Reflective functors via nearness

by

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Abstract. Using the concept of nearness for arbitrary families of sets, which is a generalization of proximity and contiguity, the well-known theorem of Taimanov on extensions of continuous functions from dense subspaces is generalized. This result is then used to show that all T_3 extensions are reflections.

1. Introduction. The concept of *proximity* or nearness between pairs of subsets was first introduced by F. Riesz [22] in 1908 but was ignored by mathematicians until the fifties when Efremovič [6] and Smirnov [25] systematically developed the now classical theory (for a compact account see Naimpally and Warrack [21]). Lodato [20] considered a generalized proximity which has proved to be of great value in solving many topological problems; in particular, it was used by Gagrat and Naimpally [8] to obtain a generalization of the well-known theorem of Taimanov [27] concerning extensions of continuous functions from dense subspaces. Another type of generalization was discovered by Herrlich [12] who used topological bases for closed sets.

In considering T_1-extensions Ivanova and Ivanov [16] generalized the concept of proximity to *contiguity* in which nearness of finitely many sets is postulated. In this paper we further generalize the concept to *nearness* of families of sets of arbitrary cardinality. This concept includes as special cases topologies, proximities, uniformities etc. and an account is written by Herrlich [13]. His axioms are different from ours. Our purpose here is to prove, in a certain sense, an ultimate generalization of Taimanov's theorem which yields a general theory of reflective functors. In our work here, a near structure is generated in either of the two ways: (i) that induced on X by a super space αX in which X is dense and (ii) that induced on X by separating bases of Steiner [26]. From our general result it will follow that for *strict* Hausdorff extensions the best we can do is to get θ-continuous extensions (Fomin [7], Hunsaker and Naimpally [14], Rudolf [23]). In case of *simple* Hausdorff extensions and all T_3 extensions we get continuous extensions which shows that they are all reflections.

We then show that from our results we can handle even non-compact extensions such as Banaschewski T_2-minimal extension [3], Katětov extension ([9], [17]), Liu's αX [18], Liu and Strecker's ϱX [19] etc. We also generalize a recent result of Hunsaker and Sharma [15] concerning the Harris regular-closed extension [11]. This should be compared with EF-proximities and contiguities which are useful in dealing with Hausdorff compactifications and T_1-compactifications respectively. Also we note that the concept of nearness is related to the theory of structures of Harris [10].

We have given a fairly representative bibliography on reflective functors and the interested reader will find further references in the items included here.

2. Near structure. It is well known that every EF-proximity (see 2.3 (v)) on a Tychonoff space X is induced by the EF-proximity δ_0 on a suitable compact Hausdorff space αX, where $A\,\delta_0\,B$ iff their closures in αX intersect. The most general situation that we can think of is: we are given a topological space X which is dense in a topological space αX; in this case we may say that a family $\mathcal{A}$ of subsets of X is *near* iff the closures of the members of $\mathcal{A}$ in αX have a common point. This then provides a motivation for defining a *near structure* on X; indeed with certain additional assumptions every near structure on X is obtained in the above manner from some superspace αX; see Thron [28] for a similar result concerning LO-proximities and Herrlich [13].

We now axiomatize the concept of nearness and the reader will find the axioms natural if the above example is kept in view. For notation we write "$\mathcal{A}$ is near" by "$\eta\mathcal{A}$" and "$\mathcal{A}$ is not near" by "$\bar{\eta}\mathcal{A}$". We write $B\eta\mathcal{A}$ for $\eta\{\{B\}\cup\mathcal{A}\}$ and $\mathrm{Cl}_\eta A = \{x \in X\colon \{x\}\eta A\}$. P^nX denotes the power set $P^{n-1}X$ for $n \in N$, where $P^0X = X$. $\mathcal{A}$, $\mathcal{B}$ denote subsets of PX. $\mathcal{A}\vee\mathcal{B} = \{A\cup B\colon A \in \mathcal{A},\ B \in \mathcal{B}\}$.

