n-movable compacta and ANR-systems

by

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Abstract. K. Borsuk recently introduced a new shape invariant for metric compacta called n-movability. In this paper we give an alternate description in terms of ANR-systems and generalize the notion to (Hausdorff) compacta. Using this approach we answer two questions raised by Borsuk. We also show that an n-dimensional n-movable compactum is movable.

1. Introduction. K. Borsuk [1] introduced a new shape invariant for metric compacta called n-movability. In this paper we give an alternate description in terms of ANR-systems and generalize the notion to (Hausdorff) compacta. We then apply this method to answer two questions raised by Borsuk in [1]. It is also shown that an n-dimensional n-movable compactum is movable.

A compactum X lying in the Hilbert cube I^∞ is said to be *n-movable* (in the sense of Borsuk) if for every neighborhood U of X in I^∞ there exists a neighborhood V of X (in I^∞) such that for every compactum $C \subset V$ with $\dim C \leqslant n$ and every neighborhood W of X (in I^∞) there exists a homotopy $\Phi\colon C \times [0,1] \to U$ satisfying both conditions: $\Phi(x, 0) = x$ and $\Phi(x, 1) \in W$ for every point $x \in C$. Borsuk [1] showed that n-movability is a shape invariant for metric compacta.

An ANR-system is an inverse system $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$, where each X_α is an ANR, i.e., a compact ANR for metric spaces and $p_{\alpha\alpha'}$, $\alpha \leqslant \alpha'$, $\alpha, \alpha' \in A$, are maps from $X_{\alpha'}$ into X_α; $(A, \leqslant)$ is a closure-finite directed set [8]. If $X = \operatorname{Invlim} \underline{X}$, we say that $\underline{X}$ *is associated with* X and we denote by $p_\alpha\colon X \to X_\alpha$ the natural projections. A map of systems $\underline{f}\colon \underline{X} \to \underline{Y} = \{Y_\beta, q_{\beta\beta'}, B\}$ consists of an increasing function $f\colon B \to A$ and of a collection $\{f_\beta, B\}$ of maps $f_\beta\colon X_{f(\beta)} \to Y_\beta$ such that $\beta \leqslant \beta'$ implies the homotopy relation

$$f_\beta p_{f(\beta)f(\beta')} \simeq q_{\beta\beta'} f_{\beta'}\,.$$

The identity map $\underline{1}_{\underline{X}}\colon \underline{X} \to \underline{X}$ is given by $1(\alpha) = \alpha$, $1_\alpha = 1_{X_\alpha}$. The composition of maps $\underline{f}\colon \underline{X} \to \underline{Y}$, $\underline{g}\colon \underline{Y} \to \underline{Z} = \{Z_\gamma, r_{\gamma\gamma'}, C\}$ is the map $\underline{h} = \underline{g}\underline{f}\colon \underline{X} \to \underline{Z}$ given by $h(\gamma) = fg(\gamma)$ and $h_\gamma = g_\gamma f_{g(\gamma)}$. Two maps of systems $\underline{f}, \underline{g}\colon \underline{X} \to \underline{Y}$

are said to be *homotopic*, $\underline{f} \simeq \underline{g}$, provided for every $\beta \in B$ there is an index $\alpha \in A$, $\alpha \geqslant f(\beta)$, $g(\beta)$ such that $f_\beta p_{f(\beta)\alpha} \simeq g_\beta p_{g(\beta)\alpha}$. ANR-systems $\underline{X}$ and $\underline{Y}$ are said to be of the *same homotopy type*, $\underline{X} \simeq \underline{Y}$, provided there exists maps of systems $\underline{f}\colon \underline{X} \to \underline{Y}$, $\underline{g}\colon \underline{Y} \to \underline{X}$, such that $\underline{gf} \simeq \underline{1}_{\underline{X}}$, $\underline{fg} \simeq \underline{1}_{\underline{Y}}$. If the second homotopy relation $\underline{fg} \simeq \underline{1}_{\underline{Y}}$ holds, then we say that $\underline{X}$ (shape) *dominates* $\underline{Y}$. Any two ANR-systems $\underline{X}$, $\underline{X}'$ associated with a compactum X are of the same homotopy type [8]. Therefore, if $\underline{Y}$ is associated with a compactum Y and $\underline{X}$ dominates $\underline{Y}$, then so does $\underline{X}'$. Similarly, we see if $\underline{Y}'$ is also associated with Y, then $\underline{X}$ dominates $\underline{Y}'$, so we can say X dominates Y.

