Relations between some general nth-order derivatives

by

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Abstract. Let $f_r, \overline{f}_r, \overline{f}_r^+, \overline{f}_{r.ap}$ etc. denote various rth-order Peano derivatives of f and $\Delta_r f$, $\overline{\Delta}_r f$, $\overline{\Delta}_r^+ f$, $\overline{\Delta}_{r.ap}^+ f$ the various rth-order Riemann derivatives defined using equally spaced rth-order difference operators. Using an extension of a decomposition theorem due to Marcinkiewicz, [6], various properties of these derivatives are obtained. It is shown that if f is r-concave on a measurable set E with f_{r-1} existing and finite there, then $f_{r.ap}$ and $(f_{r-1})'_{ap}$ exist, are finite, and equal almost everywhere on E. If $f_{r.ap}(x)$ exists and a finite $\Delta_{r.ap} f(x)$ exists with the same value; further if $\overline{f}_r < \infty$ on a measurable set E, then $\overline{\Delta}_r^+ f = (f_{r-1})'_{ap} = f_{r.ap} = \overline{\Delta}_r^- f$ almost everywhere on E; hence $\{x;\, f_r(x) = \pm\infty\}$ is of measure zero. These last two results had been obtained for Cesàro derivatives by Sargent [11].

1. Introduction. There is essentially only one first order derivative; others are derived from it by changing the limit concept used; for example, by using upper limits, one sided limits, symmetric limits or approximate limits. The relationships between these derivatives are trivial at a point but deep results arise once we consider existence on sets of positive measure [10, p. 152]. Moving to derivatives of higher order, the situation is much more complex, since there are several natural extensions of the first order derivative and each can be varied by changing the limit concept as mentioned above. The extension of the first order results to these higher order derivatives is far from complete, although some very important results are already classical [4, 6, 7, 8]. It is the intention of this paper to obtain further relations between higher derivatives of several types.

2. Preliminaries. Let f be a real function defined in the closed interval $[a, b]$. If $x_0, x_1, \ldots, x_r$ be any $r+1$ distinct points in $[a, b]$, then the pth divided difference of f at these points is defined by

$$V_r(f;\, x_k) = V_r(f;\, x_0, x_1, \ldots, x_r) = \sum_{k=0}^{r} \frac{f(x_k)}{w'(x_k)}, \tag{2.1}$$

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where

$$w(x) = \prod_{k=0}^{r} (x - x_r).$$

Clearly

(2.2) $$V_r(f;\ x_k) = \frac{V_{r-1}(f;\ x_0, x_1, \ldots, x_{r-1}) - V_{r-1}(f;\ x_1, x_2, \ldots, x_r)}{x_0 - x_r}.$$

Let x_0 be any fixed point in $[a, b]$ and let $x_1, x_2, \ldots, x_r$ be any set of r distinct points in $[a, b]$ different from x_0 with the property

$$0 < |x_0 - x_1| < |x_0 - x_2| < \ldots < |x_0 - x_r|.$$

If the iterated limit

(2.3) $$\lim_{x_r \to x_0} \lim_{x_{r-1} \to x_0} \ldots \lim_{x_1 \to x_0} r! V_r(f;\ x_k)$$

exists (possibly infinite), then this limit is called the *generalized derivative of f at x_0 of order r* and is denoted by $D_r f(x_0)$. Taking lim sup (resp. lim inf) at each stage in (2.3) we get the upper (resp. lower) derivates $\overline{D}_r f(x_0)$ (resp. $\underline{D}_r f(x_0)$). The one sided derivates $\overline{D}_r^+ f(x_0)$, $\underline{D}_r^- f(x_0)$ etc. are obtained in the usual way by taking all the points $x_1, x_2, \ldots, x_r$ on the same side of x_0.

With the point x_0 fixed in $[a, b]$, if in a certain neighbourhood of the point x_0, if f has the representation

(2.4) $$f(x_0 + h) = \sum_{k=0}^{r} a_k \cdot \frac{h^k}{k!} + o(h^r), \quad \text{as} \quad h \to 0,$$

where $a_0 = f(x_0)$ and all of a_k, $0 \leqslant k \leqslant r$, are independent of h, then a_k is called the *k-th Peano derivative* [8] or the *k-th de la Vallée-Poussin derivative* [7] *of f at x_0* and is denoted by $f_k(x_0)$. The definition is such that if $f_n(x_0)$ exists, then $f_m(x_0)$ also exists for $0 \leqslant m \leqslant n$.