2.1. Definition. Let X be a non-empty set and $\eta \subset P^2X$. Then η is called a *Čech near structure* or *Čech nearness* on X iff

(a) $\bigcap\{A\colon A \in \mathcal{A}\} \neq \emptyset$ implies $\eta\mathcal{A}$.

(b) $\bar{\eta}\mathcal{A}$ and $\bar{\eta}\mathcal{B}$ implies $\bar{\eta}(\mathcal{A}\vee\mathcal{B})$.

(c) $\eta\mathcal{A}$ and for all $B \in \mathcal{B}$ there is an $A \in \mathcal{A}$ such that $A \subset B$ implies $\eta\mathcal{B}$.

(d) $\emptyset \in \mathcal{A}$ implies $\bar{\eta}\mathcal{A}$.

The pair (X, η) is called a *Čech near space*. If X is a topological space and $\mathrm{Cl}_\eta A \supset A^-$ (the topological closure) for each $A \subset X$, then η is said to be *associated* with the space X. If further $A^- = \mathrm{Cl}_\eta A$, then η is said to be *compatible* with the space X.

2.2. Remarks. The term "Čech nearness" is an obvious analogue of "Čech proximity" studied by Thron and Warren [29]. The most important Čech near structure η_0 associated with a topological space X is

defined as follows:

$$\eta_0\mathcal{A} \quad \text{iff} \quad \bigcap\{A^-\colon A \in \mathcal{A}\} \neq \emptyset\,.$$

Further if X is an R_0-space (i.e. $x \in \{y\}^-$ implies $y \in \{x\}^-$ for all x, y in X), then η_0 is compatible with X.

Notation. $\mathcal{A}^- = \{A^-\colon A \in \mathcal{A}\}$.

2.3. Definitions. A Čech near structure η on X is called:

(i) LO iff $\eta\mathcal{A} \Leftrightarrow \eta\mathcal{A}^-$.

(ii) T_1 iff η is LO and $x \neq y \Rightarrow \{x\}\,\bar{\eta}\,\{y\}$.

(iii) H iff $x \neq y$ implies the existence of an $E \subset X$ such that $\{x\}\,\bar{\eta}\,E$ and $(X-E)\,\bar{\eta}\,\{y\}$.

(iv) R iff η is T_1 and $x \notin A^-$ implies the existence of an $E \subset X$ such that $\{x\}\,\bar{\eta}\,E$ and $(X-E)\,\bar{\eta}\,A$. (Note that this is patterned after Harris' R-proximity [11]; however, an R-proximity need not be a LO-proximity.)

(v) EF iff η is T_1 and $A\,\bar{\eta}\,B$ implies the existence of an $E \subset X$ such that $A\,\bar{\eta}\,E$ and $(X-E)\,\bar{\eta}\,B$.

Obviously $\mathrm{EF} \Rightarrow R \Rightarrow H \Rightarrow T_1 \Rightarrow \mathrm{LO} \Rightarrow$ Čech.

2.4. Definition. Let k be an arbitrary infinite cardinal. A Čech near structure η is called a *k-Čech near structure* iff $\bar{\eta}\mathcal{A}$ implies there exists $\mathcal{B} \subset \mathcal{A}$ such that $|\mathcal{B}| < k$ and $\bar{\eta}\mathcal{B}$, and k is the smallest cardinal having this property. The contiguity of Ivanova and Ivanov [16] is an $\aleph_0$-T_1 near structure.

2.5. Example. Another important method of constructing compatible near structures on T_1-spaces is through the separating bases of Steiner [26]. A *separating* or S_1 *base* $\mathfrak{L}$ on a T_1-space X is a ring of closed subsets of X such that whenever $x \notin A$, a closed subset of X, there exist $L_x, L_A \in \mathfrak{L}$ such that $x \in L_x$, $A \subset L_A$ and $L_x \cap L_A = \emptyset$. An S_1-base is (i) S_2 iff $x \neq y$ implies the existence of $L'_x, L'_y \in \mathfrak{L}$ such that $x \notin L'_x$, $y \notin L'_y$ and $L'_x \cup L'_y = X$ (ii) S_3 iff $x \notin A$ a closed subset of X implies the existence of L'_x, L'_A such that $x \notin L'_x$, $A \cap L'_A = \emptyset$ and $L'_x \cup L'_A = X$ (iii) S_4 (or normal) iff $L_1, L_2 \in \mathfrak{L}$ and $L_1 \cap L_2 = \emptyset$ implies the existence of $L'_1, L'_2 \in \mathfrak{L}$ such that $L_1 \cap L'_1 = \emptyset$, $L_2 \cap L'_2 = \emptyset$ and $L'_1 \cup L'_2 = X$.

