Some remarks on shape properties of compacta

by

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Abstract. It is shown that the shape of a compactum X lying in a space $M \in \mathrm{AR}$ is trivial if and only if for every neighborhood U of X there is a map $f\colon M \to M$ such that $f(M) \subset U$ and $f(x) = x$ for every point $x \in X$.

Moreover it is shown that the shape of X is trivial if and only if X is movable and approximatively n-connected for every $n = 0, 1, \ldots$

Also some properties of the approximatively n-connected spaces are proved. In particular it is shown that a movable pointed compactum (X, x_0) is approximatively n-connected if and only if the nth fundamental group $\underline{\pi}_n(X, x_0)$ is trivial.

The aim of this note is to prove two theorems characterizing compacta with trivial shape and to establish a simple relation between the property of the approximative n-connectedness and the triviality of fundamental groups.

Concerning the basic notions of the theory of shape (as *fundamental sequence*, *shape* of a compactum, *fundamental retraction*, *fundamental absolute retract* FAR, *movability*, *approximative map* of a pointed compactum (X, x_0) towards another pointed compactum (Y, y_0), *fundamental groups* $\underline{\pi}_n(X, x_0)$ and so on) see [3]. Compare also [4], where instead of compacta lying in the Hilbert cube Q, one considers compacta lying in an arbitrary $\mathrm{AR}(\mathfrak{M})$-space M.

§ 1. First theorem on compacta with trivial shape. The shape of the space consisting of only one point is said to be *trivial*. One knows ([3], p. 274) that compacta with trivial shape are the same as FAR-spaces. Some other characterizations of compacta with trivial shape are given in [2], p. 72–74 and [6], p. 38. Compare also [7], p. 91–92 and [8], p. 18. Let us prove the following

(1.1) THEOREM. *The shape of a compactum X lying in a space $M \in \mathrm{AR}$ is trivial if and only if for every neighborhood U of X (in M) there is a map $f\colon M \to M$ satisfying both conditions*:

1° $f(M) \subset U$,

2° $f(x) = x$ *for every point* $x \in X$.

Proof. If $X \subset M$ and if $\mathrm{Sh}(X)$ is trivial then there exists a fundamental retraction $\underline{f} = \{f_k, M, X\}_{M,M}$. Then for every neighborhood U

of X (in M) there is an index k such that the map $f = f_k\colon M \to M$ satisfies the conditions 1° and 2°.

On the other hand, suppose that for every neighborhood U of X (in M) there is a map $f\colon M \to M$ satisfying 1° and 2°. First let us consider the special case when M is the Hilbert cube Q, being the subset of the Hilbert space E^∞ consisting of all points $(t_1, t_2, \ldots) \in E^\infty$ with $0 \leqslant t_n \leqslant 1/n$ for $n = 1, 2, \ldots$ Setting

$$\lambda_n(t) = 1/n \quad \text{if} \quad t \geqslant 1/n\,,$$
$$\lambda_n(t) = t \quad \text{if} \quad 0 \leqslant t \leqslant 1/n\,,$$
$$\lambda_n(t) = 0 \quad \text{if} \quad t \leqslant 0\,,$$

and

$$r_0(t_1, t_2, \ldots) = \big(\lambda_1(t_1), \lambda_2(t_2), \ldots\big) \quad \text{for every } (t_1, t_2, \ldots) \in E^\infty\,,$$

one obtains a retraction $r_0\colon E^\infty \to Q$ satisfying the condition

(1.2) $$|r_0(x) - r_0(x')| \leqslant |x - x'| \quad \text{for every } x, x' \in E^\infty\,.$$

Now let us consider a sequence $U_1 \supset U_2 \supset \ldots$ of open neighborhoods of X (in Q) shrinking to X. By our hypothesis, there exists for every $k = 1, 2, \ldots$ a map

$$a_k\colon Q \to Q$$

such that $a_k(Q) \subset U_k$ and $a_k(x) = x$ for every point $x \in X$. Since $a_k(Q)$ is a compact subset of the open set U_k, there exists a positive number ε_k such that

(1.3) if $x \in Q$ and $\varrho\big(x, a_k(Q)\big) < \varepsilon_k$ then $x \in U_k$.

