(i)(iii). The proof is similar to that of (i)(ii).

(ii)(ii). In this case $r_1t_2+r_2t_1<0$ and $t_1t_2+r_1r_2\Delta>0$ are immediate, and (11.4.3) holds, so (11.4.2) follows as before, and the product is of type (ii).

(iii)(iii). The proof is similar to that of (ii)(ii).

(ii)(iii). Here $r_1^2\Delta>t_1^2$ and $r_2^2\Delta<t_2^2$, and this is used in a now familiar manner to show $(r_1t_2+r_2t_1)^2\Delta>(t_1t_2+r_1r_2\Delta)^2$; hence the product is of type (ii).

Suppose now that R is discrete ordered. Note that S is an integral domain, for if $(r_1x+s_1)(r_2x+s_2)=0$ in S, then

$$(r_1x+s_1)(r_2x+s_2)=r_1r_2(x^2+bx+c)\,,$$

so that $(r_2r_1x+r_2s_1)(r_2r_1x+s_2r_1)=r_1^2r_2^2p(x)$, which is impossible unless $r_1r_2=0$. Thus we cannot have $\Delta\ll 1$ in S. If Δ were in the same Archimedean class with 1, then $2x+b=\Delta^{1/2}$ would be in this class, but not be an integer (because $p(x)$ is irreducible), and S would not be discrete. Thus $\Delta\gg 1$.

To show the condition is not sufficient consider

(11.4) EXAMPLE. $R=Z[y]$, $p(x)=x^2+(y+4)x+2$, $S=R[x]/(p(x))$. Then $\Delta=y^2+8y+8\gg 1$ in the usual order for R, and $\delta=2x+(y+4)$. If we put $u=\delta-(y+3)=2x+1$, we find $0<u<1$ in the order defined for S by (11.3.1) to (11.3.3). Since S is an integral domain, it cannot be discrete ordered.

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Regarding arc-wise accessibility in the plane

by

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Abstract. Suppose S is a bounded relatively closed subset of the upper half-plane H, and that F is the set of all points on the x-axis which can be reached from the line $y=1$ by an arc lying in S. Question: Which sets F arise in this way? It is (with mild restrictions on S) necessary and sufficient that F be an $\mathcal{F}_{\sigma\delta}$ set. The corresponding problem where S is open is also briefly discussed. The results of the paper prove sharpness for a theorem by J. Gresser on almost arc-wise accessibility, and also demonstrate some possibilities for sectioning bounded sets.

§ 1. Closed access sets.

DEFINITIONS. Let I be the closed unit interval; we shall identify I with $I\times\{0\}$; an arc is a set that's homeomorphic with the half-open interval $[0,1)$. Unless otherwise indicated, the topologies to be considered are the relative plane topologies on I and on $I\times(0,1]$. Given a set $S\subseteq I\times(0,1]$, we shall say that a point $x\in I$ is *accessible* (or more accurately, 1-accessible) through S iff some arc in S meets $I\times\{1\}$ and touches x. x is *almost accessible* iff it is accessible through every neighborhood of S. Given $F\subseteq I$, S is an *access* set for F iff F is the set of all points in I accessible through S. S is a *parallel* access set for F iff in addition there is a disjoint family $\{a_x|\ x\in F\}$ of arcs in S such that $\forall x\in F$, a_x meets $I\times\{1\}$ and touches x. S is a *definitive* access set for F iff in addition, $\forall x\in I-F$, no arc in S touches x.

Relative to the above definitions, Gresser's Lemma 1, p. 324 of [2] is equivalent to the following:

If S is an access set for a dense subset of I, then every point of I is almost accessible through the closure of S (in $I\times(0,1]$).

The theorem's sharpness follows from Example 7 and the following result whose proof will be established through a series of lemmas:

THEOREM 5. *For any subset F of I, the following are equivalent:*

1) *F has a closed parallel access set.*

2) *F has a closed definitive access set.*

3) *F has a closed definitive parallel access set.*

4) *F is an $\mathcal{F}_{\sigma\delta}$ subset of I.*

DEFINITION. If α is an arc in $I\times(0,1]$ from $I\times\{1\}$ to $I\times\{0\}$, it will be convenient to let $C^+(\alpha)$ (respectively, $C^-(\alpha)$) be the right (left) component of $I\times(0,1]-\alpha$.