Given an S_1-base $\mathfrak{L}$ we define a Čech near structure $\eta = \eta(\mathfrak{L})$ as follows: $\bar{\eta}\mathcal{A}$ iff there is a function $f\colon \mathcal{A} \to \mathfrak{L}$ such that $A \subset f(A)$ and $\bigcap\{f(A)\colon A \in \mathcal{A}\} = \emptyset$. It is easily verified that $\eta(\mathfrak{L})$ is always a compatible T_1-near structure; in fact, $\eta(\mathfrak{L})$ is H, R or EF according as $\mathfrak{L}$ is S_2, S_3 or S_4. Conversely, every T_α-space $(\alpha = 1, 2, 3, 4)$ X has an associated S_α-base $\mathfrak{L}$ $(\alpha = 1, 2, 3, 4)$ which induces the comptible near structure $\eta(\mathfrak{L})$ on X. It is an open problem whether every EF space has a compatible S_4 base. We may define analogously k-Čech near structure $\eta_k(\mathfrak{L})$; in particular, the contiguity $\eta_c(\mathfrak{L})$.

2.6. DEFINITION. Let (X, η_1) and (Y, η_2) be two Čech near spaces. A function $f\colon X \to Y$ is called a *near map* iff for all $\mathcal{A} \subset PX$, $\eta_1 \mathcal{A}$ implies $\eta_2 f(\mathcal{A})$, where $f(\mathcal{A}) = \{f(A)\colon A \in \mathcal{A}\}$.

2.7. THEOREM. *Let X and Y be topological spaces, η_0 the associated Čech near structure on X* (see 2.2) *and η_2 be any* LO *near structure on Y. Then $f\colon X \to Y$ is a near map if f is continuous and the converse holds if η_2 is compatible with Y.*

Proof. Suppose f is continuous; then

$$\begin{aligned}\eta_0 \mathcal{A} &\Rightarrow \bigcap \{A^-\colon A \in \mathcal{A}\} \neq \emptyset\\ &\Rightarrow \bigcap \{f(A^-)\colon A \in \mathcal{A}\} \neq \emptyset\\ &\Rightarrow \bigcap \{f(A)^-\colon A \in \mathcal{A}\} \neq \emptyset\\ &\qquad \text{since } f \text{ is continuous}\\ &\Rightarrow \eta_2 f(\mathcal{A})^-\\ &\Rightarrow \eta_2 f(\mathcal{A}).\end{aligned}$$

Thus f is continuous implies f is a near map.

Conversely, if f is a near map, then

$$x \in A^- \Rightarrow \{x\}\, \eta_0\, A \Rightarrow \{f(x)\}\, \eta_2\, f(A) \Rightarrow f(x) \in f(A)^-$$

and f is continuous.

2.8. COROLLARY. (Necessity of the generalized Taimanov Theorem.) *Let X be dense in a topological space αX and let X be assigned the Čech near structure η_1 induced by η_0 on αX. Let η_2 be a compatible* LO *near structure on Y. Then a necessary condition that $f\colon (X, \eta_1) \to (Y, \eta_2)$ has a continuous extension $\bar{f}\colon \alpha X \to Y$ is that f is a near map.*

Our motivation for the introduction of the concept of a near clan is again the situation with which we began this section. Suppose X is dense in a topological space αX and that we have information only about X. In order to explore αX we must express each $x \in \alpha X$ in terms of certain objects formed from subsets of X and the most natural one is $\sigma^x = \{E \subset X\colon x \in \mathrm{Cl}_{\alpha X} E\}$. Two simple properties satisfied by σ^x in terms of the Čech near structure η induced on X by η_0 on αX results in the following definition.