Since $a_k(x) = x$ for every point $x \in X$, there exists a closed neighborhood $V_k \subset U_k$ of X (in Q) such that

(1.4) $$\varrho\big(x, a_k(x)\big) < \varepsilon_k \quad \text{for every point } x \in V_k\,.$$

It is clear that we can assume that $V_{k+1} \subset V_k$ for $k = 1, 2, \ldots$

Now let us set

$$\beta_k(x) = a_k(x) - x \quad \text{for every point } x \in V_k\,.$$

It follows by (1.4) that β_k is a map of V_k into the ball B_k consisting of all points $y \in E^\infty$ with $\varrho(y, 0) < \varepsilon_k$. Since B_k is an AR($\mathfrak{M}$)-set, the map β_k has an extension β'_k being a map of Q into B_k. Setting

$$\hat{a}_k(y) = r_0[a_k(y) - \beta'_k(y)] \quad \text{for every point } y \in Q\,,$$

one sees easily (by virtue of (1.2) and (1.3)) that $\hat{a}_k$ is a map of Q into itself such that

$$\hat{a}_k(Q) \subset U_k \text{ and } \hat{a}_k(x) = x \quad \text{for every point } x \in V_k\,.$$

By our hypothesis, there exists for every $k = 1, 2, \ldots$ a map $f_k\colon Q \to Q$ such that

$$f_k(Q) \subset V_k \text{ and } f_k(x) = x \quad \text{for every point } x \in X\,.$$

Setting

$$\varphi_k(y, t) = \hat{a}_k\big(tf_{k+1}(y) + (1-t)f_k(y)\big) \quad \text{for every } (y, t) \in Q \times \langle 0, 1\rangle\,,$$

we get a homotopy

$$\varphi_k\colon Q \times \langle 0, 1\rangle \to Q$$

with values in U_k. Moreover

$$\varphi_k(y, 0) = f_k(y) \quad \text{and} \quad \varphi_k(y, 1) = f_{k+1}(y) \quad \text{for every point } y \in Q\,,$$

because $\varphi_k(y, 0) = \hat{a}_k f_k(y)$, $\varphi_k(y, 1) = \hat{a}_k f_{k+1}(y)$ and also $f_k(y) \in V_k$, $f_{k+1}(y) \in V_{k+1} \subset V_k$ for every point $y \in Q$.

Thus we have shown that $\underline{f} = \{f_k, Q, X\}$ is a fundamental sequence. Since $f_k(x) = x$ for every point $x \in X$, we infer that $\underline{f}$ is a fundamental retraction. Hence $\mathrm{Sh}(X)$ is trivial.

Passing to the general case, we may assume that $M \cap Q = X$. Then there are two retractions

$$r\colon M \cup Q \to M \quad \text{and} \quad s\colon M \cup Q \to Q\,.$$

Let V be a neighborhood of X in Q. Then there exists a neighborhood U of X in M such that $s(U) \subset V$. By our hypothesis, there is a map $g\colon M \to M$ such that $g(M) \subset U$ and that $g(x) = x$ for every point $x \in X$. Setting

$$f(y) = sgr(y) \quad \text{for every point } y \in Q\,,$$

one gets a map $f\colon Q \to Q$ such that $f(Q) \subset V$ and that $f(x) = x$ for every point $x \in X$. By virtue of the just settled special case, one infers that $\mathrm{Sh}(X)$ is trivial. Thus the proof of Theorem (1.1) is finished.