LEMMA 1. *Every $\mathcal{F}_\sigma$ subset F of I has a closed parallel access set S such that the plane closure of S is just $F^-\cup S^-$.*

Proof. Let $M=I\times[\frac{1}{3},\frac{2}{3}]$. Suppose $F\subseteq I$ is an $\mathcal{F}_\sigma$ set. Let $\{F_n|\ n\in\omega\}$ be a nested sequence of closed sets whose union is F. For each n, let $\{G_{nk}|\ k\in\omega\}$ be a sequencing of the components of $I-F_n$. Proceeding by induction with $n\geqslant 0$, assume chosen (for $n>0$) or choose (for $n=0$) the following: a closed parallel access set S_n for F_n, where $S_0=F_0\times(0,1]$; and for each $k\in\omega$, a closed set $T_{nk}\subseteq G_{nk}\times(0,1]$ and a family

$$\{\varphi_{nk}\colon\ G_{nk}^-\times[0,1]\to T_{nk}\cup G_{nk}^-\times\{0\}|\ \ k\in\omega\}$$

of homeomorphisms (in the relative plane topologies) such that for each k,

(i) $S_n\cap\bigcup\{T_{nj}|\ j\in\omega\}=\emptyset$,

(ii) φ_{nk} is the identity map on $G_{nk}\times\{0\}$ and takes $G_{nk}^-\times\{1\}$ into $I\times\{1\}$,

(iii) T_{nk} *crosses* M at least $2n+1$ times in the sense that every arc in T_{nk} from $I\times\{1\}$ to $I\times\{0\}$ divides M into at least $2n+2$ components.

(iv) Any disjoint monotone sequence of arcs in S_n that is cofinal with a similar sequence in the complement of S_n converges to an arc.

Let C be a component of $T_{nk}\cap M$ such that C separates T_{nk}, and every arc in T_{nk} from $I\times\{1\}$ to $I\times\{0\}$ crosses M in C. Then $\varphi_{nk}^{-1}[C]$ is a relatively closed subset of $G_{nk}\times[0,1]$ separating $G_{nk}\times\{0\}$ and $G_{nk}\times\{1\}$. Let ψ_{nk} be a homeomorphism of $G_{nk}^-\times[0,1]$ into itself such that $\psi_{nk}[G_{nk}^-\times(0,1]]$ crosses $\varphi_{nk}^{-1}[C]$ three times, ψ_{nk} restricted to $G_{nk}\times\{0\}$ is the identity map, and ψ_{nk} takes $G_{nk}^-\times(0,1]$ into $G_{nk}\times(0,1]$ and takes $G_{nk}^-\times\{1\}$ into $G_{nk}\times\{1\}$. Then $\varphi_{nk}\circ\psi_{nk}[G_{nk}^-\times(0,1]]$ is a subset of T_{nk} which crosses M at least $2n+3$ times. For each $k\in\omega$, let

$$K_k=\varphi_{nk}\circ\psi_{nk}[(F_{n+1}\cap G_{nk}^-)\times(0,1]]\,.$$

Let $S_{n+1}=S_n\cup\bigcup\{K_k|\ k\in w\}$. Each K_k is closed in $I\times(0,1]$, and so is S_{n+1} for the following reasons: any sequence not eventually in a given K_k is contained in a corresponding sequence of K_k's, and is thus related to a corresponding sequence of G_{nk}'s, which contains a monotone subsequence converging to some $y\in F_n$. But then the corresponding subsequence of K_k's converges to an arc α in S_n from $I\times\{1\}$ to y, by property (iv) above. Thus the original sequence has a limit point on $\alpha\cup\{y\}$. That S_{n+1} is a parallel access set and satisfies (iv) now follows from the same properties for the K_k's. To complete the induction cycle, let $G_{n+1,k}$ be a component of $I-F_{n+1}$. $G_{n+1,k}$ is contained in some G_{nj}. Let $\varphi_{n+1,k}$ be the restriction of $\varphi_{nj}\circ\psi_{nj}$ to $G_{n+1,k}^-\times[0,1]$, and set $T_{nk}=\varphi_{n+1,k}[G_{n+1,k}^-\times(0,1]]$. Then $T_{n+1,k}$, $\varphi_{n+1,k}$, and S_{n+1} clearly satisfy conditions (i)-(iii) above.