2.9. DEFINITION. A *near clan* σ in a Čech near space (X, η) is a subset of PX satisfying:

(a) $\eta\sigma$,

(b) $(A \cup B) \in \sigma$ iff $A \in \sigma$ or $B \in \sigma$.

A near bunch is a near clan σ such that $A \in \sigma$ iff $A^- \in \sigma$.

A *near cluster* σ is a near bunch in which

(c) $A \notin \sigma$ implies the existence of a $B \in \sigma$ such that $A\, \bar{\eta}\, B$.

2.10. EXAMPLE. The most important example of a near clan is σ^x defined in the paragraph just preceeding 2.9. If αX is compact Hausdorff, then σ^x is a near cluster.

We state below several results which are either known or follow easily from our definitions:

2.11. LEMMA. *Let η be a Čech near structure associated with a topological space X. In case $X^- = \alpha X$ we assume that η is induced by η_0 on αX.*

(i) If $\mathcal{F}$ is an ultrafilter and η is a LO-contiguity, then

$$b(\mathcal{F}) = \{A \subset X\colon A^- \in \mathcal{F}\}$$

is a near bunch called *contiguity bunch*.

(ii) If $\mathcal{L}$ is a separating base on X, $\eta = \eta_c(\mathcal{L})$ is the contiguity induced by $\mathcal{L}$, and $\mathcal{F}$ is an $\mathcal{L}$-ultrafilter on X, then

$$\sigma(\mathcal{F}) = \{E \subset X\colon E\, \eta\, \mathcal{F}\},$$

is a near cluster called *contiguity cluster* in [4].

(iii) Every contiguity clan is contained in a maximal contiguity clan. If η is EF, then each contiguity bunch is contained in a unique maximal contiguity bunch which is a contiguity cluster (cf. [8]).

(iv) For each $x \in X$,

$$\sigma_x = \{E \subset X\colon x \in E^-\},$$

is a near bunch called *point near bunch*. If η is compatible with X, then σ_x is a near cluster called *point near cluster*.

(v) If η is compatible with X, σ is a near bunch and $\{x\} \in \sigma$, then $\sigma = \sigma_x$.

(vi) If σ is a near bunch, $A \in \sigma$ and $A \subset B$, then $B \in \sigma$. In particular, $X \in \sigma$.

(vii) If η is H and σ is a near bunch, then there is at most one $x \in X$ such that $\{x\}\, \eta\, \sigma$.

(viii) If αX is T_3, then for each $x \in \alpha X$, σ^x is a near cluster.

(ix) If αX is compact, then η is a contiguity.

2.12. DEFINITIONS. Suppose $X^- = \alpha X$ and the Čech near structure on X is induced by η on αX. We say that X is *relatively clan* (resp. *cluster*) *complete* in αX iff for every near clan (resp. cluster) in (X, η) there corresponds an $x \in \alpha X$ such that $\{x\}\, \eta\, \sigma$. If $X = \alpha X$ we drop the word "relatively"; note that in this case if σ is a near cluster, then $\{x\}\, \eta\, \sigma$ implies $\{x\} \in \sigma$ and $\sigma = \sigma_x$.

2.13. Remarks. The most important example of a relatively clan complete space is $X^- = \alpha X$ and αX has the Čech near structure η_0. In case αX is compact Hausdorff, X is relatively cluster complete in αX and we have the classical EF proximity theory, due to Leader [30], of Efremovič–Smirnov ([6], [25]). Another important example is: suppose $\mathfrak{L}$ is a separating base on a T_1-space X and $w(X, \mathfrak{L})$ is the corresponding Wallman compactification of X; then X is relatively cluster complete in $w(X, \mathfrak{L})$ when X is assigned the contiguity $\eta_c(\mathfrak{L})$ which is the subspace contiguity induced by η_0 on $w(X, \mathfrak{L})$ (see [4] for details). Note that $(X, \eta(\mathfrak{L}))$ is bunch-complete.