§ 2. Approximatively n-connected compacta. Let $S = S^n$ denote the boundary of the ball $B = B^{n+1}$ defined in the Euclidean $(n+1)$-space E^{n+1} as the set of all points $y \in E^{n+1}$ with $|y| \leqslant 1$. Let $a = (1, 0, \ldots, 0) \in S$. First let us prove the following proposition:

(2.1) *Suppose that x_0 is a point of a subset U_0 of a space U. A map $f\colon (S, a) \to (U_0, x_0)$ is null-homotopic in (U, x_0) if and only if there exists a map $\hat{f}\colon B \to U$ such that $\hat{f}(y) = f(y)$ for every point $y \in S$.*

Proof. If $f\colon (S, a)\to(U_0, x_0)$ is null-homotopic in (U, x_0) then there exists a homotopy $\varphi\colon S\times\langle 0, 1\rangle\to U$ such that $\varphi(y, 0)=f(y)$, $\varphi(y, 1)=x_0$ for every point $y\in S$ and $\varphi(a, t)=x_0$ for every $0\leqslant t\leqslant 1$. Setting

$$\hat{f}(y)=\varphi\left(\frac{y}{|y|}, 1-|y|\right) \quad \text{for every point } y\in B\setminus(0),$$

$$\hat{f}(0)=x_0,$$

one gets a map $\hat{f}\colon B\to U$ satisfying the required conditions.

One the other hand, if there exists a map $\hat{f}\colon B\to U$ satisfying the condition $\hat{f}(y)=f(y)$ for every point $y\in S$, then setting

$$\varphi(y, t)=\hat{f}\big(a+(1-t)(y-a)\big) \quad \text{for every } (y, t)\in S\times\langle 0, 1\rangle,$$

one gets the required homotopy $\varphi\colon S\times\langle 0, 1\rangle\to U$.

Let us recall that a compactum X lying in a space $M\in \mathrm{AR}(\mathfrak{M})$ is said to be *approximatively n-connected* if for every point $x_0\in X$ and for every neighborhood U of X (in M) there exists a neighborhood U_0 of X (in M) such that every map $f\colon (S, a)\to(U_0, x_0)$ is null-homotopic in (U, x_0). One sees readily that the choice of a space $M\in\mathrm{AR}(\mathfrak{M})$ containing X is immaterial and that the approximative n-connectedness of X implies the approximative n-connectedness of every compactum Y with $\mathrm{Sh}(Y)\leqslant\mathrm{Sh}(X)$.

Now let us prove the following

(2.2) Theorem. *A compactum $X\subset M\in\mathrm{AR}(\mathfrak{M})$ is approximatively n-connected if and only if for every neighborhood U of X in M there is a neighborhood W of X (in M) such that every map $f\colon S\to W$ is null-homotopic in U.*

Proof. Suppose that X is approximatively n-connected. Let U be a neighborhood of X (in M). Then for every point $x_0\in X$ there exists a neighborhood V_{x_0} of X (in M) such that each map $f\colon (S, a)\to(V_{x_0}, x_0)$ is null-homotopic in (U, x_0). It is clear that if x_0' is a point of V_{x_0} sufficiently close to x_0 then for every map $g\colon (S, a)\to(V_{x_0}, x_0')$ there exists a homotopy

$$\vartheta\colon S\times\langle 0, 1\rangle\to V_{x_0}$$

such that $\vartheta(a, 0)=x_0$ and $\vartheta(y, 1)=g(y)$ for every point $y\in S$. Setting

$$f(y)=\vartheta(y, 0) \quad \text{for every point } y\in S,$$

one gets a map $f\colon (S, a)\to(V_{x_0}, x_0)$. By our hypothesis, this map is null-homotopic in (U, x_0) and we infer by (2.1) that there exists a map

$$\hat{f}\colon B\to U$$

such that $\hat{f}(y)=f(y)$ for every point $y\in S$. Setting

$$\hat{g}(y)=\hat{f}(2y) \qquad \text{if} \quad |y|\leqslant\tfrac{1}{2},$$

$$\hat{g}(y)=\vartheta\left(\frac{y}{|y|}, 2|y|-1\right) \qquad \text{if} \quad \tfrac{1}{2}\leqslant|y|\leqslant 1,$$

we get a map $\hat{g}\colon B\to U$ such that $\hat{g}(y)=g(y)$ for every point $y\in S$. Applying again (2.1), we infer that g is null-homotopic in (U, x_0').

Thus we have shown that for every point $x_0\in X$ there exists a neighborhood G_{x_0} of x_0 in M such that if $x_0'\in G_{x_0}$ then every map $f\colon (S, a)\to(V_{x_0}, x_0')$ is null-homotopic in (U, x_0').