Finally, let $S=\bigcup\{S_n|\ n\in\omega\}$. It is clear from the construction that S, and thus S^- contains a disjoint family $\{\alpha_x|\ x\in F\}$ of arcs, where each α_x meets $I\times\{1\}$ and touches x. Hence the proof will be complete if we show that no arc in S^- meets $I\times\{1\}$ and touches a point of $I-F$. Suppose to the contrary that γ is such an arc. Then γ touches each given G_{nk}. Let α^- and α^+ be the left and right boundaries of $\varphi_{nk}[G_{nk}\times(0,1]]$, and let β^- and β^+ be the left and right boundaries of $\varphi_{nk}\circ\psi_{nk}[G_{nk}\times(0,1]]$. Because ψ_{nk} takes $G_{nk}^-\times(0,1]$ into $G_{nk}\times(0,1]$, $C^+(\alpha^-)\cap C^-(\beta^-)$ and $C^+(\beta^+)\cap C^-(\alpha^+)$ are connected open subsets of $I\times(0,1]-S^-$. Since γ touches G_{nk} it lies between these two sets and is thus contained in

$$\big(C^+(\beta^-)\cap C^-(\beta^+)\big)^-=\varphi_{nk}\circ\psi_{nk}[G_{nk}^-\times(0,1]\,.$$

Hence α crosses M at least $2n+3$ times. But n was arbitrary, a contradiction. ∎

LEMMA 2. *Every $\mathcal{F}_{\sigma\delta}$ set in I has a closed definitive parallel access set.*

Proof. Let K be the Cantor ternary set. Let φ be an increasing (continuous) function of I onto I which is constant on each component of $I-K$, and maps different components to different numbers. For each $n>0$, let $\{J_{nk}|\ k<2^n\}$ be the set of closed intervals left in I after the nth stage of the construction of K (so that the length of each J_{nk} is 3^{-n}). Suppose F is an $\mathcal{F}_{\sigma\delta}$ subset of I, then $E=\varphi^{-1}[F]\cap K$ is a nowhere dense $\mathcal{F}_{\sigma\delta}$ set in K. Let $\{E_n|\ n\in\omega\}$ be a nested sequence of $\mathcal{F}_\sigma$ sets in K whose intersection is E. For each $n>0$ and $k<2^n$, let $E_{nk}=E_n\cap J_{nk}$. Let S_{nk} be a closed parallel access set for E_{nk} which is contained in $J_{nk}\times(0,1]$, and is of the type constructed above. Let T_{nk} be the reflection of S_{nk} about the line $I\times\{1\}$. Notice that if α is an open arc in $(S_{nk}\cup T_{nk})^-$ from $I\times\{2\}$ to $I\times\{0\}$, then $\alpha\subseteq S_{nk}\cup T_{nk}$ since $(S_{nk}\cup T_{nk})^-\cap I\times\{0,2\}$ is nowhere dense, and thus for some $x\in E_{nk}$, α goes from $\langle x,2\rangle$ to $\langle x,0\rangle$ by parallelness. Let ψ be the homeomorphism of $I\times(0,1]$ onto itself given by

$$\psi(x,y)=\langle yx+(1-y)\varphi(x),y\rangle\,.$$

For each $n>0$, let θ_n be the increasing linear function from the interval $(0,2)$ onto the interval $\big(1/(n+1),1/n\big)$, and let $\psi_n(x,y)=\psi\big(x,\theta_n(y)\big)$. Let