2. Complexes and nerves. By a complex K is meant a finite simplicial complex K, and when the dimension of K is at most n, it is called an *n-complex*. No distinction will be made between a complex and its underlying space. We shall use some facts and conventions concerning open covers. An open cover of a space X here will be a finite collection $\mathcal{U}$ of non-empty open subsets of X whose union is X. The nerve $K(\mathcal{U})$ of $\mathcal{U}$ is the simplicial complex whose vertices are the members of $\mathcal{U}$ and whose simplices are the finite subsets s of $\mathcal{U}$ which have non-empty intersections: $\bigcap s \neq \emptyset$. If $\mathcal{V}$ refines $\mathcal{U}$, then there are projections $\pi\colon \mathcal{V} \to \mathcal{U}$ which satisfy the condition $V \subset \pi V$ for all $V \in \mathcal{V}$. Any projection defines a unique simplicial map $\pi\colon K(\mathcal{V}) \to K(\mathcal{U})$ (also a projection). A canonical map $\varphi\colon X \to K(\mathcal{U})$ is a map such that all vertices of the smallest simplex containing $\varphi(x)$ contain x (for every point $x \in X$). Any two canonical maps from X into $K(\mathcal{U})$ are homotopic.

For $f\colon X \to Y$ continuous and $\mathcal{V}$ an open cover of Y, $f^{-1}\mathcal{V} = \{f^{-1}V \mid V \in \mathcal{V}\}$. If $\mathcal{U}$ refines $f^{-1}\mathcal{V}$, then there are simplicial maps $\varrho\colon K(\mathcal{U}) \to K(\mathcal{V})$ defined by any vertex assignment $\varrho\colon \mathcal{U} \to \mathcal{V}$ satisfying $fU \subset \varrho U$ and any two such maps are contiguous. These maps are *induced* by f. When $Y = K$ is a complex and $\mathcal{V} = \mathcal{O}$ is the cover of K by open stars of vertices of K, then because K can be naturally identified with $K(\mathcal{O})$, we can speak of a map $g\colon K(\mathcal{U}) \to Y$ being induced by f. If g is induced by f and $\varphi\colon X \to K(\mathcal{U})$ is any canonical map, then $g\varphi \simeq f$.

3. n-movable ANR-systems. We now give the definition of n-movability in terms of ANR-systems.

Definition 1. An ANR-system $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ is said to be *n-movable*, provided for every $\alpha \in A$ there is an $\alpha' \in A$, $\alpha' \geqslant \alpha$, such that for every $\alpha'' \in A$, $\alpha'' \geqslant \alpha$, and every map $\varphi'\colon K \to X_{\alpha'}$ of an n-complex K into $X_{\alpha'}$ there is a map $\varphi''\colon K \to X_{\alpha''}$ with

$$p_{\alpha\alpha'}\varphi' \simeq p_{\alpha\alpha''}\varphi''. \tag{1}$$

In keeping with the view point of this definition we can also give a new formulation of movability (in the sense of Mardešić and Segal [7]).

Definition 2. An ANR-system $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ is said to be *movable*, provided for every $\alpha \in A$ there is an $\alpha' \in A$, $\alpha' \in A$, $\alpha' \geqslant \alpha$, such that for every $\alpha'' \in A$, $\alpha'' \geqslant \alpha$, and every map $\varphi'\colon K \to X_{\alpha'}$ of a complex K into $X_{\alpha'}$ there is a map $\varphi''\colon K \to X_{\alpha''}$ with

$$p_{\alpha\alpha'}\varphi' \simeq p_{\alpha\alpha''}\varphi'' \tag{2}$$

Lemma 1. *Movability in the sense of Mardešić and Segal and movability are equivalent.*

Proof. If $\underline{X}$ is movable in the sense of Mardešić and Segal, then for every $\alpha \in A$, there is an α', $\alpha' \geqslant \alpha$, and there are maps $r^{\alpha'\alpha''}\colon X_{\alpha'} \to X_{\alpha''}$ for each $\alpha'' \geqslant \alpha$ such that $p_{\alpha\alpha'} \simeq p_{\alpha\alpha''} r^{\alpha'\alpha''}$. If $\varphi'\colon K \to X_{\alpha'}$ is any map of a complex K into $X_{\alpha'}$, we define $\varphi'' = r^{\alpha'\alpha''}\varphi'$ and so (2) is obviously satisfied.

