma. *If $R \neq (0)$ is an ordered ring with well-ordered positive class then the additive group of R is infinite cyclic.*

Proof. If not, there must be a first element w such that $w > Ze$. But then $w - e > Ze$ also. Thus $R = Ze$. ■

(10.2) Theorem. *If $R \neq (0)$ is an ordered ring with well-ordered positive class, then R is (order-isomorphic to) a subring of Z, or to Z with trivial multiplication.*

Proof. By the lemma, $R = Ze$. If $e^2 = me$ then the map $ke \mapsto mk$ is an isomorphism of R onto mZ. If $e^2 = 0$, R has trivial multiplication. ■

By comparison of (10.2) with (5.3) we obtain

(10.3) Corollary. *A discrete ordered ring is Archimedean if, and only if, its positive class is well-ordered.*

(10.4) Corollary. (A) *If R is an ordered ring with unity and with well-ordered positive class, then R is isomorphic to Z.*

(B) *If R is an Archimedean discrete ordered ring with unity, then R is isomorphic to Z.*

11. Orders for $R[x]/(x^2+bx+c)$. Assume now that R has a unity, and consider the extension $S = R[x]/(x^2+bx+c)$, always assuming that x commutes with R. We may have

(i) $x^2+bx+c = (x-a)^2$,

(ii) $x^2+bx+c = (x-r)(x-s)$ where $r \neq s$, or

(iii) x^2+bx+c is irreducible in $R[x]$.

The first two cases are easily disposed of; in the third we shall place further restrictions upon R.

(11.1). Theorem. *If R is an ordered ring with unity and $a \in R$, then the order in R extends to $S = R[x]/\big((x-a)^2\big)$. If R is discrete, so is S.*

Proof. We may write each element of S in the form $r+sx = r'+sx'$, where $r, s \in R$ and $r' = r+sa$, $x' = x-a$. Since

$$(x')^2 = 0, \quad (r_1+s_1x')(r_2+s_2x') = r_1r_2+(r_1s_2+s_1r_2)x',$$

and S may be ordered lexicographically: $r+sx' > 0$ if $r > 0$, or $r = 0$ and $s > 0$. If R has an atomic element e, then ex' is atomic in S. ■

(11.2) Theorem. *If R is a ring with unity and $r \neq s$ in R, then $S = R[x]/\big((x-r)(x-s)\big)$ is not an $\mathcal{O}$-ring.*

Proof. Since $(x-r)(x-s) = 0$ in S, the square of the smaller factor in any ordering of S would be 0. But

$$(x-r)^2 = x^2-2rx+r^2 = (r+s)x-rs-2rx+r^2$$

cannot be 0 in S when $r \neq s$; similarly $(x-s)^2 \neq 0$. ■

Suppose now that x^2+bx+c is irreducible in $R[x]$. If $\Delta = b^2-4c < 0$, the order in R does not extend to S, since $b^2-4c = (2x+b)^2$. If $\Delta = 0$, then $4(x^2+bx+c) = (2x+b)^2$ is reducible, so if $b = 2b'$ for some b' in R, we are back to (11.1). We shall not consider $\Delta = 0$ otherwise.

For the case $\Delta > 0$ we make some further assumptions, and the next theorem summarizes our results.

(11.3) THEOREM. *Suppose that R is a commutative ordered integral domain with unity, and $p(x) = x^2+bx+c$ a polynomial over R such that for each $d \neq 0$ in R, $d^2p(x)$ is irreducible in $R[x]$, and $\Delta = b^2-4c > 0$. Let $S = R[x]/(p(x))$ and let P_S be the set of elements $rx+s$ in S such that one of the following three conditions holds*:

(11.3.1) $\quad r \geqslant 0$ *and* $2s-rb \geqslant 0$;

(11.3.2) $\quad r > 0$, $2s-rb < 0$ *and* $r^2\Delta > (2s-rb)^2$;

(11.3.3) $\quad r < 0$, $2s-rb > 0$ *and* $r^2\Delta < (2s-rb)^2$.

Then P_S is a positive class in S. If R is discrete ordered, a necessary but not sufficient condition that S be discrete ordered is that $\Delta \geqslant 1$.

Proof. Note first that if $r \neq 0$, $r^2(b^2-4c)$ is not a square in R, since otherwise $4r^2p(x) = (2rx-rb)^2-r^2(b^2-4c)$ would be reducible. Thus $P_S \cap -P_S = (0)$, and it is clear that $P_S \cup -P_S = R$, so to show that P_S is a positive class it remains to show that it is closed under addition and multiplication. For $j = 1, 2$, let $u_j = r_jx+s_j$. Then $2u_j = r_j\delta+t_j$, where $\delta = 2x+b$ and $t_j = 2s_j-r_jb$, and $\delta^2 = \Delta$. The definition of P_S divides it into three subclasses, giving us six types of sums and six types of products to consider. We label these sums (i)+(i), (i)+(ii), etc., according as u_1 and u_2 both satisfy (11.3.1), u_1 satisfies (11.3.1) and u_2 (11.3.2), etc., and in a similar way label the products (i)(i), (i)(ii), etc.

