Closure-preserving covers

by

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Abstract. Let X be a paracompact space. If X is σ-locally compact (σ-discrete), then X has a closure-preserving cover consisting of compact (resp. finite) sets. If X has a closure-preserving cover by finite sets, then X is totally paracompact. These and some more general results are established by considering order locally finite and order star-finite covers.

The present paper is concerned with covering properties of topological spaces. We shall study the properties

(1) the space X has a closure-preserving cover consisting of compact subsets of X,

(2) the space X has a closure-preserving cover consisting of finite subsets of X, and some related ones.

H. Tamano [10] put forward the question whether (1) implies the paracompactness of X. This question has been answered negatively by H. B. Potoczny [7]. In paper [8] he describes the remarkable structural characteristics of spaces possessing property (1). We dealt with property (1) and its related property in [12]. Paper [14] (announced in [13]) contains several results concerning (1) and (2) established by game-theoretical methods.

The topological terminology is that of [2]. Each space is assumed to be completely regular. Natural numbers are denoted by the letters $m, n, k, \ldots$ and ordinal numbers are denoted by the letters $\alpha, \beta, \gamma, \ldots$ $\ldots, \xi, \eta, \zeta, \ldots$

Recall that a collection $\{A_i\colon i \in I\}$ of subsets of a space X is said to be *order locally finite* if we can introduce a well-ordering $<$ in the index set I so that for each $i \in I$ the family $\{A_j\colon j < i\}$ is locally finite at each point of A_i.

Since every well-ordered set is order-isomorphic to an initial segment of ordinal numbers, we shall use the notation $\{A_\xi\colon \xi < \alpha\}$ instead of $\{A_i\colon i \in I\}$. Order locally finite covers were introduced and studied by Y. Katuta [3].

We say that a collection $\{A_\xi\colon \xi < \alpha\}$ of subsets of a space X is *order star-finite* if for each $\xi < \alpha$ the set A_ξ meets at most finitely many A_η with $\eta < \xi$.

Clearly, every order star-finite collection of open sets in X is order locally finite.

LEMMA 1. *Let X be a paracompact space with* $\dim X = 0$. *If X has two order locally finite covers $\{E_\xi\colon \xi<\alpha\}$ and $\{U_\xi\colon \xi<\alpha\}$, where E_ξ is closed in X and U_ξ is an open nbhd of E_ξ for each $\xi<\alpha$, then X has two order star-finite covers $\{F_\eta\colon \eta<\beta\}$ and $\{V_\eta\colon \eta<\beta\}$, where $\{F_\eta\colon \eta<\beta\}$ refines $\{E_\xi\colon \xi<\alpha\}$, $\{V_\eta\colon \eta<\beta\}$ refines $\{U_\xi\colon \xi<\alpha\}$, and F_η is closed in X and V_η is an open nbhd of F_η for each $\eta<\beta$.*

Proof. We define, by induction with respect to ξ, two discrete families $\{E_{\xi,i}\colon i\in I_\xi\}$ and $\{U_{\xi,i}\colon i\in I_\xi\}$ of subsets of X so that $E_\xi = \bigcup\{E_{\xi,i}\colon i\in I_\xi\}$ and $E_{\xi,i}\subseteq U_{\xi,i}\subseteq U_\xi$ for each $i\in I_\xi$. We set $\{E_{0,i}\colon i\in I_0\} = \{E_0\}$ and $\{U_{0,i}\colon i\in I_0\} = \{U_0\}$. Let us assume that for some $\xi<\alpha$ and for each $\eta<\xi$ the families $\{E_{\eta,i}\colon i\in I_\eta\}$ and $\{U_{\eta,i}\colon i\in I_\eta\}$ are defined. For each point $x\in E_\xi$ there is an open nbhd U_x, where $U_x\subseteq U_\xi$ and U_x meets at most finitely many U_η with $\eta<\xi$. Thus the set $K_x = \{\eta<\xi\colon U_x\cap U_\eta\neq 0\}$ is finite. Let $\eta\in K_x$. The family $\{U_{\eta,i}\colon i\in I_\eta\}$ is discrete (in X), therefore there exists an open nbhd $U_{x,\eta}$ of x, where $U_{x,\eta}\subseteq U_x$ and $U_{x,\eta}$ meets at most one $U_{\eta,i}$ with $i\in I_\eta$. Let us put $V_x = U_x$ if $K_x = 0$, and $V_x = \bigcap\{U_{x,\eta}\colon \eta\in K_x\}$ if $K_x\neq 0$. Then V_x is an open nbhd of x and V_x meets at most finitely many sets $U_{\eta,i}$ with $i\in I_\eta$ and $\eta<\xi$. Since $\{V_x\colon x\in E_\xi\}$ is an open cover of E_ξ and X is a paracompact space with $\dim X = 0$, there is a discrete family $\{U_{\xi,i}\colon i\in I_\xi\}$ of closed-open subsets of X, where $\{U_{\xi,i}\colon i\in I_\xi\}$ covers E_ξ and refines $\{V_x\colon x\in E_\xi\}$. Let us put $E_{\xi,i} = E_\xi\cap U_{\xi,i}$ for each $i\in I_\xi$. Now the families $\{E_{\xi,i}\colon i\in I_\xi\}$ and $\{U_{\xi,i}\colon i\in I_\xi\}$ have the desired properties. Thus $\{E_{\xi,i}\colon i\in I_\xi$ and $\xi<\alpha\}$ and $\{U_{\xi,i}\colon i\in I_\xi$ and $\xi<\alpha\}$ are defined. Let $<_\xi$ be a well-ordering of I_ξ, where $\xi<\alpha$. Then we define a well-ordering $\precsim$ of $\{(\xi,i)\colon i\in I_\xi$ and $\xi<\alpha\}$ as follows: $(\xi,i)\precsim(\eta,j)$ if and only if $\xi<\eta$, or $\xi=\eta$ and $i<_\xi j$. Hence the covers $\{E_{\xi,i}\colon i\in I_\xi$ and $\xi<\alpha\}$ and $\{U_{\xi,i}\colon i\in I_\xi$ and $\xi<\alpha\}$ are order star-finite. Thus we may write $E_{\xi,i} = F_\eta$ and $U_{\xi,i} = V_\eta$, where $i\in I_\xi$, $\xi<\alpha$, $\eta<\beta$ and $\{(\xi,i)\colon i\in I_\xi$ and $\xi<\alpha\}$ is order-isomorphic to $\{\eta\colon \eta<\beta\}$. The covers $\{F_\eta\colon n<\beta\}$ and $\{V_\eta\colon \eta<\beta\}$ satisfy our requirements.