Since X is compact, there exists a finite system of points $x_1, x_2, \ldots \ldots, x_m\in X$ such that $G=G_{x_1}\cup G_{x_2}\cup\ldots\cup G_{x_m}$ is a neighborhood of X in M. We infer that setting

$$W_U=V_{x_1}\cap V_{x_2}\cap\ldots\cap V_{x_m}\cap G,$$

we get a neighborhood W_U of X (in M) such that every map $f\colon S\to W_U$ is null-homotopic in U.

On the other hand, if for every neighborhood U of X (in M) there exists a neighborhood W_U of X (in M) such that every map $f\colon S\to W_U$ is null-homotopic in U, then we infer by (2.1) that for every point $x_0\in X$ every map $f\colon (S, a)\to(W_U, x_0)$ is null-homotopic in (U, x_0). Hence X is approximatively n-connected and the proof of Theorem (2.2) is finished.

§ 3. Second theorem on compacta with trivial shape. By a *polyhedral pair* (R, R_0) we understand a pair consisting of a finite polyhedron R and of a polyhedron R_0 being the union of some simplexes of a triangulation of R. First let us prove the following

(3.1) Lemma. *Suppose that X is a compactum lying in a space $M\in\mathrm{AR}$ and n is an integer $\geqslant 0$. If X is approximatively k-connected for every $k=0, 1, \ldots, n$ then for every neighborhood U of X in M there exists a neighborhood $U_0\subset U$ of X in M such that for every polyhedral pair (R, R_0) with $\dim(R\setminus R_0)\leqslant n$ and for every map $f_0\colon R_0\to U_0$ there exists a map $f\colon R\to U$ satisfying the condition $f(x)=f_0(x)$ for every point $x\in R_0$.*

Proof. Let T be a triangulation of R such that R_0 is the union of some simplexes belonging to T. If $n=0$ then $R\setminus R_0$ consists of a finite number of points and one can get a required map f assigning to each of these points a point of X.

Assume now that $n>0$ and that the statement holds true if $\dim(R\setminus R_0)\leqslant n-1$. Let $R^{(n-1)}$ denote the $(n-1)$-dimensional skeleton of the triangulation T, i.e. the union of all simplexes of T with dimensions $\leqslant n-1$. Let U be a neighborhood of X in M. Since X is approximatively connected in dimensions $\leqslant n$, we infer by Theorem (2.2) that there exists a neighborhood $\hat{U} \subset U$ of X (in M) such that for every map g of the boundary $\dot{\Delta}$ of any n-dimensional simplex Δ into $\hat{U}$ there is a map g': $\Delta \to U$ satisfying the condition $g'(y) = g(y)$ for every point $y \in \dot{\Delta}$. By the hypothesis of induction, there exists a neighborhood $\hat{U}_0$ of X (in M) such that for every map f_0: $R_0 \to \hat{U}_0$ there is a map $\hat{f}$: $R_0 \cup R^{(n-1)} \to \hat{U}$ such that $\hat{f}(y) = f_0(y)$ for every point $y \in R_0$. Then for every n-dimensional simplex Δ of T there is a map g_Δ: $\Delta \to U$ satisfying the condition $g_\Delta(y) = \hat{f}(y)$ for every point $y \in \dot{\Delta}$. Setting

$$f(y) = \hat{f}(y) \quad \text{for every point } y \in R_0 \cup R^{(n-1)},$$

$$f(y) = g_\Delta(y) \quad \text{for every point } y \in \Delta \in T \text{ with } \Delta \setminus \dot{\Delta} \subsetneq R \setminus (R_0 \cup R^{(n-1)}),$$

one gets a map f: $R \to U$ satisfying the required conditions.

(3.2) THEOREM. *The shape of a compactum X is trivial if and only if X is movable and approximatively n-connected for every $n = 0, 1, \ldots$*

Proof. It is well known that every compactum X with trivial shape is movable and approximatively n-connected for every $n = 0, 1, \ldots$ Thus it remains to prove the converse.