$$S=\bigcup\{\psi_n[S_{nk}\cup T_{nk}]|\ \ 0\leqslant k<n\in\omega\}\,.$$

First, each given $x\in F$ is accessible through S^- from $I\times\{1\}$: Choose $u\in E$ so that $\varphi(u)=x$. For each $n>0$, there is a unique k_n such that $u\in J_{nk_n}$; let α_n be an arc in S_{nk_n} from $I\times\{1\}$ to u, and let β_n be its reflection in T_{nk_n}. Let $\alpha=\bigcup\{\psi_n[\alpha_n\cup\beta_n]|\ n>0\}\cup\{\psi_n(u,2)|\ n>0\}$. Then α is the union of a sequence of arcs. The points $\psi_n(u,2)$ all lie in a straight line from $\langle u, 1\rangle$ to $\langle x, 0\rangle$. Hence α will be an arc in S^- from $I\times\{1\}$ to x provided it is locally connected at x. But the trapezoids with bases $\psi_n[J_{nk_n}\times\{2\}]$ and $\varphi[J_{nk_n}]$ converge to x, and α is eventually in each such trapezoid. Now suppose that $n>0$ and α is an arc in S^- which meets $I\times\{1/n\}$ and touches some point $x\in I\times\{0\}$. The parallelness of the sets $S_{nk}\cup T_{nk}$ clearly carries over to S^-, so that $\forall m>n$, $\alpha\cap I\times(1/m+1, 1/m)$ has just one component, which belongs to some set of the form $\psi_m[S_{mk}\cup T_{mk}]$. For each $m>n$, let α_m be the inverse image of α in S_{mk}. Then α_m is an access arc in S_{mk} for some point $u\in E_{mk}$. Consequently, by parallelness, α goes through all points of the form $\psi_m(u, 2)$, for $m>n$, and $x=\varphi(u)$. But then u apparently belongs to each E_m for $m>n$, and thus to E since the E_m's are nested. Hence $x\in F$. Therefore S^- is a closed definitive parallel access set. ■

LEMMA 3. *Suppose S is a closed access set for F, $G\subseteq I$ is a $\mathcal{G}_\delta$ set, and $\{\alpha_x\mid x\in F\cap G\}$ is a disjoint family of arcs in S such that each α_x goes from $I\times\{1\}$ to x; then $F\cap G$ is an $\mathcal{F}_{\sigma\delta}$ set. In particular, if F has a closed parallel access set, it is an $\mathcal{F}_{\sigma\delta}$ set.*

Proof. For each connected set C that meets $I\times\{1\}$ and touches $I\times\{0\}$, and each $q>0$, let $l_q(C)$ be the least rational number p/q such that some connected subset C' of C meets $I\times\{1\}$, touches $I\times\{0\}$, and is the union of p connected sets of diameter less than $1/q$ — provided such is possible, otherwise let $l_q(C)=\infty$. Let L be the set of all 2-sided limit points of $F\cap G$ belonging to $F\cap G$. For each $y\in L$, let

$$\zeta_y=\bigcap\{C^+(\alpha_x)\mid x<y\}\cap\bigcap\{C^-(\alpha_x)\mid x>y\}\,.$$

Let $A=\{y\in L\mid \zeta_y$ has non-empty interior$\}$. Notice that A and $F\cap G-L$ are countable. Let $(\ )^*$ be the closure operator on $I\times I$.

Step I. For each $y\in L-A$, ζ_y^* contains a unique sub-continuum ζ_y which is irreducible between $I\times\{1\}$ and $I\times\{0\}$.

Proof. By Proposition 11.2, p. 17 of [4], we may let H, K be irreducible sub-continua of ζ_y^* between $I\times\{1\}$ and $I\times\{0\}$. If $H\neq K$, then $H\cap K$ isn't connected, and thus $H\cup K$ separates $I\times I$ into 3 components as a result of Theorem 22, p. 175 of [3]. But then from the definition of ζ_y, ζ_y^* must contain the middle component, a contradiction.