(i)+(i). The sum is clearly again of type (i).

(i)+(ii). Here $(t_1+t_2)^2 \leqslant t_2^2 < r_2^2\Delta < (r_1+r_2)^2\Delta$, so $2(u_1+u_2) = (r_1+r_2)(2x+b)+(t_1+t_2)$ is of type (ii) if $t_1+t_2 < 0$ (and of type (i) otherwise).

(i)+(iii). If $r_1+r_2 \geqslant 0$, the sum is of type (i); if $r_1+r_2 < 0$, then $r_2 < r_1+r_2 < 0$, so $(r_1+r_2)^2\Delta < r_2^2\Delta < t_2^2 \leqslant (t_1+t_2)^2$, and the sum is of type (iii).

(ii)+(ii). Since $t_1^2t_2^2 < r_1^2r_2^2\Delta^2$, we have $0 < t_1t_2 < r_1r_2\Delta$, and hence $(r_1+r_2)^2\Delta = r_1^2\Delta+r_2^2\Delta+2r_1r_2\Delta > t_1^2+t_2^2+2t_1t_2 = (t_1+t_2)^2$, so the sum is of type (ii).

(iii)+(iii) is proved similarly.

(ii)+(iii). Here we must consider four subcases.

(a) If $r_1+r_2 \geqslant 0$ and $t_1+t_2 \geqslant 0$, the sum is type (i).

(b) If $r_1+r_2 > 0$ and $t_1+t_2 < 0$, we show the sum is type (ii). From $0 < -r_2 < r_1$ and $0 < t_2 < -t_1$, it follows that

(11.3.4) $$0 < r_2^2\Delta < t_2^2 < t_1^2 < r_1^2\Delta .$$

Thus, $r_2^2\Delta t_1^2 < t_2^2r_1^2\Delta$, and hence $|r_2t_1| < |t_2r_1|$. Then

$$|r_1t_1|+|r_2t_1|-|r_1t_2|-|r_2t_2| < |r_1t_1|+|r_1t_2|-|r_2t_1|-|r_2t_2|;$$

i.e.,

$$(|r_1|+|r_2|)(|t_1|-|t_2|) < (|r_1|-|r_2|)(|t_1|+|t_2|);$$

i.e.,

(11.3.5) $$(r_1-r_2)(-t_1-t_2) < (r_1+r_2)(-t_1+t_2),$$

and each factor here is positive.

From (11.3.4) and (11.3.5) we infer that

$$\begin{aligned}(r_1+r_2)^2\Delta(r_1-r_2)(-t_1+t_2) &= (r_1^2-r_2^2)\Delta(r_1+r_2)(-t_1+t_2)\\ &> (t_1^2-t_2^2)(r_1-r_2)(-t_1-t_2)\\ &= (t_1+t_2)^2(-t_1+t_2)(r_1-r_2),\end{aligned}$$

and hence $(r_1+r_2)^2\Delta > (t_1+t_2)^2$, as desired.

(c) If $r_1+r_2 < 0$ and $t_1+t_2 > 0$, a similar argument shows that $(r_1+r_2)^2\Delta < (t_1+t_2)^2$.

(d) If $r_1+r_2 < 0$ and $t_1+t_2 < 0$, then $0 < r_1 < -r_2 \Rightarrow 0 < r_1^2\Delta < r_2^2\Delta$, and $0 < t_2 < -t_1 \Rightarrow 0 < t_2^2 < t_1^2$, while $t_1^2 < r_1^2\Delta < r_2^2\Delta < t_2^2$, a contradiction. Thus this case does not occur.

For the product, note that

$$(r_1\delta+t_1)(r_2\delta+t_2) = (r_1t_2+r_2t_1)\delta+(t_1t_2+r_1r_2\Delta).$$

(i)(i). The product is clearly again of type (i).

(i)(ii). (a) If $r_1t_2+r_2t_1 \geqslant 0$ and $t_1t_2+r_1r_2\Delta \geqslant 0$, the product is of type (i).

(b) Suppose $r_1t_2+t_1r_2 \leqslant 0$; i.e., $0 < t_1r_2 \leqslant r_1|t_2|$. We show that then

(11.3.6) $$t_1t_2+r_1r_2\Delta > 0,$$

and

(11.3.7) $$(r_1t_2+t_1r_2)^2\Delta < (t_1t_2+r_1r_2\Delta)^2,$$

so the product is of type (iii) or (i).

Since $0 < t_1^2r_2^2 \leqslant r_1^2t_2^2 < r_1^2r_2^2\Delta$, we have

(11.3.8) $$t_1^2 < r_1^2\Delta \quad \text{and} \quad t_2^2 < r_2^2\Delta .$$

Then $|t_1t_2| < r_1r_2\Delta$, so (11.3.6) holds. Now (11.3.7) $\Leftrightarrow t_1^2t_2^2+r_1^2r_2^2\Delta^2 > r_1^2t_2^2\Delta + r_2^2t_1^2\Delta \Leftrightarrow (r_1^2\Delta-t_1^2)(r_2^2\Delta-t_2^2) > 0$, and the last inequality follows from (11.3.8).