One knows ([1], p. 240) that for every compactum X there exists an infinite polyhedron P disjoint to X and such that $M = X \cup P$ is an AR-space. If X is movable, then for every open neighborhood U of X in M there exists a neighborhood $U_0 \subset U$ of X in M such that the inclusion-map of U_0 in U is in U homotopic to a map with all values in an arbitrarily given neighborhood V of X (in M).

Let us observe that there exists a finite polyhedron R, being the union of some simplexes of a given triangulation T of P, such that $M \setminus R \subset U_0$. Let $\dot{R}$ denote the boundary of R in M, that is $\dot{R} = R \cap \overline{M \setminus R}$. It is clear that $\dot{R}$ is a finite polyhedron lying in U_0. By Lemma (3.1), there exists a neighborhood $V \subset U_0$ of X (in M) such that for every map g: $\dot{R} \to V$ there is a map $\hat{g}$: $R \to U_0$ satisfying the condition $\hat{g}(y) = g(y)$ for every point $y \in \dot{R}$.

Consider now the inclusion map j: $\dot{R} \to U_0$. By the definition of U_0, there exists a homotopy φ: $\dot{R} \times \langle 0, 1\rangle \to U$ joining j with a map g with all values in V. Setting

$$f'(y) = g(y) \quad \text{for every point } y \in \dot{R},$$

$$f'(x) = x \quad \text{for every point } x \in X,$$

one gets a map f': $X \cup \dot{R} \to V$ homotopic in U to the inclusion map j': $X \cup \dot{R} \to U$. Since the inclusion map $\hat{j}$: $\overline{X \setminus R} \to U$ is an extension of j' and since U (as an open subset of M) is an ANR$(\mathfrak{M})$-space, we infer by the homotopy extension theorem that there exists a map

$$\hat{f}\colon \overline{X \setminus R} \to U$$

being an extension of f'. Moreover, we know already that there exists a map $\hat{g}$: $R \to U_0$ satisfying the condition $\hat{g}(y) = g(y) = f'(y) = \hat{f}(y)$ for every point $y \in \dot{R}$. Setting

$$f(y) = \hat{f}(y) \quad \text{for every point } y \in \overline{M \setminus R},$$

$$f(y) = \hat{g}(y) \quad \text{for every point } y \in R,$$

one gets a map f: $M \to M$ such that $f(M) \subset U$ and that $f(x) = x$ for every point $x \in X$. By virtue of Theorem (1.1), the shape of X is trivial. Thus the proof of Theorem (3.2) is finished.

§ 4. Approximative n-connectedness of movable pointed compacta and the fundamental groups. It is clear that for every approximatively n-connected, pointed compactum (X, x_0) the group $\underline{\pi}_n(X, x_0)$ is trivial. Hence:

(4.1) *If X is an approximatively n-connected compactum, then for every point $x_0 \in X$ the group $\underline{\pi}_n(X, x_0)$ is trivial.*

For arbitrary compacta the converse is not true (because if X is a solenoid of van Dantzig, then $\underline{\pi}_1(X, x_0)$ is trivial, but X is not approximatively 1-connected). However the following theorem (compare [3], p. 271) holds true:

(4.2) THEOREM. *A movable, pointed compactum (X, x_0) is approximatively n-connected if and only if the group $\underline{\pi}_n(X, x_0)$ is trivial.*

Proof. By (4.1) we have only to show that if X is not approximatively n-connected then $\underline{\pi}_n(X, x_0)$ is not trivial. Assume that X lies in an AR$(\mathfrak{M})$-space M. Since (X, x_0) is movable, there exists a sequence $U_1 \supset U_2 \supset \ldots$ of neighborhoods of X in M shrinking to X and such that for every $k = 1, 2, \ldots$ there is a homotopy

$$\varphi_k\colon U_{k+1} \times \langle 0, 1\rangle \to U_k$$

such that $\varphi_k(y, 0) = y$, $\varphi_k(y, 1) \in U_{k+2}$ for every point $y \in U_{k+1}$, and $\varphi_k(x_0, t) = x_0$ for every $0 \leqslant t \leqslant 1$.