Remark 1. Lemma 1 will remain true if we replace "X is paracompact and $\dim X = 0$" by "X is strongly paracompact" or by "X is paracompact and E_ξ is a locally compact closed subset of X for each $\xi<\alpha$".

Recall that a collection $\{A_i\colon i\in I\}$ of subsets of a space X is said to be *closure-preserving* if for each $J\subseteq I$ we have $\operatorname{cl}\bigcup\{A_i\colon i\in J\} = \bigcup\{\operatorname{cl}A_i\colon i\in J\}$.

LEMMA 2. *If X has two order star-finite covers $\{E_\xi\colon \xi<\alpha\}$ and $\{U_\xi\colon \xi<\alpha\}$, where E_ξ is closed in X and U_ξ is an open nbhd of E_ξ for each $\xi<\alpha$, then there is a family $\{T_\xi\colon \xi<\alpha\}$ of finite subsets of $\{\xi\colon \xi<\alpha\}$ so that $\{\bigcup\{E_\eta\colon \eta\in T_\xi\}\colon \xi<\alpha\}$ is a closure-preserving cover of X.*

Proof. Let $\{E_\xi\colon \xi<\alpha\}$ and $\{U_\xi\colon \xi<\alpha\}$ be the order star-finite covers. We set $T_\xi^0 = \{\xi\}$ and $T_\xi^1 = \{\eta<\xi\colon U_\eta\cap U_\xi\neq 0\}$ for each $\xi<\alpha$. Assume that T_ξ^n is defined for some $n\geqslant 0$. Then we set $T_\xi^{n+1} = \bigcup\{T_\eta^1\colon \eta\in T_\xi^n\}$. It is easy to prove (by induction with respect to n) that each T_ξ^n is a finite set. Now let us put $T_\xi = \bigcup\{T_\xi^n\colon n\geqslant 0\}$. It is also easy to prove (by induction with respect to ξ) that each T_ξ is a finite set. Let us put $F_\xi = \bigcup\{E_\eta\colon \eta\in T_\xi\}$, where $\xi<\alpha$. We shall now prove that the family $\{F_\xi\colon \xi<\alpha\}$ is closure-preserving. Let $S\subseteq\{\xi\colon \xi<\alpha\}$, let

$$x\in\operatorname{cl}\bigcup\{F_\xi\colon \xi\in S\}$$

and let $\beta = \inf\{\xi\colon x\in E_\xi\}$. Since $x\in E_\beta\subseteq U_\beta$, it follows that $U_\beta\cap\bigcup\{F_\xi\colon \xi\in S\}\neq 0$. However,

$$\bigcup\{F_\xi\colon \xi\in S\} = \bigcup\{E_\eta\colon \eta\in T_\xi^n,\ n\geqslant 0 \text{ and } \xi\in S\}.$$

Thus the set $R = \{\eta\colon U_\beta\cap E_\eta\neq 0,\ \eta\in T_\xi^n,\ n\geqslant 0 \text{ and } \xi\in S\}$ is non-void.

Case 1. $\sup R\leqslant\beta$. Then R is finite, because the family $\{U_\xi\colon \xi<\alpha\}$ is order star-finite. Hence $x\in\bigcup\{E_\eta\colon \eta\in R\}\subseteq\bigcup\{F_\xi\colon \xi\in S\}$.

Case 2. $\sup R>\beta$. Then there exist a $\eta\in R$ with $\eta>\beta$. Hence there exist a $n\geqslant 0$ and $\xi\in S$ such that $\eta\in T_\xi^n$ and $\beta\in T_\xi^{n+1}$, because $U_\beta\cap U_\eta\neq 0$ and $\beta<\eta$. Since $x\in E_\beta$, $\beta\in T_\xi$ and $\xi\in S$, thus $x\in\bigcup\{F_\xi\colon \xi\in S\}$. Therefore $\{F_\xi\colon \xi<\alpha\}$ is closure-preserving.