Conversely, assume $\underline{X}$ is movable. For a given $\alpha \in A$ choose α' as in Definition 2. Since the compact ANR $X_{\alpha'}$ is dominated by a complex K, there are maps $\varphi'\colon K \to X_{\alpha'}$ and $\psi\colon X_{\alpha'} \to K$ such that $\varphi'\psi \simeq 1_{X_{\alpha'}}$. Also by Definition 2 for any $\alpha'' \geqslant \alpha$ there is a map $\varphi''\colon K \to X_{\alpha''}$ such that $p_{\alpha\alpha'}\varphi' \simeq p_{\alpha\alpha''}\varphi''$. Let $r^{\alpha'\alpha''} = \varphi''\psi$. Then

$$p_{\alpha\alpha''} r^{\alpha'\alpha''} = p_{\alpha\alpha''}\varphi''\psi \simeq p_{\alpha\alpha'}\varphi'\psi \simeq p_{\alpha\alpha'},$$

i.e., $\underline{X}$ is movable in the sense of Mardešić and Segal.

Remark 1. It follows from Definition 2 that if $\underline{X}$ is movable, then it is n-movable, $n = 1, 2, \ldots$ Moreover, if $\underline{X}$ is n-movable, then $\underline{X}$ is $(n-1)$-movable.

Example 1. Every ANR-system $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ with finite sets X_α and onto bonding maps is movable because for every $\alpha'' \geqslant \alpha$ there is a map $r^{\alpha\alpha''}\colon X_\alpha \to X_{\alpha''}$ satisfying $p_{\alpha\alpha''} r^{\alpha\alpha''} = 1_{X_\alpha} = p_{\alpha\alpha}$. But then the above remark implies $\underline{X}$ is n-movable, $n = 1, 2, \ldots$

Theorem 1. *Let $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ and $\underline{Y} = \{Y_b, q_{\beta\beta'}, B\}$ be ANR-systems. If $\underline{X}$ dominates $\underline{Y}$ and $\underline{X}$ is n-movable, then $\underline{Y}$ is also n-movable.*

Proof. Let $\underline{f}\colon \underline{X} \to \underline{Y}$ and $\underline{g}\colon \underline{Y} \to \underline{X}$ be maps of ANR-systems such that $\underline{fg} \simeq \underline{1}_{\underline{Y}}$. We need to show that for every $\beta \in B$ there is a $\beta' \geqslant \beta$ such that for every $\beta'' \geqslant \beta$ and every map $\psi'\colon K \to Y_{\beta'}$ of an n-complex K into $Y_{\beta'}$ there is a map $\psi''\colon K \to Y_{\beta''}$ with

$$q_{\beta\beta'}\psi' \simeq q_{\beta\beta''}\psi''. \tag{3}$$

From the n-movability of $\underline{X}$ we have that there is an $\alpha' \geqslant \alpha = f(\beta)$ such that for every $\alpha'' \geqslant \alpha$ and every map $\varphi'\colon K \to X_{\alpha'}$ there is a map $\varphi''\colon K \to X_{\alpha''}$ satisfying (1).

Since $fg \simeq 1_{\underline{Y}}$, there exists a $\bar{\beta} \geqslant \beta$, $g(\alpha)$ such that

$$f_\beta g_\alpha q_{g(\alpha)\bar{\beta}} \simeq q_{\beta\bar{\beta}} \cdot \tag{4}$$

Since g increases, $\alpha \leqslant \alpha'$ implies $g(\alpha) \leqslant g(\alpha')$. B being directed, there exists a $\beta' \geqslant g(\alpha'), \bar{\beta}$. Just as in [7, Theorem 1] we have the relation

$$f_\beta p_{\alpha\alpha'} g_{\alpha'} q_{g(\alpha')\beta'} \simeq q_{\beta\beta'} \cdot \tag{5}$$