Suppose that $f_{r-1}(x_0)$ exists finitely. Then the upper (resp. lower) rth Peano derivative of f at x_0 is defined as the upper (resp. lower) limit of

(2.5) $$\frac{r!}{h^r}\left\{f(x_0 + h) - \sum_{k=0}^{r-1} \frac{h^k}{k!} f_k(x_0)\right\}$$

as h approaches 0 and will be denoted by $\overline{f}_r(x_0)$ (resp. $\underline{f}_r(x_0)$). The one sided Peano derivatives are obtained by suitably restricting h while taking limits in (2.5) and these will be denoted by $\overline{f}_r^+(x_0)$, $\underline{f}_r^-(x_0)$, etc. If all the Peano derivates of f at x_0 are equal (possibly infinite), then this value is the Peano derivative of f at x_0. From (2.4) and (2.5) it is clear that even when $f_r(x_0)$ is infinite, all the previous derivatives $f_k(x_0)$, $0 \leqslant k < r$, must be finite. The approximate Peano derivative $f_{r,ap}(x_0)$ is the limit of (2.5) when h is restricted to a set having 0 as a point of density. The ordinary approximate derivative of a function φ at x_0 is denoted by $\varphi'_{ap}(x_0)$.

We shall denote by $\Delta_r(f, x, h)$ a finite difference of order r for the function f at x with increment h, defined as follows:

(2.6) $$\Delta_0(f;\ x, h) = f(x)$$
$$\Delta_r(f;\ x, h) = \Delta_{r-1}(f;\ x+h, h) - \Delta_{r-1}(f;\ x, h), \quad r \geqslant 1.$$

Clearly

(2.7) $$\Delta_r(f;\ x, h) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x + kh)$$

and

(2.8) $$V_r(f;\ x, x+h, \ldots, x+rh) = \frac{\Delta_r(f;\ x, h)}{r!\,h^r}.$$

It follows from (2.7) and (2.4) that if $f_r(x)$ exists finitely, then

(2.9) $$\Delta_r(f;\ x, h) = h^r f_r(x) + o(h^r), \quad \text{as} \quad h \to 0$$

and hence

(2.10) $$\lim \frac{\Delta_r(f;\ x, h)}{h^r} = f_r(x).$$

We denote by $\Delta_r f(x)$ the limit of (2.10) when this limit exists. Thus if $f_r(x_0)$ exists finitely $\Delta_r f(x_0)$ also exists and have the same value. But the converse is not true. For, consider the function $f(x) = |x|$. Clearly $\Delta_2(f, 0, u) = 0$ and hence $\Delta_2 f(0) = 0$, although $f_2(0)$ does not exist. Thus $\Delta_r f(x)$ may be called a *generalized derivative of f at x of order r*. The derivates $\overline{\Delta}_r^+ f(x_0)$, etc. are defined in the usual way.

Let $E \subset [a, b]$. If for all choices of $r+1$ distinct points $x_0, x_1, \ldots, x_r$ in E, we have $V_r(f;\ x_0, x_1, \ldots, x_r) \geqslant 0$, then f is called *strongly r-convex*, or *simply r-convex* (in the sense of Popoviciu [9]) *in E*. If $-f$ is r-convex in E, then f is called *r-concave in E*. Clearly for $r = 2$, the case of 2-convexity of f coincides with the definition of usual convex functions.

As usual $\mu(E)$ will denote the Lebesgue measure of the measurable set E and $f^{(k)}$ will denote the ordinary kth derivative of f, except that the first derivative of f will be denoted by f' istead of $f^{(1)}$; lim *ap* will indicate the approximate limit.

3. An auxiliary theorem. The following theorem, needed later, is an extension of an interesting result of Marcinkiewicz [6], used by many authors [2, 12, 13] (see also Zygmund [14, II, p. 73]).