LEMMA 3 (K. Nagami [6]). *For each paracompact space X there is a paracompact space X' with* $\dim X' = 0$, *so that there is a perfect map f from X' onto X.*

THEOREM 1. *If X has two order locally finite covers $\{E_\xi\colon \xi<\alpha\}$ and $\{U_\xi\colon \xi<\alpha\}$, where E_ξ is compact and U_ξ is an open nbhd of E_ξ for each $\xi<\alpha$, then X has a closure-preserving cover by compact sets.*

Proof. By a theorem of Y. Katuta [3] the space under the assumption of Theorem 1 is paracompact. Thus, by Lemma 3, there is a paracompact space X' with $\dim X' = 0$, so that there is a perfect map f from X' onto X. It is easy to verify that $\{f^{-1}(E_\xi)\colon \xi<\alpha\}$ and $\{f^{-1}(U_\xi)\colon \xi<\alpha\}$ are order locally finite covers of X', where $f^{-1}(E_\xi)$ is compact and $f^{-1}(U_\xi)$ is an open nbhd of $f^{-1}(E_\xi)$ for each $\xi<\alpha$. Applying Lemma 1, we get two order star-finite covers $\{F_\eta\colon \eta<\beta\}$ and $\{V_\eta\colon \eta<\beta\}$ of X', where F_η is compact and V_η is an open nbhd of F_η for each $\eta<\beta$. Applying Lemma 2, we get the family $\{T_\eta\colon \eta<\beta\}$ of finite subsets of $\{\eta\colon \eta<\beta\}$ so that the family $\{H_\eta\colon \eta<\beta\}$, where $H_\eta = \bigcup\{F_\xi\colon \xi\in T_\eta\}$, is closure-preserving and covers X'. Since f is continuous and closed, $\{f(H_\eta)\colon \eta<\beta\}$ is a closure-preserving cover of X by compact sets (see [12], Theorem 2).

EXAMPLE 1. A paracompact scattered space X^* having a closure-preserving cover by finite sets and having no two order locally finite covers $\{E_\xi: \xi < \alpha\}$ and $\{U_\xi: \xi < \alpha\}$, where E_ξ is compact and U_ξ is an open nbhd of E_ξ for each $\xi < \alpha$.

At first we shall construct a space X by a modification of the construction which has been used by H. B. Potoczny [7]. We set

$$X = \{(\alpha, \beta): \alpha \leqslant \beta < \omega_2\}, \quad D = \{(\alpha, \alpha): \alpha < \omega_2\},$$

and

$$G_\alpha = \{(\alpha, \alpha)\} \cup \{(\alpha, \xi): \alpha < \xi < \omega_2\} \cup \{(\xi, \alpha): \xi < \alpha\} \quad \text{for each } \alpha < \omega_2.$$

We define a topology of X as follows: if $(\alpha, \beta) \in X - D$, then the singleton $\{(\alpha, \beta)\}$ is a basic open set; if $(\alpha, \alpha) \in D$, then $G_\alpha - H_\alpha$, where H_α is a countable subset of $X - D$, is a basic open set. It is clear that the sets described above constitute a basis for a topology of X.

Every point of $X - D$ is an isolated point and $G_\alpha \cap D = \{(\alpha, \alpha)\}$ for each $\alpha < \omega_2$. Therefore D is a closed discrete subset of X. Thus the space X is scattered.

Every countable subset H of X is closed, because

$$X - H = \bigcup \{G_\alpha - H: \alpha < \omega_2\}.$$

The basic open sets in X are simultaneously closed, because

$$X - \{(\alpha, \beta)\} = \bigcup \{G_\gamma - \{(\alpha, \beta)\}: \gamma < \omega_2\} \quad \text{whenever } \alpha < \beta < \omega_2$$

and

$$X - (G_\alpha - H_\alpha) = H_\alpha \cup \bigcup \{G_\beta - \{(\beta, \alpha)\}: \alpha < \beta\} \cup \\ \cup \bigcup \{G_\beta - \{(\alpha, \beta)\}: \alpha < \beta < \omega_2\} \quad \text{for each } \alpha < \omega_2.$$

X is a Hausdorff space, because the basic open sets separate the points.

Since each countable subset of X is closed, there is no infinite compact subset in X; i.e. each compact subset of X is finite.

Since each set G_α has the Lindelöf property, X is a locally Lindelöf space.

$\{G_\alpha: \alpha < \omega_2\}$ is a point-finite cover of X, because $G_\alpha \cap G_\beta = \{(\alpha, \beta)\}$, where $\alpha < \beta < \omega_2$.

The space X is metacompact, because it has the point-finite cover $\{G_\alpha: \alpha < \omega_2\}$ by the closed-open Lindelöf sets G_α.