Now let $\beta'' \geqslant \beta$. Since f is increasing, we have $\alpha'' = f(\beta'') \geqslant f(\beta) = \alpha$. Therefore, by the definition of α', for every map $\varphi'\colon K \to X_{\alpha'}$ there is a map $\varphi''\colon K \to X_{\alpha''}$ satisfying (1). So letting

$$\varphi' = g_{\alpha'} q_{g(\alpha')\beta'} \psi' \tag{6}$$

we have there exists a map $\varphi''\colon K \to X_{\alpha''}$ such that

$$p_{\alpha\alpha'} \varphi' \simeq p_{\alpha\alpha''} \varphi''. \tag{7}$$

Finally we define $\psi''\colon K \to Y_{\alpha''}$ to be

$$\psi'' = f_{\beta''} \varphi''. \tag{8}$$

Then using (5), (6), (7), the fact that $\underline{f}\colon \underline{X} \to \underline{Y}$, and (8) we get

$$\begin{aligned} q_{\beta\beta'} \psi' &\simeq f_\beta p_{\alpha\alpha'} g_{\alpha'} q_{g(\alpha')\beta'} \psi' \\ &= f_\beta p_{\alpha\alpha'} \varphi' \\ &\simeq f_\beta p_{\alpha\alpha''} \varphi'' \\ &\simeq q_{\beta\beta''} f_{\beta''} \varphi'' \\ &= q_{\beta\beta''} \psi''. \end{aligned}$$

This is the desired relation (3) so $\underline{Y}$ is n-movable.

COROLLARY 1. *Let $\underline{X}$ and $\underline{X}'$ be ANR-systems associated with a compactum X. If $\underline{X}$ is n-movable, then so is $\underline{X}'$.*

Proof. Since $\underline{X}$ and $\underline{X}'$ are of the same homotopy type each dominates the other.

DEFINITION 3. A compactum X is said to be *n-movable* provided there is an n-movable ANR-system $\underline{X}$ associated with it. It follows from Corollary 1 that the n-movability of a compactum is independent of the choice of the associated ANR-system $\underline{X}$. As an immediate consequence of Definition 3 and Theorem 1 we have

EXAMPLE 2. Every 0-dimensional compactum is n-movable, $n = 1, 2, \ldots$, because there is an n-movable ANR-system $\underline{X}$, as in Example 1, associated with X (see also [8, Section 11]).

THEOREM 2. *Let X and Y be compacta. If X dominates Y and X is n-movable, then Y is also n-movable.*

Since we prove in the next section that our definition of n-movability agrees with that of Borsuk on metric compacta, Theorem 2 is a generalization to the non-metric case of Borsuk's result [1, Theorem (2.1)].

4. n-movability in the metric case. Every compact metric space X admits an associated ANR-sequence, i.e., an inverse sequence $\underline{X} = \{X_m, p_{mm'}\}$ of ANR's X_m such that $X = \operatorname{Inv\,lim} \underline{X}$. X is n-movable if and only if it admits an associated ANR-sequence $\underline{X}$ which is n-movable. In particular, if X is embedded in the Hilbert cube I^∞, we can define a decreasing sequence of ANR's $X_m \subset I^\infty$ such that each X_m is a neighborhood of X and $X = \bigcap_{m=1}^{\infty} X_m$. Then X is the inverse limit of the sequence $\underline{X} = \{X_m, i_{mm'}\}$, where $i_{mm'}\colon X_{m'} \to X_m$, $m \leqslant m'$, is the inclusion map. We call such an $\underline{X}$ an inclusion ANR-sequence for X and note that X is n-movable if and only if the sequence is n-movable.

THEOREM 3. *A metric compactum $X \subset I^\infty$ is n-movable if and only if it is n-movable in the sense of Borsuk.*

Proof. First assume that X is n-movable and choose an inclusion ANR-sequence $\underline{X} = \{X_m, i_{mm'}\}$ for X which is n-movable by Theorem 1. If U is a neighborhood of X in I^∞, there is an m with $X_m \subset U$. By Definition 1 there is an $m' \geqslant m$ with the property that for every $m'' \geqslant m$ and every map $\varphi'\colon K \to X_{m'}$ of an n-complex K into $X_{m'}$ there is a map $\varphi''\colon K \to X_{m''}$ with $i_{mm'}\varphi' \simeq i_{mm''}\varphi''$. Let $V = X_{m'}$, and let $\varrho\colon N \to V$ be a retraction of an open neighborhood N of V onto V. If W is a neighborhood of X, there is an $m'' \geqslant m$ such that $X_{m''} \subset W$. Let $C \subset V$ be a compactum with $\dim C \leqslant n$. Let $\varepsilon = \operatorname{dist}(V, I^\infty - N)$ and let $\mathfrak{U}$ be a cover of C by open sets in C of diameter $< \varepsilon$ such that $K(\mathfrak{U})$ has dimension $\leqslant n$. For each vertex U of $K(\mathfrak{U})$ choose a point $u \in U$ and define the map $\varphi\colon K(\mathfrak{U}) \to I^\infty$ linearly on the simplices of $K(\mathfrak{U})$: if $U_0, U_1, \ldots, U_r$ are the vertices of a simplex of $K(\mathfrak{U})$, then