Step II. For any $y\in L-A$, ζ_y contains an arc from $I\times\{1\}$ to $I\times\{0\}$ iff $\forall q>0$, $l_q(\zeta_y)$ is finite.

Proof. The condition on ζ_y is equivalent to Whyburn's property S, and the assertion follows directly from Proposition 15.7, p. 23, and Theorem 5.1, p. 36 of [4].

Step III. Suppose ζ_y doesn't contain an arc and $y\in L-A$; then there is an integer $q_0>0$ such that for each $q>q_0$ and each sequence $\{x_n\mid n\in\omega\}$ in $F\cap G$ converging to y, $\lim\limits_{n\to\infty} l_q(\alpha_{x_n})=\infty$.

Proof. Notice that to find the desired integer q_0, it suffices to first find an integer q_1 which works for increasing sequences and a second q_2 for decreasing sequences, and then set $q_0=\max\{q_1, q_2\}$. q_1 for example may be found as follows: Let ξ_y be the intersection of ζ_y^* and the boundary of $\bigcup\{C^-(\alpha_x)\mid x<y\}$. A moment's thought shows that ξ_y is connected between $I\times\{1\}$ and $I\times\{0\}$. But it isn't locally connected since it doesn't contain an arc. Thus by Theorem 12.1, p. 18 of [4], we can find a point $a\in\xi_y$ and a number $\varepsilon>0$ so that if R is the ε-disk about a, we may then choose a sequence $\{C_j\mid j\in\omega\}$ of disjoint components of $\xi_y\cap R^*$ converging to a continuum containing a. It suffices to set $q_1=6/\varepsilon$: Pick $q>6/\varepsilon$, and let $\{x_n\mid n\in\omega\}$ be an increasing sequence in $F\cap G$ converging to y. Let p be any positive integer; let j_0 be such that $\forall j>j_0$, a is less than $\varepsilon/3$ from C_j. For each j with $j_0\leqslant j\leqslant j_0+p$, choose $a_j\in C_j$ so that $\operatorname{dist}(a, a_j)<\varepsilon/3$. For any given C_i and C_j with $i\neq j$, there is a neighborhood V of ξ_y such that C_i and C_j are in different components of $V\cap R^*$; for if not, then for each neighborhood V of ξ_y, let U_V be the component of $V\cap R^*$ containing C_i (and thus C_j also). But then the intersection of the U_V's is a continuum in $\xi_y\cap R^*$ containing C_i and C_j, a contradiction. So let V be a neighborhood of ξ_y such that for $i\neq j$, with $j_0\leqslant i$, $j\leqslant j_0+p$, C_i and C_j lie in different components of $V\cap R^*$. For each i, let V_i be the component of $V\cap R^*$ containing C_i. By definition of ξ_y and a compactness argument, we may now choose n_0 so that $\forall n>n_0$, $\alpha_{x_n}\subseteq V$. Choose $n_1\geqslant n_0$ so that for each $n>n_1$ and each j with $j_0\leqslant j\leqslant j_0+p$, the intersection of V_j with the $\varepsilon/3$-neighborhood of a_j meets α_{x_n}. Pick $n>n_1$. For each j, with $j_0\leqslant j\leqslant j_0+p$, let β_j be a component of $\alpha_{x_n}\cap V_j$ which comes within $\varepsilon/3$ of a_j. Each β_j meets the boundary of R, so that its diameter is greater than $\varepsilon/3$. Consequently, α_{x_n} contains $p+1$ disjoint arcs of diameter greater than $\varepsilon/3$. Clearly, if $q>6/\varepsilon$, then $l_q(\alpha_{x_n})\geqslant p/q$.

Steep IV. Finally, $F\cap G$ is an $\mathcal{F}_{\sigma\delta}$ set.