For each countable subset H of X there is a closed-open Lindelöf subset U of X such that $H \subseteq U$ and $U \cap D = H \cap D$. Namely, we set $U = (H - D) \cup \bigcup \{G_\alpha: (\alpha, \alpha) \in H \cap D\}$. Since $H - D$ is a countable set of isolated points, $H - D$ is a closed-open set. Since $\bigcup \{G_\alpha: (\alpha, \alpha) \in H \cap D\}$ is open, it remains to show that the set is closed. If $(\beta, \beta) \in D - H$, then $G_\beta \cap \bigcup \{G_\alpha: (\alpha, \alpha) \in H \cap D\}$ is a countable subset of $X - D$. Thus $G_\beta - \bigcup \{G_\alpha: (\alpha, \alpha) \in H \cap D\}$ is a basic nbhd of (β, β). Clearly,

$$X - \bigcup \{G_\alpha: (\alpha, \alpha) \in H \cap D\} \\ = \bigcup \{G_\beta - \bigcup \{G_\alpha: (\alpha, \alpha) \in H \cap D\}: (\beta, \beta) \in D - H\}$$

and hence the assertion follows.

We set $F_{\alpha,\beta} = \{(\alpha, \alpha), (\alpha, \beta), (\beta, \beta)\}$ for each $(\alpha, \beta) \in X - D$. The family $\{F_{\alpha,\beta}: (\alpha, \beta) \in X - D\}$ covers X. We shall prove that the family is a closure-preserving one. Let $S \subseteq X - D$ and let $x \in \operatorname{cl} \bigcup \{F_{\alpha,\beta}: (\alpha, \beta) \in S\}$. If x is an isolated point of X, then obviously $x \in \bigcup \{F_{\alpha,\beta}: (\alpha, \beta) \in S\}$. Thus we may assume that $x = (\gamma, \gamma) \in D$. Since G_γ is a basic nbhd of (γ, γ), we have $G_\gamma \cap \bigcup \{F_{\alpha,\beta}: (\alpha, \beta) \in S\} \neq 0$. Hence $G_\gamma \cap F_{\alpha,\beta} \neq 0$ for some $(\alpha, \beta) \in S$. Since $F_{\alpha,\beta}$ is a three-point set, we shall consider three cases.

Case 1. $(\alpha, \alpha) \in G_\gamma$. Then $\alpha = \gamma$ and so $(\gamma, \gamma) \in F_{\alpha,\beta}$.

Case 2. $(\alpha, \beta) \in G_\gamma$. Since $\alpha < \beta$, we have $\alpha = \gamma$. Thus $(\gamma, \gamma) \in F_{\alpha,\beta}$.

Case 3. $(\beta, \beta) \in G_\gamma$. Then $\beta = \gamma$ and so $(\gamma, \gamma) \in F_{\alpha,\beta}$.

The space X is not normal, because the sets $\{(\alpha, \alpha): \alpha \leqslant \omega_1\}$ and $\{(\alpha, \alpha): \omega_1 < \alpha < \omega_2\}$ cannot be separated by open sets. In order to prove this, let U be an open nbhd of $\{(\alpha, \alpha): \alpha \leqslant \omega_1\}$ in X. For each $\alpha \leqslant \omega_1$ there is a basic open set U_α with $(\alpha, \alpha) \in U_\alpha \subseteq U$. For each $\alpha \leqslant \omega_1$ we have $U_\alpha = G_\alpha - H_\alpha$, where H_α is a countable subset of $X - D$. Let us set

$$S_1 = \{\xi < \omega_2: (\eta, \xi) \in H_\alpha \text{ for some } \eta < \omega_2 \text{ and for some } \alpha \leqslant \omega_1\},$$

$$S_2 = \{\xi < \omega_2: (\xi, \eta) \in H_\alpha \text{ for some } \eta < \omega_2 \text{ and for some } \alpha \leqslant \omega_1\}$$

and

$$S = \{\xi: \xi \leqslant \omega_1\} \cup S_1 \cup S_2.$$

Then $\operatorname{card} S < \aleph_2$. Hence there is an ordinal $\gamma < \omega_2$ for which $\sup S < \gamma$. Clearly $(\gamma, \gamma) \notin \bigcup \{U_\alpha: \alpha \leqslant \omega_1\}$. Since $\gamma \notin S$, we have $(\alpha, \gamma) \notin H_\alpha$ for each $\alpha \leqslant \omega_1$. Thus $(\alpha, \gamma) \in U_\alpha$ for each $\alpha \leqslant \omega_1$. Finally, we shall show that $(\gamma, \gamma) \in \operatorname{cl} \bigcup \{U_\alpha: \alpha \leqslant \omega_1\}$. Let $U_\gamma = G_\gamma - H_\gamma$ be any basic nbhd of (γ, γ). Since H_γ is countable and $\{(\alpha, \gamma): \alpha \leqslant \omega_1\}$ is an uncountable subset of G_γ, there exists an $\alpha \leqslant \omega_1$ with $(\alpha, \gamma) \in U_\gamma$. Thus $U_\gamma \cap U_\alpha \neq 0$. It follows that $(\gamma, \gamma) \in \operatorname{cl} U$.

We define X^* as a one-point extension $X \cup \{p\}$ of X as follows: U is a basic open nbhd of $x \neq p$ in X^* if and only if U is a basic open nbhd of x in X; U is a basic open nbhd of p in X^* if and only if $X^* - U$ is a closed-open Lindelöf set in X.

It follows from the definition of X^* that the basic open sets are simultaneously closed and they separate points. Thus X^* is a regular space.

Let U be a basic open set in X^*. Then U has the Lindelöf property if and only if $p \notin U$; $X^* - U$ has the Lindelöf property if and only if $p \in U$. Thus X^* is a Lindelöf space. Since X^* is a regular Lindelöf space, it is paracompact (see [2], p. 211).