$$\varphi\Big(\sum_{j=0}^{r} t_j U_j\Big) = \sum_{j=0}^{r} t_j u_j\,,$$

where $t_j \geqslant 0$ for $j = 0, 1, \ldots, r$ and $\sum_{j=0}^{r} t_j = 1$. Since $U_0 \cap U_1 \cap \ldots \cap U_r \neq \emptyset$, there is a point x in C common to all the sets U_j $(j = 0, 1, \ldots, r)$. Then the distance of the image point $\sum_{j=0}^{r} t_j u_j$ from x is

$$\Big\| \sum_{j=0}^{r} t_j x - t_j u_j \Big\| \leqslant \sum_{j=0}^{r} t_j \|x - u_j\| < \varepsilon\,,$$

where $\|p\| = \sqrt{\sum_{k=1}^{\infty} p_k^2}$ for $p = (p_1, p_2, \ldots) \in I^\infty$; hence $\varphi(K(\mathfrak{U})) \subset N$.

Let $\eta: C \to K(\mathfrak{U})$ be any canonical map. For any $x \in C$ the image $\varphi(\eta(x))$ has the form $\sum_{j=0}^{r} t_j u_j$ as above, where $x \in U_0 \cap U_1 \cap \ldots \cap U_r$; hence as above $d(x, \varphi(\eta(x))) < \varepsilon$. Since the segment with endpoints x and $\varphi(\eta(x))$ lies in N, the map

$$g_t: C \times I \to I^\infty$$

defined by

$$g_t(x) = (1-t)x + t\varphi(\eta(x))$$

is a homotopy in N between the inclusion map $C \to N$ and $\varphi\eta$.

Now there is a map $\psi'': K(\mathfrak{U}) \to X_{m''} \subset W$ such that $\psi'' \simeq \varrho\varphi$ in U; hence $\psi''\eta \simeq \varrho\varphi\eta$ in U. Since ϱg_t gives a homotopy in V between the inclusion $C \to V$ and $\varrho\varphi\eta$, it follows that there is a deformation $h_t: C \to U$ such that $h_0(x) = x$ and $h_1(C) \subset W$. So X is n-movable in the sense of Borsuk.

Now assume X is n-movable in the sense of Borsuk, and let m be given. Put $U = X_m$, and find an open neighborhood $V \subset U$ of X such that for every neighborhood W of X any compactum $C \subset V$ with $\dim C \leqslant n$ can be deformed in U into W. Choose m' so that $m' \geqslant m$ and $X_{m'} \subset V$. If $m'' \geqslant m$, put $W = X_{m''}$ and consider any map $\varphi': K \to X_{m'}$ of an n-complex K into $X_{m'}$. Let $\varepsilon = \operatorname{dist}(X_{m'}, I^\infty - V)$. Since φ' is uniformly continuous, there is a $\delta > 0$ such that any two points of K whose distance is less than δ have images under φ' whose distance is less than $\varepsilon/6$. Let K' be a subdivision of K, every simplex of which has diameter $< \delta$. For each vertex v of K' let $\varphi(v)$ be a point of V whose distance from $\varphi'(v)$ is less than $\varepsilon/6$. We assume the choice of $\varphi(v)$ made so that all the points $\varphi(v)$, where v is a vertex of K', are in general position. Then the vertex assignment $v \to \varphi(v)$ extends linearly on each simplex of K' to give a homeomorphism φ of K' into I^∞. The diameter of the φ-image of any simplex of K' is less than $\varepsilon/2$, because the distance between the φ-images of any two vertices of a simplex of K' is less than $\varepsilon/2$. The diameter of the φ'-images of any simplex of K' is less than $\varepsilon/6$. Since $\|\varphi(v) - \varphi'(v)\| < \varepsilon/6$ for every vertex of K', it then follows that $\|\varphi(p) - \varphi'(p)\| < \varepsilon$ for every point of K'. By the choice of ε the homotopy