2.14. Definitions. Let η be a Čech near structure associated with a topological space X and let Σ_X denote the set of all near clans in X. For $\underset{\sim}{P} \subset \Sigma_X$ and $A \subset X$ we say that A *absorbs* $\underset{\sim}{P}$ iff $A \in \sigma'$ for each $\sigma' \in \underset{\sim}{P}$. It is easy to verify that, "$\sigma \in \mathrm{Cl}\underset{\sim}{P}$ iff whenever A absorbs $\underset{\sim}{P}$, $A \in \sigma$," defines a Kuratowski closure operator on Σ_X. The resulting topology is called the *absorption* or *A-topology*.

2.15. Theorem. *Let $X^- = \alpha X$ and let η be the Čech near structure on X induced by η_0 on αX. The map $\Phi\colon \alpha X \to \Sigma_X$ defined by $\Phi(x) = \sigma^x = \{E \subset X\colon \{x\}\, \eta_0\, E\}$ is continuous.*

Proof. Let $x \in \alpha X$, $E \subset \alpha X$ and $\Phi(x) \notin \Phi(E)$. Then there is an $A \subset X$ which absorbs $\Phi(E)$ and $A \notin \Phi(x)$. This implies that $E \subset \mathrm{Cl}_{\alpha X} A$ and $x \notin \mathrm{Cl}_{\alpha X} A$, i.e. $x \notin \mathrm{Cl}_{\alpha X} E$, showing thereby that Φ is continuous.

2.16. Corollary. *If αX is T_1 and $\{\mathrm{Cl}_{\alpha X} A\colon A \subset X\}$ is a base for closed sets of αX (i.e. αX has the strict extension topology, see Banaschewski* [2]), *then Φ is a homeomorphism.*

Proof. Φ is obviously one-to-one. We show that Φ is closed. Suppose $x \in \mathrm{Cl} E$; then there is an $A \subset X$ such that $x \notin \mathrm{Cl}_{\alpha X} A$ and $\mathrm{Cl} E \subset \mathrm{Cl}_{\alpha X} A$. So $A \notin \sigma^x$ and A absorbs $\Phi(E)$, i.e. $\Phi(x) \notin \mathrm{Cl}\Phi(E)$.

2.17. Theorem. *Let η_1, η_2 be two Čech near structures associated with topological spaces X and Y respectively. If $f\colon X \to Y$ is a near map, then the function $f_\Sigma\colon \Sigma_X \to \Sigma_Y$ defined by*

$$f_\Sigma(\sigma) = \{E \subset Y\colon f^{-1}(E) \in \sigma\},$$

is continuous.

Proof. It is easily checked that $f_\Sigma(\sigma) \in \Sigma_Y$. Let $\sigma \in \Sigma_X$, $\underset{\sim}{P} \subset \Sigma_X$ and $f_\Sigma(\sigma) \notin \mathrm{Cl} f_\Sigma(\underset{\sim}{P})$. Then there is an $A \subset Y$ such that A absorbs $f_\Sigma(\underset{\sim}{P})$ and $A \notin f_\Sigma(\sigma)$. Clearly $f^{-1}(A)$ absorbs $\underset{\sim}{P}$ and $f^{-1}(A) \notin \sigma$, i.e. $\sigma \notin \mathrm{Cl}\underset{\sim}{P}$, thus showing that f_Σ is continuous.

2.18. Remark. If in the above theorem, f is also continuous, then it is easily checked that

$$f_\Sigma(\sigma_x) = \sigma_{f(x)},$$

and we may consider f_Σ to be a continuous extension of f.

We recall that a function $f\colon X \to Y$ is called θ-continuous (Fomin [7]) iff for each $x \in X$, $f(x) \in V$ open in Y implies the existence of an open set U in X such that $x \in U$ and $f(U^-) \subset V^-$. An important problem in Topology is to find conditions for θ-continuous extensions of continuous functions (see Rudolf [23]).