Each countable subset of X^* is closed. To prove this, let H be a countable subset of X^*. Since $H-\{p\}$ is a countable subset of X, there exists a closed-open Lindelöf nbhd U of $H-\{p\}$ in X. Since $H-\{p\}$ is closed in U and U is closed in X^*, the set $H-\{p\}$ is closed in X^*. Thus $H = \{p\} \cup (H-\{p\})$ is closed in X^*.

Since each countable subset of X^* is closed, there is no infinite compact set in X^*; i.e. each compact subset of X^* is a finite set.

Since $\{F_{\alpha,\beta}\colon (\alpha,\beta) \in X-D\}$ is a closure-preserving cover of X, $\{\{p\} \cup F_{\alpha,\beta}\colon (\alpha,\beta) \in X-D\}$ is a closure-preserving cover of X^* (by four-point sets).

Finally, let us suppose that X^* has two order locally finite covers $\{E_\xi\colon \xi<\alpha\}$ and $\{U_\xi\colon \xi<\alpha\}$ where E_ξ is compact and U_ξ is an open nbhd of E_ξ in X^* for each $\xi<\alpha$. Since E_ξ is finite, $E_\xi-\{p\}$ is compact. Hence the space X has two order locally finite covers $\{E_\xi-\{p\}\colon \xi<\alpha\}$ and $\{U_\xi-\{p\}\colon \xi<\alpha\}$ where $E_\xi-\{p\}$ is compact and $U_\xi-\{p\}$ is an open nbhd of $E_\xi-\{p\}$ in X for each $\xi<\alpha$. Thus X is paracompact by a theorem of Y. Katuta [3]. This is a contradiction, because X is not normal (see [2], p. 207).

Recall that X is said to be *σ-locally compact* if X has a countable cover $\{X_n\colon n \geqslant 0\}$ where each X_n is a locally compact closed subset of X. These spaces were studied by K. Morita [5] and A. H. Stone [9].

LEMMA 4 (Y. Katuta [3]). *If X is paracompact and σ-locally compact, then X has two order locally finite covers $\{E_\xi\colon \xi<\alpha\}$ and $\{U_\xi\colon \xi<\alpha\}$ where E_ξ is compact and U_ξ is an open nbhd of E_ξ in X for each $\xi<\alpha$.*

THEOREM 2. *If X is paracompact and σ-locally compact, then X has a closure-preserving cover by compact sets.*

Proof. By Lemma 4 the space X has two order locally finite covers $\{E_\xi\colon \xi<\alpha\}$ and $\{U_\xi\colon \xi<\alpha\}$ where E_ξ is compact and U_ξ is an open nbhd of E_ξ for each $\xi<\alpha$. Thus, by Theorem 1, the space X has a closure-preserving cover by compact sets.

Recall that X is said to be *σ-discrete* if X has a countable cover $\{X_n\colon n \geqslant 0\}$ where each X_n is a discrete closed subset of X.

EXAMPLE 2. A σ-discrete, locally compact, scattered, non-normal space X with $\operatorname{ind} X = 0$ that has no closure-preserving cover by compact sets. As an example of the space X it suffices to take the space defined in [0], p. 167. This space X is zero-dimensional, locally compact, σ-discrete, separable and contains a closed discrete uncountable subset F; $X-F$ is countable, isolated and dense in X.

LEMMA 5. *Each closure-preserving family of pairwise disjoint closed sets is discrete.*

The proof is immediate.

Recall that X is said to be *C-scattered* if for each non-void closed subset E of X there is a point $x \in E$ and an open nbhd U of x for which $E \cap \operatorname{cl} U$ is compact. C-scattered spaces were studied in [11].

THEOREM 3. *Let X be a paracompact space. If X has two order locally finite covers $\{E_\xi\colon \xi<\alpha\}$ and $\{U_\xi\colon \xi<\alpha\}$ where E_ξ is a C-scattered closed subset of X and U_ξ is an open nbhd of E_ξ for each $\xi<\alpha$, then X has a countable cover $\{X_n\colon n>0\}$ where X_n is a C-scattered closed subset of X for each $n>0$.*

Proof. By Lemma 3 there are a paracompact space X' with $\dim X' = 0$ and a perfect map f from X' onto X. The set $E'_\xi = f^{-1}(E_\xi)$ is C-scattered (see [11], Theorem 1.3) and closed in X' and $U'_\xi = f^{-1}(U_\xi)$ is an open nbhd of E'_ξ in X' for each $\xi<\alpha$. It is easy to check that $\{E'_\xi\colon \xi<\alpha\}$ and $\{U'_\xi\colon \xi<\alpha\}$ are order locally finite covers of X'. By Lemma 1 there exist two order star-finite covers $\{F_\eta\colon \eta<\beta\}$ and $\{V_\eta\colon \eta<\beta\}$ where $\{F_\eta\colon \eta<\beta\}$ refines $\{E'_\xi\colon \xi<\alpha\}$, $\{V_\eta\colon \eta<\beta\}$ refines $\{U'_\xi\colon \xi<\alpha\}$, and F_η is closed in X' and V_η is an open nbhd of F_η for each $\eta<\beta$. Since F_η is a closed subset of some E'_ξ, it is C-scattered. By Lemma 2 there is a collection $\{T_\eta\colon \eta<\beta\}$ of finite subsets of $\{\eta\colon \eta<\beta\}$ such that $\{H_\eta\colon \eta<\beta\}$, where $H_\eta = \bigcup\{F_\xi\colon \xi \in T_\eta\}$, is a closure-preserving cover of X'. Since H_η is the union of a finite family of C-scattered sets, it is also C-scattered (see [11], Theorem 1.1). Let us remark that $\operatorname{card} T_\eta \geqslant 1$, because $\eta \in T_\eta$. For each $n>0$ we set