Proof. For each $p, q>0$, let F_{pq} be the closure of $\{x\in L-A\mid l_q(\zeta_x)\leqslant p/q\}$. Let $F_q=\bigcup\{F_{pq}\mid 0<p<\infty\}$, and let $M=\bigcap\{F_q\mid q>0\}$. Then $G\cap M\cap L$ is and $\mathcal{F}_{\sigma\delta}$; since $F\cap G$ and $F\cap G\cap L-A$ differ by a countable set, it suffices to show that $F\cap G\supseteq G\cap M\cap L\supseteq F\cap G\cap L-A$. First, if $x\in F\cap G\cap L-A$, then $\alpha_x\subseteq\zeta_x$, so that each $l_q(\zeta_x)$ is finite and $x\in F_q$; hence $x\in M$, and $x\in G\cap M\cap L$. Now suppose $y\in G\cap M\cap L$, but $y\notin F\cap G$. Then for some given q_0, $\forall q>q_0$, $l_q(\zeta_y)$ is infinite by Step II and the fact $y\notin F$. Pick $q>q_0$; since $y\in M$, y belongs to some F_{pq}. Hence there is a sequence $\{x_n\mid n\in\omega\}$ in $L\cap F_{pq}$ converging to y. But then $\forall n$, $l_q(\alpha_{x_n})\leqslant p$. This contradicts Step III. Hence $F\cap G\supseteq G\cap M\cap L\supseteq F\cap G\cap L-A$. ■

LEMMA 4. *If S is a closed definitive access set for F, then F is an $\mathcal{F}_{\sigma\delta}$.*

Proof. The first step is to establish the following: If x is a 2-sided limit point of F, and $x \notin F$, then $\exists n > 0$, x "separates" $S \cap I \times (0, 1/n]$ in the sense that if γ_1, γ_2 are arcs in $S \cap I \times (0, 1/n]$ from $I \times \{1/n\}$ to points $y_1, y_2 \in I$, with $y_1 < x < y_2$, then $\gamma_1 \cap \gamma_2 = \emptyset$.

Proof. Suppose to the contrary that x is a 2-sided limit point of F and that $\forall n > 0$, there is an open arc in $S \cap I \times (0, 1/n]$ from $y \in F$ to $y' \in F$ with $y < x < y'$. We can show that x is a 1-access point of F. Let $\{y_n \mid n \in \omega\}$ and $\{y_n' \mid n \in \omega\}$ be sequences in F converging to x such that $y_n < x < y_n'$ and $\forall n > 0$, $|y_n - y_n'| < 1/n$. For each n, let β_n and β_n' be arcs in S from $I \times \{1\}$ to y_n and y_n'; let γ_n be an open arc in $S \cap I \times (0, 1/n]$ which touches F on both sides of x. Then, with some thought, one can see that there is an arc in S made from pieces of the β_n's, β_n''s, and γ_n's which goes from $I \times \{1\}$ to x.

Now let L be the set of all 2-sided limit points of F; $F^- - L$ is countable. For each n, let G_n be those points of $L - F$ which "separate" $S \cap I \times \times (0, 1/n]$ in the above sense. Notice that

$$\begin{aligned} F \cup (F^- - L) &= F^- \cap \big(F \cup (I - L)\big) \\ &\supseteq F^- \cap \big(F \cup (I - \bigcup \{G_n^- \mid n \in \omega\})\big) \\ &\supseteq F . \end{aligned}$$

It suffices to show that $F \cup (I - \bigcup \{G_n^- \mid n \in \omega\})$ is an $\mathcal{F}_{\sigma\delta}$ set since then $F^- \cap \big(F \cup (I - \bigcup \{G_n \mid n \in \omega\})\big)$ is also, and this differs from F by a countable set. But

$$\begin{aligned} F \cup (I - \bigcup \{G_n^- \mid n \in \omega\}) &= \bigcap \{F \cup (I - G_n^-) \mid n \in \omega\} \\ &= \bigcap \{(I - G_n^-) \cup (F \cap G_n^-) \mid n \in \omega\} . \end{aligned}$$