$$\varphi_t(p) = (1-t)\varphi'(p) + t\varphi(p), \quad p \in K'$$

occurs in V. Put $C = \varphi(K')$. By Borsuk's definition there is a deformation

$$h_t: C \to U = X_m$$

such that $h_1(C) \subset W$. Define $\varphi'': K' \to X_{m''}$ by $\varphi'' = h_1\varphi$. Then in X_m we have $\varphi'' \simeq h_0\varphi = \varphi \simeq \varphi'$.

5. Finite dimensional movable compacta. In this section we show that an n-movable compactum of dimension $\leqslant n$ is movable.

LEMMA 2. *Let X be the inverse limit of the inverse system $\{X_\alpha, p_{\alpha\alpha'}, A\}$ of compacta, and let $f: X \to Y$ and $f_\alpha: X_\alpha \to Y$ (for each $\alpha \in A$) be maps into a compact* ANR *Y such that $f = f_\alpha p_\alpha$, for all $\alpha \in A$, where $p_\alpha: X \to X_\alpha$ is the natural projection. If the (covering) dimension of X is at most n, then there exist an n-complex K, an index $\beta \in A$, and maps $h: X_\beta \to K$, $g: K \to Y$ such that $gh \simeq f_\beta$.*

Proof. Since Y is a compact ANR, there is a finite complex L and maps $p: Y \to L$ and $q: L \to Y$ such that $qp \simeq 1_Y$. If the Lemma is true when Y is replaced by L and f, f_α replaced by pf, pf_α, respectively, then it is also true for Y, because we take map the $\bar{g}: K \to L$ given by the modified Lemma and define $g = q\bar{g}$. Since $\bar{g}h \simeq pf_\beta$, $gh = q\bar{g}h \simeq qpf_\beta \simeq f_\beta$. Therefore it suffices to prove the Lemma for the case when Y is a complex L.

Let $\mathcal{O}$ be the open star cover of L and let $\mathfrak{U}$ be a cover of X refining $f^{-1}\mathcal{O}$ whose nerve $K(\mathfrak{U})$ is an n-complex. Take $K = K(\mathfrak{U})$, φ any canonical map $X \to K(\mathfrak{U})$, and g any map $K(\mathfrak{U}) \to L$ induced by f.

By Lemma 3.8 of [2, p. 263] there exist an index $\beta \in A$, an open cover $\mathfrak{V}$ of X_β such that $p_\beta^{-1}\mathfrak{V}$ refines $\mathfrak{U}$, and a p_β-induced map $\varrho: K(p_\beta^{-1}\mathfrak{V}) \to K(\mathfrak{U})$ which is a simplicial isomorphism. Let $\pi: K(p_\beta^{-1}\mathfrak{V}) \to K(\mathfrak{U})$ be a projection and $\psi: X_\beta \to K(\mathfrak{V})$ a canonical map, and define $h = \pi\varrho^{-1}\psi$.

Since $p_\beta^{-1}\mathfrak{V}$ refines $f^{-1}\mathcal{O} = p_\beta^{-1}f_\beta^{-1}\mathcal{O}$, $\mathfrak{V}$ refines $f_\beta^{-1}\mathcal{O}$. Let $g_\beta: K(\mathfrak{V}) \to L$ be induced by f_β. It is easily seen that both $g\pi$ and $g_\beta\varrho$ are maps $K(p_\beta^{-1}\mathfrak{V}) \to L$ which are induced by $f = f_\beta p_\beta$; hence $g\pi \simeq g_\beta\varrho$. Therefore $gh = g\pi\varrho^{-1}\psi \simeq f_\beta$.