$$A_n = \{\eta<\beta\colon \operatorname{card} T_\eta \leqslant n\}, \quad X'_n = \bigcup\{H_\eta\colon \eta \in A_n\} \quad \text{and} \quad X_n = f(X'_n).$$

Since $\bigcup\{A_n\colon n>0\} = \{\eta\colon \eta<\beta\}$, it follows that $\bigcup\{X'_n\colon n>0\} = X'$. Thus $\bigcup\{X_n\colon n>0\} = X$. Since the family $\{H_\eta\colon \eta<\beta\}$ is closure-preserving and each H_η is closed, the set X'_n is closed for each $n>0$. Thus each X_n is closed, because f is a closed map. By Theorem 1.3 of [11] X_n is C-scattered if and only if X'_n is C-scattered. Now we prove (by induction with respect to n) that X'_n is C-scattered.

$$X'_1 = \bigcup\{H_\eta\colon \eta \in A_1\} = \bigcup\{F_\eta\colon \eta \in A_1\}.$$

The family $\{F_\eta\colon \eta \in A_1\}$ is closure-preserving and its members are pairwise disjoint. Therefore, by Lemma 5, it is a discrete family. Hence X'_1 is locally C-scattered and therefore C-scattered. Assume that for some $n>0$ the set X'_n is C-scattered. Let us set $B_{n+1} = \{\eta<\beta\colon \operatorname{card} T_\eta = n+1\}$. Then $A_{n+1} = A_n \cup B_{n+1}$. Let us set

$$Y'_n = \bigcup\{F_\eta\colon \eta \in B_{n+1}\} \quad \text{and} \quad Z'_n = \bigcup\{F_\eta\colon \eta \in T_\xi - \{\xi\} \text{ and } \xi \in B_{n+1}\}.$$

Then $X'_{n+1} = X'_n \cup Y'_n \cup Z'_n$. We claim that $Z'_n \subseteq X'_n$. If $\eta \in T_\xi - \{\xi\}$ and $\xi \in B_{n+1}$, then $\eta<\xi$. Hence $T_\eta \subseteq T_\xi$ and $\operatorname{card} T_\eta < \operatorname{card} T_\xi = n+1$. Thus

$\eta \in A_n$ and $F_\eta \subseteq H_\eta \subseteq X'_n$. We claim that Y'_n is C-scattered. Since F_η with $\eta \in B_{n+1}$ is C-scattered, it suffices to prove that the family $\{V_\eta: \eta \in B_{n+1}\}$ consists of pairwise disjoint sets. Let us suppose that we have ξ and η in B_{n+1} with $\xi < \eta$ and $V_\xi \cap V_\eta \neq 0$. Then $T_\xi \subseteq T_\eta$ and so $\operatorname{card} T_\eta \geqslant n+2$. This is a contradiction. Hence Y'_n is C-scattered. Since X'_{n+1} is the union of C-scattered sets X'_n and Y'_n, it is also C-scattered (see [11], Theorem 1.1).

Remark 2. If $\{E_n: n \geqslant 0\}$ is a countable cover of X with E_n closed, then X has two order locally finite covers $\{E_n: n \geqslant 0\}$ and $\{U_n: n \geqslant 0\}$ where $U_n = X$ for each $n \geqslant 0$.

Remark 3. Theorem 3 is surprising if we look at Theorem 2.5 of [11]; roughly speaking, the countable covers are sufficient to define the class of spaces.

As a corollary of Theorem 3 we have

THEOREM 4. *If X has two order locally finite covers $\{E_\xi: \xi < \alpha\}$ and $\{U_\xi: \xi < \alpha\}$ where E_ξ is compact and U_ξ is an open nbhd of E_ξ for each $\xi < \alpha$, then X has a countable cover $\{X_n: n \geqslant 0\}$ where X_n is a C-scattered closed subset of X for each $n \geqslant 0$.*

THEOREM 5. *Each paracompact σ-discrete space has a closure-preserving cover by finite sets.*