2.19. Theorem. *Let X be dense in αX and let η_1 be a near structure on X induced by an H near structure η on αX. Let X be relatively clan complete in αX. Then the map*

$$\psi\colon \Sigma_X \to \alpha X,$$

where $\psi(\sigma) = x$, the unique point such that $\{x\}\, \eta\, \sigma$, is θ-continuous.

Proof. Suppose U'_x is an open nbhd. of x in αX and let $U_x = U'_x \cap X$. Then $\{x\}\, \bar{\eta}\, F$, where $F = X - U_x$ and $F \notin \sigma$ implies $\sigma \in \Sigma_X - F^*$, where $F^* = \{b \in \Sigma_X\colon F \in b\}$. We must show that if $\sigma_1 \in \mathrm{Cl}(\Sigma_X - F^*)$, then $x_1 \in \mathrm{Cl}\, U'_x$, where $\{x_1\}\, \eta\, \sigma_1$. Suppose $x_1 \in U'_{x_1}$ open in αX; $U_{x_1} = U'_{x_1} \cap X$ and $F_1 = X - U_{x_1}$. $\{x_1\}\, \bar{\eta}\, F_1$ implies

$$\begin{aligned} \sigma_1 \notin F_1^* &\Rightarrow \sigma_1 \in \Sigma_X - F_1^* \\ &\Rightarrow (\Sigma_X - F_1^*) \cap (\Sigma_X - F^*) \neq \emptyset \\ &= F_1^* \cup F^* \neq X \\ &= U_{x_1} \cap U_x \neq \emptyset \\ &= x_1 \in \mathrm{Cl}\, U'_x . \end{aligned}$$

2.20. Remark. Herrlich's Example 3 in § 2 [12] shows that we cannot hope to get continuity of ψ.

2.21. Corollary. *If η is R, then the map ψ is continuous.*

2.22. Theorem. (*Generalized Taimanov Theorem.*) *Suppose X is dense in a topological space αX and has an associated Čech near structure η_1 induced by η_0 on αX. Let Y be relatively bunch complete in λY and let η_2 on Y be induced by an R near structure η on λY. Then a necessary and sufficient condition that a continuous function $f\colon X \to Y$ has a continuous extension $\bar{f}\colon \alpha X \to \lambda Y$ is that f is a near map.*

Proof. The result follows from the following:

$$\begin{tikzcd} \alpha X \arrow[r, "\varphi", "(2.15)"'] & \Sigma_X \arrow[r, "f_\Sigma", "(2.17)"'] & \Sigma_Y \arrow[r, "\psi", "(2.21)"'] & \lambda Y \\ & X \arrow[ul] \arrow[r, "f", "\bar{f} = \psi f_\Sigma \varphi."'] & Y \arrow[ur] & \end{tikzcd}$$

2.23. Remark. The above result includes, as special cases, several previously known generalizations of Taimanov's theorem; this is what we propose to show below.

First suppose that $\eta = \eta_0$ and λY is compact. Then from 2.11 (ix) it follows that η_2 is a contiguity and we may take f to be a *contiguity map*. If further λY is Hausdorff, then η_2 is EF and λY is the Smirnov compactification of (Y, δ_2), where δ_2 is the natural proximity associated with η_2. We now show that in this case we may further reduce the map f to a *proximally continuous* map. We need only show that if $f\colon X \to Y$ is proximally continuous, then f is a contiguity map. Suppose $\bar{\eta}_2$ $(A_i\colon 1 \leqslant i \leqslant n)$, $A_i \subset Y$. Then $\bigcap_{i=1}^{n} \mathrm{Cl}_{\lambda Y} A_i = \emptyset$ and so for each $p \in \lambda Y$, there corresponds a $j_p \in \{1, \ldots, n\}$ such that $p \notin \mathrm{Cl}_{\lambda Y} A_{j_p}$. Since η is EF and hence R, there is a nbhd. U_p of p such that $U_p \bar{\eta} A_{j_p}$. Since λY is compact, there are $p_i \in \lambda Y$, $1 \leqslant i \leqslant n$ such that $\lambda Y = \bigcup_{i=1}^{n} U_{p_i}$. If $V_i = U_{p_i} \cap Y$, then $Y = \bigcup_{i=1}^{n} V_i$ and $V_i \bar{\eta}_2 A_{j_i}$ (where we write A_{j_i} for $A_{j_{p_i}}$). Clearly $X = \bigcup_{i=1}^{n} f^{-1}(V_i)$ and $f^{-1}(V_i) \bar{\eta}_1 f^{-1}(A_{j_i})$. Since X is dense in aX, it follows that $\bar{\eta}_1 \{f^{-1}(A_{j_i})\colon 1 \leqslant i \leqslant n\}$ and so f is a contiguity map.