THEOREM 4. *If X is an n-movable compactum of (covering) dimension $\leqslant n$, then X is movable.*

Proof. Let $\underline{X} = \{X_\alpha, p_{\alpha\alpha'}, A\}$ be an ANR-system associated with X such that each X_α is a compact ANR (see [8, Theorem 7]). We shall show that for any $\alpha \in A$ there is a $\beta \geqslant \alpha$ such that for each $\alpha'' \geqslant \alpha$ there is a map $r^{\beta\alpha''}: X_\beta \to X_{\alpha''}$ with $p_{\alpha\alpha''}r^{\beta\alpha''} \simeq p_{\alpha\beta}$. For a given α choose α' as in the definition of n-movable. By taking $Y = X_{\alpha'}$, $f = p_{\alpha'}$, $f_\gamma = p_{\alpha'\gamma}$, and the inverse system associated with X consisting of those X_α with $\gamma \geqslant \alpha'$ we find by Lemma 2 that there exist $\beta \geqslant \alpha'$, an n-complex K and maps $g: K \to X_{\alpha'}$, $h: X_\beta \to K$ such that $gh \simeq p_{\alpha'\beta}$. By the definition of n-movability for every $\alpha'' \geqslant \alpha$ there is a map $\varphi: K \to X_{\alpha''}$ with $p_{\alpha\alpha''}\varphi \simeq p_{\alpha\alpha'}g$. Then taking $r^{\beta\alpha''} = \varphi h$ yields the desired relation

$$p_{\alpha\alpha''}r^{\beta\alpha''} = p_{\alpha\alpha''}\varphi h \simeq p_{\alpha\alpha'}gh \simeq p_{\alpha\alpha'}p_{\alpha'\beta} = p_{\alpha\beta}.$$

Remark 2. It would have been convenient if the compactum X of dimension $\leqslant n$ in Theorem 4 could have been represented as the inverse limit of an inverse system of polyhedra of dimension $\leqslant n$. However, this is not possible in general (see [6]).

6. n-movability and sphere-like continua. Let $Q = (q_1, q_2, \ldots)$ be a sequence of primes and let $S_Q^n = \{X_m, p_{mm+1}\}$ be an inverse sequence of n-spheres $X_m = S^n$, where the bonding maps are maps of S^n into S^n of degree q_m. Then the shape of the inverse limit S_Q^n of $\underline{S}_Q^n$ is completely determined. It was proved in [8, Theorem 19] that every metric S^n-like continuum X is of the shape of a point, S^n or some S_Q^n. The first two shapes are obviously movable but the third is not [7]. We will show that the S_Q^n is $(n-1)$-movable but not n-movable. This gives a positive answer to problem (4.6) of [1]. S_Q^n is $(n-1)$-movable: since all mappings of a space of dimension $<n$ into S^n are inessential [4, Theorem VI.4] they are homotopic. By Theorem 4 S_Q^n is not n-movable it is n-dimensional but not movable [7].

7. n-movability and an example of Kahn. In [3] it was shown that a continuum $X = \operatorname{Invlim}\{X_m, p_{mm'}\}$ described by Kahn in [5] is not movable. (Actually a family of such continua was described.) Here we show that X is n-movable for $n = 1, 2, \ldots$ This yields a positive answer to problem 4.7 of [1].

Given any positive integer m, there is an $m' \geqslant m$ such that any map φ' of any polyhedron P of dimension $\leqslant n$ into $X_{m'}$ is inessential. (Since the connectivity of X_m increases as m does we just choose m' large enough so that the connectivity of $X_{m'}$ is greater than n.) So $p_{mm'}\varphi'$ is also inessential. For any $m'' > m'$ let $\varphi''\colon P \to X_{m''}$ be any constant map. Then

$$p_{mm'}\varphi' \simeq p_{mm''}\varphi''.$$

Hence X is n-movable for $n = 1, 2, \ldots$

Added in proof. The referee has pointed out that some of these results are included in the paper of A. Kodama and T. Watanabe, *A note on Borsuk's n-movability*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. (to appear).

References

[1] K. Borsuk, *On the n-movability*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 20 (1972), pp. 859–864.

[2] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton 1952.

[3] D. Handel and J. Segal, *An acyclic continuum with non-movable suspensions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 21 (1973), pp. 171–172.

[4] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton 1948.

[5] D. S. Kahn, *An example in Čech cohomology*, Proc. Amer. Math. Soc. 16 (1956), p. 584.

[6] S. Mardešić, *On covering dimension and inverse limits of compact spaces*, Illinois J. Math. 4 (1960), pp. 278–291.

[7] — and J. Segal, *Movable compacta and ANR-systems*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 18 (1970), pp. 649–654.

[8] — — *Shapes of compacta and ANR-systems*, Fund. Math. 72 (1972), pp. 41–59.

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