Hence it suffices to show that for each n, $F \cap G_n^-$ is an $\mathcal{F}_{\sigma\delta}$. Pick n; let L_n be the set of 2-sided limit points of G_n. Then $G_n^- - L_n$ is countable, and thus L_n is a $\mathcal{G}_\delta$ set. Let $\{\alpha_x \mid x \in F \cap L_n\}$ be a family of arcs in S with each α_x from $I \times \{1\}$ to x, and let β_x be the component of $\alpha_x \cap I \times (0, 1/n]$ that touches x. Then, by the definition of G_n and L_n, $\{\beta_x \mid x \in F \cap L_n\}$ is a disjoint family in $S \cap I \times (0, 1/n]$, and $S \cap I \times (0, 1/n]$ is in effect a closed access set for F, since S is a definitive access set for F. Thus, by the previous lemma, $F \cap L_n$ is an $\mathcal{F}_{\sigma\delta}$. But then so is $F \cap G_n^-$, since $G_n^- - L_n$ is countable. ∎

§ 2. Open access sets. If S is an open simply connected access set for F, it seems likely that F must be an $\mathcal{F}_{\sigma\delta}$ set in view of the similar Theorem 9.10 of [1]. At any rate, the following examples due to J. Gresser show that both the rationals and the irrationals have open access sets. The symmetries of the construction in Example 7 are such that it will be convenient there to work within $[-1, 1] \times I$ instead of in $I \times I$.

EXAMPLE 6 (of an open access set for the set Q of rational numbers in the unit interval). Let K be a Cantor set, and let $\{C_q \mid q \in Q\}$ be the set of components of $I - K$, indexed so that $\forall p, q \in Q, p < q$ iff lub C_p $<$ lub C_q. Let S be the union of all open triangles with base $C_q \times \{1\}$ and tip $\langle q, 0\rangle$, for $q \in Q$. Then S is the required open access set for Q. ∎

EXAMPLE 7 (of a simply connected open access set S for a set $F \subseteq [-1, 1]$ such that S^- is a definitive closed access set for F, and $[-1, 1] - F$ is a countable dense subset of $[-1, 1]$). Let L_0 be the closure of the graph of the polar equation

$$\theta = \pi/8\,[3 + r\sin(\pi/\sin(\pi/r))] \quad \text{for} \quad 0 < r < 2/3 .$$

Let $L = L_0 \cup \{0\} \times (2/3, 1]$. Then L is irreducibly connected between $\langle 0, 1\rangle$ and $\langle 0, 0\rangle$, converges to $\langle 0, 0\rangle$, and fails to contain any arc that touches the origin. Let D_0 be the intersection of the closed unit disk with the upper half-plane H. For each integer $j > 0$, let D_j be the intersection of H with the closed unit disk of radius $1/2^{j+1}$ centered at $3/2^{j+1}$. The sets D_j are mutually disjoint, and their plane closures cover I. Moreover, each lies below the line $\theta = \pi/6$. For each integer $j > 0$, let p_j be the top point of D_j. Let B_0 be the disk of radius $1/9$ about $\langle 0, 1\rangle$. Let G be the interior of the right component of $D_0 - L \cup \bigcup \{D_j \mid j > 0\}$. Since G is analytically equivalent to a circle, one may construct a disjoint family $\{\alpha_j \mid j > 0\}$ of open arcs such that each α_j goes from $\langle 0, 1\rangle$ to p_j, and $\lim_{j \to \infty} \alpha_j = L$. Then let $\{\beta_j \mid j > 0\}$ be a family of open neighborhoods of the α_j's such that each β_j is contained in G and $\{\beta_i^* - B_0 \mid j > 0\}$ is also disjoint, where $(\ )^*$ denotes plane-closure. Finally, let $M_+ = B_0 \cup \cup \bigcup \{\beta_j \mid j > 0\}$, and let N_+ be the closure (in D_0) of $M_+ \cup L \cup \cup \bigcup \{D_j \mid j > 0\}$. For M_+, N_+, and each α_j and D_j, let M_-, N_-, α_{-j}, and D_{-j} be the reflection of the corresponding set about the y-axis. Our initial construction is then completed by setting $M = M_- \cup M_+$, and $N = N_- \cup N_+$. Notice that the origin is not accessible in N. This property will be carried over to the diadic rationals in $[-1, 1]$ through a series of transformations, and the final set S will be the union of the images of M under these maps.