If (X, x_0) is not approximatively n-connected, then there is a neighborhood U of X (in M) such that for every neighborhood V of X (in M) there exists a map

$$f\colon (S, a) \to (V, x_0)$$

which is not null-homotopic in (U, x_0). Let k_0 be an index such that $U_{k_0} \subset U$. Then there is a map

$$\xi_1\colon (S, a) \to (U_{k_0+2}, x_0)$$

which is not null-homotopic in (U_{k_0}, x_0). Assume that for a natural index m a map

$$\xi_m\colon (S, a) \to (U_{k_0+m+1}, x_0)$$

is defined, which is not null-homotopic in (U_{k_0}, x_0). Setting

$$\xi_{m+1}(y) = \varphi_{k_0+m}(\xi_m(y), 1) \quad \text{for every point } y \in S,$$

we get a map

$$\xi_{m+1}\colon (S, a) \to (U_{k_0+m+2}, x_0)$$

homotopic to ξ_m in (U_{k_0+m}, x_0). Hence ξ_{m+1} is not null-homotopic in (U_{k_0}, x_0). Thus we obtain a sequence of maps $\xi_1, \xi_1, \ldots$ of (S, a) into (U_{k_0}, x_0) such that $\underline{\xi} = \{\xi_k\colon (S, a) \to (X, x_0)\}$ is an approximative map, being a representative of an element of the group $\underline{\pi}_n(X, x_0)$, different from zero. Hence $\underline{\pi}_n(X, x_0)$ is not trivial and the proof of Theorem (4.2) is finished.

Theorems (3.2) and (4.2) give the following

(4.3) Corollary. *If (X, x_0) is a movable pointed compactum for which all groups $\underline{\pi}_n(X, x_0)$ are trivial then* Sh(X) *is trivial.*

The question whether every movable compactum X with trivial fundamental groups $\underline{\pi}_n(X, x_0)$ for every $x_0 \in X$ is approximatively n-connected remains open.

§ 5. Components of an approximatively n-connected compactum. Let us prove the following

(5.1) Theorem. *A compactum X is approximatively n-connected $(n > 0)$ if and only if every component of it is approximatively n-connected.*

Proof. Assume that X lies in a space $M \in \mathrm{AR}(\mathfrak{M})$ and let Y be a component of X. Consider a neighborhood V of Y. Then there is a neighborhood U of X (in M) such that the component $\hat{U}$ of U containing Y lies in V.

If X is approximatively n-connected, then we infer by Theorem (2.2) that there is a neighborhood $U_0 \subset U$ of X such that for every map

$$f\colon S^n \to U_0$$

there exists apoint $x_0 \in U$ and a homotopy

$$\varphi\colon S^n \times \langle 0, 1\rangle \to U$$

such that $\varphi(z, 0) = f(z)$, $\varphi(z, 1) = x_0$ for every point $z \in S^n$.

Let V_0 denote the component of U_0 containing Y. If all values of f belong to V_0 then the set $\varphi(S^n \times \langle 0, 1\rangle)$ lies in $\hat{U} \subset V$. Thus we have assigned to every neighborhood V of Y (in M) a set V_0 (being a neighborhood of Y in M) such that every map of S^n with values in V_0 is null-homotopic in V. It follows, by Theorem (2.2), that Y is approximatively n-connected.

Now let us assume that every component of X is approximatively n-connected. Consider a point $x_0 \in X$ and let Y be the component of X containing x_0. If U is a neighborhood of X (in M), then U is also a neighborhood of Y, hence there exists a neighborhood V_0 of Y (in M) such that every map of (S^n, a) (where $a \in S^n$) into (V_0, x_0) is null-homotopic in (U, x_0). Moreover, there exists a neighborhood U_0 of X (in M) such that the component $\hat{U}_0$ of U_0 containing Y lies in V_0. Then, for every map

$$f\colon (S^n, a) \to (U_0, x_0)$$

the values of f belong to $\hat{U}_0$ and we infer that f is null-homotopic in (U, x_0). Hence X is approximatively n-connected and the proof of Theorem (5.1) is finished.