Thus we have shown:

2.24. Lemma. *If in Theorem 2.22, λY is compact Hausdorff and $\eta = \eta_0$ on λY, then f is a near map if and only if f is proximally continuous.*

The main result of [8] viz. Theorem 5.1 now follows from 2.22 and 2.24; in fact, we get a slightly more general result since we do not assume aX to be R_0. It was shown in [8] how the above mentioned result includes as special cases several of the known results on extensions of continuous functions from dense subspaces. However, proximities as they deal with only *two* sets at a time, were found to be rather awkward in dealing with Lindelöf or real compact extensions, where *countably* many sets occur (see e.g. Theorem (6.1) of [8]). The concept of nearness is, on the other hand, capable of handling such cases, as we show below.

Suppose λY is T_3 Lindelöf (and hence normal) and $\eta = \eta_0$ on λY. Clearly in this case η_2 is EF and if $\bar{\eta}_2 \mathcal{A}$, then there exist $\mathcal{B} = \{A_n\colon n \in N\} \subset \mathcal{A}$ such that $\bar{\eta}_2 \mathcal{B}$. Thus η_2 is ω-contiguity (see [5]) or c-nearness and $f\colon X \to Y$ is a near map iff f is c-near. Proceeding in a similar manner, we may take a strong delta normal base $\mathfrak{L}$ on Y and take λY to be the $\mathfrak{L}$-realcompactification of Y (see Alò and Shapiro [1]). In this case, again $\eta_2 = \eta_c(\mathfrak{L})$ and f is a near map iff f is c-near.

3. Reflective functors. In this section we show how, in contrast to the main result (5.1) of [8], our Theorem 2.22 enables us to handle non-compact extensions with ease. In particular, some of the results of [14] follow easily.

In this section we suppose that all topological spaces are *Hausdorff* and that each space X is dense in some prescribed extension aX. Further each aX is assigned the H-near structure η_0 and each X is assigned the near structure induced by η_0. An important class of problems is to find necessary and sufficient conditions on $f\colon X \to Y$ to have a continuous extension $\bar{f}\colon aX \to aY$; in other words, to find a class of maps for which aX is a reflection of X.

3.1. Strict extensions. If each aX is a strict extension of X, then Theorems 2.15, 2.17, 2.19 show that each near map $f\colon X \to Y$ has a θ-continuous extension $\bar{f}\colon aX \to aY$. Examples of this type are the Banaschewski T_2-minimal extension [3] and the Fomin extension [7]. In this connection we note that every λ-map of [14] is a near map (the proof of this is similar to that of "p-map implies near map" proved below) and so our present theory includes some of the results of [14].

3.2. Simple extensions. If each extension aX is simple, then Theorems 2.15, 2.17, 2.19 show that a necessary and sufficient condition that $f\colon X \to Y$ has a continuous extension $\bar{f}\colon aX \to aY$ is that f is a near map. We now discuss the case of Katětov extension [17] and relate our results to those of Harris [9]; we do not consider here aX of Liu, ϱX of Liu and Strecker [19] which are discussed in [14].