(c) Suppose $t_1t_2+r_1r_2\Delta < 0$. Then from (b) we see that $r_1t_2+t_1r_2 > 0$, and we must show that $(r_1t_2+t_1r_2)^2\Delta > (t_1t_2+r_1r_2\Delta)^2$, i.e. the product is type (ii). But now $|t_1t_2| > r_1r_2\Delta$, so $t_1^2t_2^2 > r_1^2r_2^2\Delta^2 > r_1^2\Delta t_2^2$; hence $r_1^2\Delta < t_1^2$ and $r_2^2\Delta > t_2^2$, and the desired inequality follows as in (b).

(i)(iii). The proof is similar to that of (i)(ii).

(ii)(ii). In this case $r_1t_2+r_2t_1<0$ and $t_1t_2+r_1r_2\Delta>0$ are immediate, and (11.4.3) holds, so (11.4.2) follows as before, and the product is of type (ii).

(iii)(iii). The proof is similar to that of (ii)(ii).

(ii)(iii). Here $r_1^2\Delta>t_1^2$ and $r_2^2\Delta<t_2^2$, and this is used in a now familiar manner to show $(r_1t_2+r_2t_1)^2\Delta>(t_1t_2+r_1r_2\Delta)^2$; hence the product is of type (ii).

Suppose now that R is discrete ordered. Note that S is an integral domain, for if $(r_1x+s_1)(r_2x+s_2)=0$ in S, then

$$(r_1x+s_1)(r_2x+s_2)=r_1r_2(x^2+bx+c)\,,$$

so that $(r_2r_1x+r_2s_1)(r_2r_1x+s_2r_1)=r_1^2r_2^2p(x)$, which is impossible unless $r_1r_2=0$. Thus we cannot have $\Delta\ll 1$ in S. If Δ were in the same Archimedean class with 1, then $2x+b=\Delta^{1/2}$ would be in this class, but not be an integer (because $p(x)$ is irreducible), and S would not be discrete. Thus $\Delta\gg 1$.

To show the condition is not sufficient consider

(11.4) EXAMPLE. $R=Z[y]$, $p(x)=x^2+(y+4)x+2$, $S=R[x]/(p(x))$. Then $\Delta=y^2+8y+8\gg 1$ in the usual order for R, and $\delta=2x+(y+4)$. If we put $u=\delta-(y+3)=2x+1$, we find $0<u<1$ in the order defined for S by (11.3.1) to (11.3.3). Since S is an integral domain, it cannot be discrete ordered.

References

[1] L. Fuchs, *Partially ordered algebraic systems*, Reading, Mass. 1963.

[2] Ya. V. Hion, *Archimedean ordered rings*, Uspehi Mat. Nauk 9 (1954), pp. 237–242.

Reçu par la Rédaction le 20. 6. 1972

Regarding arc-wise accessibility in the plane

by

James Williams (Bowling Green, Ohio)

Abstract. Suppose S is a bounded relatively closed subset of the upper half-plane H, and that F is the set of all points on the x-axis which can be reached from the line $y=1$ by an arc lying in S. Question: Which sets F arise in this way? It is (with mild restrictions on S) necessary and sufficient that F be an $\mathcal{F}_{\sigma\delta}$ set. The corresponding problem where S is open is also briefly discussed. The results of the paper prove sharpness for a theorem by J. Gresser on almost arc-wise accessibility, and also demonstrate some possibilities for sectioning bounded sets.

§ 1. Closed access sets.

DEFINITIONS. Let I be the closed unit interval; we shall identify I with $I\times\{0\}$; an arc is a set that's homeomorphic with the half-open interval $[0,1)$. Unless otherwise indicated, the topologies to be considered are the relative plane topologies on I and on $I\times(0,1]$. Given a set $S\subseteq I\times(0,1]$, we shall say that a point $x\in I$ is *accessible* (or more accurately, 1-accessible) through S iff some arc in S meets $I\times\{1\}$ and touches x. x is *almost accessible* iff it is accessible through every neighborhood of S. Given $F\subseteq I$, S is an *access* set for F iff F is the set of all points in I accessible through S. S is a *parallel* access set for F iff in addition there is a disjoint family $\{\alpha_x|\ x\in F\}$ of arcs in S such that $\forall x\in F$, α_x meets $I\times\{1\}$ and touches x. S is a *definitive* access set for F iff in addition, $\forall x\in I-F$, no arc in S touches x.

Relative to the above definitions, Gresser's Lemma 1, p. 324 of [2] is equivalent to the following:

If S is an access set for a dense subset of I, then every point of I is almost accessible through the closure of S (in $I\times(0,1]$).

The theorem's sharpness follows from Example 7 and the following result whose proof will be established through a series of lemmas:

THEOREM 5. *For any subset F of I, the following are equivalent:*
1) *F has a closed parallel access set.*
2) *F has a closed definitive access set.*
3) *F has a closed definitive parallel access set.*
4) *F is an $\mathcal{F}_{\sigma\delta}$ subset of I.*