Proof. If X is paracompact and σ-discrete, then $\dim X = 0$ (see [2], p. 274). X is σ-discrete, i.e. $X = \bigcup\{X_n: n \geqslant 0\}$ where each X_n is a discrete closed subset of X. Remark that $X_n - \bigcup\{X_k: k < n\}$ is closed in X. Hence we may assume, without loss of generality, that $X_m \cap X_n = 0$ for $m \neq n$. There is a well-ordering $\{x_\xi: \xi < \alpha\}$ of X where $\xi < \eta$ holds whenever $x_\xi \in X_m$, $x_\eta \in X_n$ and $m < n$. Let us put $E_\xi = \{x_\xi\}$ for each $\xi < \alpha$. The family $\{E_\xi: x_\xi \in X_n\}$ is discrete for each $n \geqslant 0$. Since X is paracompact, it is collectionwise normal (see [2], p. 214). Thus there is a discrete family $\{U_\xi: x_\xi \in X_n\}$ of open sets in X with $E_\xi \subseteq U_\xi$. It is easy to check that the covers $\{E_\xi: \xi < \alpha\}$ and $\{U_\xi: \xi < \alpha\}$ of X are order locally finite. By Lemma 1 we get two order star-finite covers $\{F_\eta: \eta < \beta\}$ and $\{V_\eta: \eta < \beta\}$ of X where F_η contains at most one point and V_η is an open nbhd of F_η for each $\eta < \beta$. By Lemma 2 there is a family $\{T_\eta: \eta < \beta\}$ of finite subsets of $\{\eta: \eta < \beta\}$ for which $\{\bigcup\{F_\xi: \xi \in T_\eta\}: \eta < \beta\}$ is a closure-preserving cover of X. Thus X has a closure-preserving cover by finite sets.

LEMMA 6. *For any family $\{E_i: i \in I\}$ of non-void sets there exists a $J \subseteq I$ for which the family $\{E_j: j \in J\}$ is pairwise disjoint and where for each $i \in I$ there exists a $j \in J$ with $E_i \cap E_j \neq 0$.*

The proof is a standard application of the Kuratowski-Zorn Lemma.

THEOREM 6. *If X has a closure-preserving cover by finite sets, then X has a countable cover $\{X_n: n \geqslant 0\}$ where X_n is a scattered closed subset of X and it is the union of n discrete (not necessarily closed) subsets for each $n \geqslant 0$.*

Proof. Let $\{E_i: i \in I\}$ be a closure-preserving cover of X, where E_i is finite for each $i \in I$. Let us set $I_n = \{i \in I: \operatorname{card} E_i \leqslant n\}$ and $X_n = \bigcup\{E_i: i \in I_n\}$ for each $n \geqslant 0$. Then X_n is a closed subset of X, because $\{E_i: i \in I\}$ is a closure-preserving family of closed subsets of X. Now it suffices to prove the following auxiliary statement $S(n)$: If X has a closure-preserving cover $\{E_i: i \in I\}$ with $\operatorname{card} E_i \leqslant n$ for each $i \in I$, then X is scattered and it is the union of n discrete subsets. Clearly, $S(0)$ holds. By Lemma 5, $S(1)$ holds. Assume that $S(n)$ holds for some $n > 0$. We shall prove that $S(n+1)$ also holds. So let X be a space with a closure-preserving cover $\{E_i: i \in I\}$ where $\operatorname{card} E_i \leqslant n+1$ for each $i \in I$. We may assume, without loss of generality, that $E_i \neq 0$ for each $i \in I$. By Lemma 6 there exists a $J \subseteq I$ for which the family $\{E_j: j \in J\}$ is constituted by pairwise disjoint sets and where for each $i \in I$ there exists a $j \in J$ with $E_i \cap E_j \neq 0$. Let us put $Y = \bigcup\{E_j: j \in J\}$. Since $\{E_i: i \in I\}$ is a closure-preserving family of closed sets, Y is a closed subset of X. By Lemma 5 the family $\{E_j: j \in J\}$ is discrete. Thus Y is a discrete closed subset of X. Let us fix $i \in I$. Since $E_i \cap Y \neq 0$, $\operatorname{card}(E_i - Y) \leqslant n$. Thus $X - Y$ has the closure-preserving cover $\{E_i - Y: i \in I\}$ with $\operatorname{card}(E_i - Y) \leqslant n$ for each $i \in I$. Hence, by $S(n)$, $X - Y$ is a scattered space and it is the union of n discrete subsets. Thus $X = (X - Y) \cup Y$ is also scattered, because the union of two scattered spaces is a scattered space (see [4], p. 79). Clearly, X is the union of $n+1$ discrete subsets.

PROBLEM 1. Let X be a paracompact space with a closure-preserving cover by compact sets. Does X have a countable cover by C-scattered closed subsets?

Remark 4. We may consider Theorem 4 (according to Theorem 1) and Theorem 6 as partial solutions of Problem 1.

EXAMPLE 3. A compact scattered space X that has no closure-preserving cover by finite sets. We take $X = \{\xi: \xi \leqslant \omega_1\}$ endowed with the order (interval) topology. X is a compact scattered space. Assume that $\{E_i: i \in I\}$ is a cover of X by finite sets. The set $\{i \in I: \operatorname{card} E_i = 1\}$ is finite, because X is compact. Let us set $I' = \{i \in I: \operatorname{card} E_i > 1\}$. We define $f: I' \to X$ as follows: $f(i) = \max\{\alpha < \omega_1: \alpha \in E_i\}$. Clearly, there is a sequence $\{i_n: n \geqslant 0\} \subseteq I'$ for which $f(i_0) < f(i_1) < f(i_2) < \ldots$ Let $\alpha = \sup\{f(i_n): n \geqslant 0\}$. Then $\alpha \notin \bigcup\{E_{i_n}: n \geqslant 0\}$ and $\alpha \in \operatorname{cl}\bigcup\{E_{i_n}: n \geqslant 0\}$. Hence $\{E_i: i \in I\}$ is not closure-preserving.