Let $aX = \tau X$ the Katětov extension of X and let η_τ be the near structure induced on X by η_0 on τX. The following is obvious:

3.3. Theorem. *For $\mathcal{A} \subset PX$, $\eta_\tau \mathcal{A}$ iff there is a p-filter $\mathcal{F}$ in X such that $A \cap F \neq \emptyset$ for each $A \in \mathcal{A}$ and each $F \in \mathcal{F}$ iff for every p-cover a of X there is an $E \in a$ such that $E \cap A \neq \emptyset$ for each $A \in \mathcal{A}$. Further $\bar{\eta}_\tau\, \mathcal{A}$ implies the existence of a finite subset $\{A_i\colon 1 \leqslant i \leqslant n\}$ of $\mathcal{A}$ such that $\bigcap_{i=1}^{n} A_i^{-0} = \emptyset$.*

3.4. Theorem. *A function $f\colon (X, \eta_1) \to (Y, \eta_2)$ is a near map if and only if f is a p-map. (Here η_1, η_2 are induced by η_0 on τX, τY respectively.)*

Proof. Suppose f is a p-map and $\eta_1 \mathcal{A}$ for $\mathcal{A} \subset PX$. Then there exists a p-filter $\mathcal{F}$ on X such that $\mathcal{A} \cap \mathcal{F} \neq \emptyset$ (i.e. $A \cap F \neq \emptyset$ for all $A \in \mathcal{A}$, $F \in \mathcal{F}$). Clearly $f(\mathcal{A}) \cap f(\mathcal{F}) \neq \emptyset$ and so $f(\mathcal{A}) \cap f^0(\mathcal{F}) \neq \emptyset$, where $f^0(\mathcal{F}) = \{E \subset Y\colon E$ open and $f^{-1}(E) \in \mathcal{F}\}$ is the p-filter in Y (see Harris [9]). Thus $\eta_2 f(\mathcal{A})$ and f is a near map.

Conversely if f is a near map and $\mathcal{F}$ is a p-filter in X, then $\eta_1 \mathcal{F}$ and so $\eta_2 f(\mathcal{F})$. Hence there is a p-filter $\mathcal{G}$ in Y such that $f(\mathcal{F}) \cap \mathcal{G} \neq \emptyset$. Hence $f^0(\mathcal{F})$ is a p-filter in Y and f is a p-map.

3.5. T_3 extensions. If each aX is T_3, then Theorem 2.22 shows that, irrespective of whether aX is simple or strict, $f\colon X \to Y$ has a continuous extension $\bar{f}\colon aX \to aY$ if and only if f is a near map. Thus the category of objects $\{aX\}$ and continuous functions as morphisms is the largest epireflective subcategory of T_3 spaces and near maps. We will now show

how this can be used to obtain a recent solution to Problem II of Harris [11] by Hunsaker and Sharma [15] (also see [24]).

First consider a slightly general situation. Let αX consist of some distinguished or d-clusters in X with the absorption topology. Then an argument similar to Theorem 2 of [4] shows that $\bar{f}$: $\alpha X \to \alpha Y$ is a continuous extension of f: $X \to Y$ iff for each $\sigma \in \alpha X$ $f(\sigma) \subset \bar{f}(\sigma)$, i.e.

(*) for each $\sigma \in \alpha X$ there is a unique $\sigma' \in \alpha Y$ such that $f(\sigma) \subset \sigma'$.

Let a proximally continuous map satisfying (*) be called *strongly p-continuous*. Obviously, every near map is strongly p-continuous. Conversely every strongly p-continuous map is a near map: $\eta_1 \mathcal{A}$ for $\mathcal{A} \subset PX$ implies there is a $\sigma \in \alpha X$ such that $\mathcal{A} \subset \sigma$. Since f is strongly p-continuous it satisfies (*) and so $f(\mathcal{A}) \subset f(\sigma) \subset \sigma'$ and $\eta_2 f(\mathcal{A})$. (We note $\eta_1 \mathcal{A}$ iff there is a $\sigma \in \alpha X$ such that $\mathcal{A} \subset \sigma$.) In particular, if αX is the Harris regular-closed extension of an RC-regular space X, then αX is the space of all contractive clusters (Hunsaker and Sharma [15]) and hence αX is an epireflection.

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