For each integer j, let f_j be the afine transformation which takes D_0 onto D_j. Let $\mathfrak{Z}$ be the set of all finite sequences of integers, and for each $\boldsymbol{j} = \langle j_1, j_2, \ldots, j_n\rangle \in \mathfrak{Z}$, let $f_{\boldsymbol{j}} = f_{j_1} \circ f_{j_2} \circ \ldots \circ f_{j_n}$. Let

$$S = \bigcup \{f_{\boldsymbol{j}}[M] \mid \boldsymbol{j} \in \mathfrak{Z}\} \cap [-1, 1] \times (0, 1] .$$

Then S is open and simply connected. The set $F = \{f_{\boldsymbol{j}}(0) \mid \boldsymbol{j} \in \mathfrak{Z}\}$ is the set of diadic rationals in the interval $(-1, 1)$.

For the sake of argument, for each $\boldsymbol{j} \in \mathfrak{Z}$, let $D_{\boldsymbol{j}} = f_{\boldsymbol{j}}[D_0]$ and let $q_{\boldsymbol{j}} = f_{\boldsymbol{j}}(0)$ be its center. Since the origin isn't accessible in N, it follows that q_j isn't accessible in $D_j \cap S^- \subseteq f_j(N)$, so that q_j isn't an accessible point of S^-. On the other hand, if $x \in [-1, 1]-F$, then there is an infinite sequence $\{j_n|\ n > 0\}$ of non-zero integers such that x is in the closure of each D_{j_n}, for $j_n = \{j_k|\ k \leqslant n\}$. In this case, $\alpha = \bigcup \{f_{j_n}(\alpha_{j_n})|\ n \in \omega\}$ is an arc from $\langle 0, 1\rangle$ to x contained in S. ∎

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Semigroups which admit few embeddings

by

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Abstract. $S(X)$ is the semigroup, under composition, of all continuous selfmaps of the topological space X. Two classes of spaces are given such that if X is from the first and Y is from the second and φ is any isomorphism from $S(X)$ into $S(Y)$, then there is a unique idempotent v of $S(Y)$ and a unique homeomorphism h from the range of v onto X such that $\varphi(f) = h \circ f \circ h^{-1} \circ v$ for each f in $S(X)$. It follows from this that there is a fairly extensive class of spaces such that the semigroup of precisely three spaces from the class can be embedded in $S(I)$ and the semigroups of precisely five can be embedded in $S(R)$ where I and R denote respectively the closed unit interval and the space of real numbers.

1. Introduction. The symbol $S(X)$ is used to denote the semigroup, under composition, of all continuous selfmaps of the topological space X. It is well known that there exist semigroups $S(X)$ into which many other such semigroups may be embedded. In fact, given any collection of semigroups, one need only choose a set X whose cardinality is not less than that of any of the semigroups and then each semigroup of the collection can be embedded in $S(X)$ where X is given the discrete topology. In this case, $S(X)$ is, of course, simply the full transformation semigroup on X. The problem is made a bit more difficult by requiring that X satisfy various topological conditions and when we discuss some examples, we will see that for each collection of semigroups, one can produce an arcwise connected metric space X so that each semigroup of the collection can be embedded in $S(X)$. However, such semigroups are really not our main concern here. We are much more interested in semigroups at the other end of the spectrum, that is, in semigroups of continuous functions into which very few other such semigroups can be embedded.

The main theorem of the paper is proven in section 4 and it gives two classes of spaces such that if X is from the first and Y is from the second, then for each monomorphism φ from $S(X)$ into $S(Y)$, there exists a unique idempotent v of $S(Y)$ and a unique homeomorphism h from X onto the range of v such that $\varphi(f) = h \circ f \circ h^{-1} \circ v$ for each f in $S(X)$. We then look at some special cases in more detail and to give some idea of the type of result we get, we mention the essential ingredients of a result on $S(I)$ and one on $S(R)$ where I is the closed unit interval and