Remark 5. Let X be the Aleksandrov compactification of the space defined in Example 2. Then X is another example of a compact scattered space that has no closure-preserving cover by finite sets.

Recall that X is said to be *totally paracompact* if each open basis of X contains a locally finite cover of X.

THEOREM 7. *If X is paracompact and if X has a closure-preserving cover by finite sets, then X is totally paracompact.*

Proof. By Theorem 6 the space X has a countable cover by its scattered closed subsets. By Theorem 3.1 of [11] each paracompact scattered space is absolutely paracompact. By Theorem 1.7 of D. Curtis [1] a paracompact space having a countable cover by its closed absolutely paracompact subsets is totally paracompact.

THEOREM 8. *If X has two order locally finite covers $\{E_\xi\colon \xi < \alpha\}$ and $\{U_\xi\colon \xi < \alpha\}$ where E_ξ is compact and U_ξ is an open nbhd of E_ξ for each $\xi < \alpha$, then X is totally paracompact.*

Proof. By a theorem of Y. Katuta [3] the space X is paracompact. By Theorem 4 the space X has a countable cover by its C-scattered closed subsets. By Theorem 3.1 of [11] and Theorem 1.7 of [1] the space X is totally paracompact if it is paracompact and if it has a countable cover by its C-scattered closed subsets.

PROBLEM 2. Let X be a paracompact space with a closure-preserving cover by compact sets. Is X totally paracompact?

Remark 6. Theorem 7 and Theorem 8 (according to Theorem 1) can be considered as partial solutions of Problem 2.

Recall that a family $\{E_i\colon i \in I\}$ is said to be *σ-closure-preserving* if I is the union of such a countable family $\{I_n\colon n \geqslant 0\}$ that $\{E_i\colon i \in I_n\}$ is closure-preserving for each $n \geqslant 0$.

EXAMPLE 4. A paracompact scattered space X for which the following holds: (a) X has no closure-preserving cover by compact sets, but yet (b) X has a σ-closure-preserving cover by compact sets. X is the space S_0 defined in [11], p. 71. This space X is paracompact (even Lindelöf) and scattered. It is uncountable and has a countable dense open subset N. Each compact subset of X is finite. Hence (a) follows. The family $\{\{p, p_\alpha\}\colon \alpha < \omega_1\}$ is closure-preserving and covers $X - N$. Obviously, N has a cover by singletons. Hence (b) follows.

Remark 7. Let X be the Aleksandrov compactification of the space defined in Example 2. Then X is an example of a compact space satisfying (a) and (b).

Added in proof. Example 1 provides an improvement of the result of H. B. Potoczny [7] in the following sense. The auxiliary space X, which is constructed in Example 1, has a closure-preserving cover consisting of three-point sets but X is even not subparacompact.

Let us recall that X is said to be *subparacompact* if each open cover of X has a σ-locally finite closed refinement (D. K. Burke, *On subparacompact spaces*, Proc. Amer. Math. Soc. 23 (1969), pp. 655–663). It is easy to check that (a) each F_σ-set in X is closed; (b) if each F_σ-set is closed, then each σ-locally finite family is closure-preserving. Thus, if X were subparacompact, then it would be paracompact by Theorem 1 of E. Michael, *Another note on paracompact spaces*, Proc. Amer. Math. Soc. 8 (1957), pp. 822–828. But X is even not normal. Hence the assertion follows.

References

[0] A. V. Arhangel'skiĭ, *Mappings and spaces*, Uspehi Mat. Nauk 21 (1966), 4 (130), pp. 133–184 (Russian).

[1] D. Curtis, *Total and absolute paracompactness*, Fund. Math. 77 (1973), pp. 277–283.

[2] R. Engelking, *Outline of General Topology*, Amsterdam 1968.

[3] Y. Katuta, *A theorem on paracompactness of product spaces*, Proc. Japan Acad. 43 (1967), pp. 615–618.

[4] K. Kuratowski, *Topology*, Vol. I, New York–London–Warszawa 1966.

[5] K. Morita, *On the product of paracompact spaces*, Proc. Japan Acad. 39 (1963), pp. 559–563.

[6] K. Nagami, *A note on Hausdorff spaces with the star-finite property*, II, Proc. Japan Acad. 37 (1961), pp. 189–192.

[7] H. B. Potoczny, *A nonparacompact space which admits a closure-preserving cover of compact sets*, Proc. Amer. Math. Soc. 32 (1972), pp. 309–311.

[8] — *Closure-preserving families of compact sets*, General Topology and its Applications 3 (1973), pp. 243–248.

[9] A. H. Stone, *Absolute F_σ-spaces*, Proc. Amer. Math. Soc. 13 (1962), pp. 495–499.

[10] H. Tamano, *A characterization of paracompactness*, Fund. Math. 72 (1971), pp. 189–201.

[11] R. Telgársky, *C-scattered and paracompact spaces*, Fund. Math. 73 (1971), pp. 59–74.

[12] — *Concerning two covering properties*, Proc. Japan Acad. (in press).

[13] — *On topological properties defined by games*, Colloquia Mathematica Societatis János Bolyai 8, Topics in Topology, Keszthely (Hungary), 1972, pp. 617–624.

[14] — *Spaces defined by topological games*, Fund. Math. (in press).

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