§ 6. Components of a pointed movable compactum. It is clear that the movability of a pointed compactum (X, x_0) implies the movability of X. The following theorem gives a condition characterizing the movability of (X, x_0):

(6.1) Theorem. *Let Y be a component of a compactum X and let $x_0 \in Y$. In order (X, x_0) be movable it is necessary and sufficient that X and (Y, x_0) be movable.*

Proof. Assume that X lies in a space $M \in \mathrm{AR}(\mathfrak{M})$. Let V be a neighborhood of Y (in M). Then there exists a neighborhood U of X (in M) such that the component $\hat{U}$ of U containing Y lies in V.

If (X, x_0) is movable then X is movable. Moreover, there exists a neighborhood U_0 of X such that for every neighborhood W of X (in M) there is a homotopy

$$\varphi\colon U_0 \times \langle 0, 1\rangle \to U$$

such that $\varphi(x, 0) = x$, $\varphi(x, 1) \in W$ for every point $x \in U_0$ and $\varphi(x_0, t) = x_0$ for every $0 \leqslant t \leqslant 1$.

Consider an arbitrarily given neighborhood W_0 of Y in M. Then the neighborhood W of X can be selected so that its component $\hat{W}$ containing Y lies in W_0. Let $\hat{U}_0$ denote the component of U_0 containing Y. Then $\hat{U}_0$ is a connected neighborhood of Y (in M) and $\varphi(\hat{U}_0 \times \langle 0, 1\rangle) \subset \hat{U} \subset V$. Consequently the restriction $\varphi|(\hat{U}_0 \times \langle 0, 1\rangle)$ is a homotopy joining in (V, x_0) the inclusion-map $j\colon (\hat{U}, x_0) \to (V, x_0)$ with a map having all values in the set $\hat{W} \subset W_0$. Hence (Y, x_0) is movable.

Now let us assume that X and (Y, x_0) are movable. Consider an open neighborhood U of X (in M). The movability of X implies that there exists a neighborhood U_0 of X (in M) such that for every neighborhood W of X (in M) there is a homotopy

$$\varphi\colon U_0\times\langle 0,1\rangle\to U$$

such that $\varphi(x, 0)=x$ and $\varphi(x,1)\in W$ for every point $x\in U_0$.

Let W_0 denote the component of W containing Y. The movability of (Y, x_0) implies (because U and W are neighborhoods of Y in M) that there exists a neighborhood V_0 of Y in M such that there is a homotopy

$$\psi\colon V_0\times\langle 0,1\rangle\to U$$

satisfying the conditions: $\psi(x,0)=x$, $\psi(x,1)\in W$ for every point $x\in V_0$ and $\psi(x_0,t)=x_0$ for every $0\leqslant t\leqslant 1$.

Since the neighborhood U_0 of X can be replaced by any neighborhood of X contained in U_0, we may assume that U_0 is open in M and that the component $\hat{U}$ of U_0 containing Y lies in V_0. Setting $\hat{U}'=U_0\setminus\hat{U}$, we get two disjoint open (in M) sets. It suffices to set

$$\chi(x,t)=\varphi(x,t)\quad\text{if}\quad (x,t)\in\hat{U}'\times\langle 0,1\rangle,$$

$$\chi(x,t)=\psi(x,t)\quad\text{if}\quad (x,t)\in\hat{U}\times\langle 0,1\rangle$$

in order to obtain a homotopy $\chi\colon U_0\times\langle 0,1\rangle\to U$ such that $\chi(x,0)=x$, $\chi(x,1)\in W$ for every $0\leqslant t\leqslant 1$. Hence (X, x_0) is movable and the proof of Theorem (6.1) is finished.

(6.2) Remark. It is known ([5], p. 140) that there exists a movable compactum X with a non-movable component Y. If $x_0\in Y$ then we infer by Theorem (6.1) that (X, x_0) is not movable. Consequently the movability of X does not imply the movability of (X, x_0) for every point $x_0\in X$.

(6.3) Problem *Is it true that the movability of a continuum X implies the movability of (X, x_0) for every point $x_0\in X$?*

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