(3.1) THEOREM. *If f is measurable and f_k exists finitely on a measurable set $E\subset[a,b]$, then there exists a perfect set $E_0\subset E$ such that $\mu(E_0-E)$ is arbitrarily small and two functions g and h such, that*

$$f=g+h,$$

where $g^{(k)}$ exists and is continuous on $[a,b]$ with the property that if $(x_i, x_i+\delta_i)$ is any contiguous interval of E_0 and if $0\leqslant t\leqslant\delta_i$, then

$$g^{(k)}(x_i+t)-g^{(k)}(x_i)=o(t), \quad \text{as} \quad t\to 0,$$

$$g^{(k)}(x_i+\delta_i)-g^{(k)}(x_i+t)=o(\delta_i-t) \quad \text{as} \quad t\to\delta_i$$

and

$$h_r(x)=0, \quad \text{for} \quad x\in E_0,\ r=0,1,2,\ldots,k.$$

In the proof of the theorem we shall very often have to consider the above property of $g^{(k)}$, so we state it separately.

(3.2) Given a perfect set $P\subset[a,b]$, a function φ is said to *satisfy the property* $W(P)$ if $\varphi^{(k)}$ exists and is continuous in $[a,b]$ and if $(x_i, x_i+\delta_i)$ is any contiguous interval of P and if $0\leqslant t\leqslant\delta_i$, then

$$\varphi^{(k)}(x_i+t)-\varphi^{(k)}(x_i)=o(t), \quad \text{as} \quad t\to 0,$$

$$\varphi^{(k)}(x_i+\delta_i)-\varphi^{(k)}(x_i+t)=o(\delta_i-t), \quad \text{as} \quad t\to\delta_i.$$

If φ satisfy the property $W(P)$ we shall write $\varphi^{(k)}\in W(P)$.

To prove the theorem we consider the following lemmas.

(3.3) LEMMA. *Under the hypotheses of Theorem* (3.1) *there exists a perfect set $E_0\subset E$ such that $\mu(E-E_0)$ is arbitrarily small and two functions g and h such that*

$$f=g+h,$$

where $g^{(k)}\in W(E_0)$ and

$$h_k(x)=0 \quad \text{for} \quad x\in E_0.$$

Proof. Since f_k exists finitely on E by (2.9) there exists a perfect set $E_0\subset E$ such that $\mu(E-E_0)$ is arbitrarily small and all the derivatives f_k $(r=0,1,\ldots,k)$ are continuous on E_0 and

(3.4) $$f_r(x)=\lim_{u\to 0}\frac{\Delta_r(f,x,u)}{u^r}\,[\text{unif.}], \quad x\in E,\ r=1,2,\ldots,k.$$

Define a function λ in $[a,b]$ such that

$$\lambda(x)=f_k(x), \quad x\in E_0$$

and if $(x_i, x_i+\delta_i)$ is any contiguous interval of E_0 and if $0\leqslant t\leqslant\delta_i$, then

$$\lambda(x_i+t)=\lambda(x_i)+[\lambda(x_i+\delta_i)-\lambda(x_i)]\,\omega\left(\frac{t}{\delta_i}\right),$$

where ω is a polynomial such that

$$\omega(0)=0, \quad \omega(1)=1,$$

$$\omega'(0)=\omega'(1)=0.$$

Let g be the indefinite integral of λ over $[a,b]$ of order k. Letting $h=f-g$, it can be verified that g and h have the desired property.

We suppose that for fixed r_0, $0<r_0\leqslant k$, the following lemma is true.

(3.5) LEMMA. *Under the hypotheses of Theorem* (3.1) *there exists a perfect set $E_0\subset E$ such that $\mu(E-E_0)$ is arbitrarily small and two functions g and h such that*

$$f=g+h,$$

where $g^{(k)}\in W(E_0)$ and

$$h_r(x)=0 \quad \text{for} \quad x\in E_0,\ r_0\leqslant r\leqslant k.$$

We now prove

(3.6) LEMMA. *Under the hypotheses of Theorem* (3.1) *there exists a perfect set $E_0\subset E$ such that $\mu(E-E_0)$ is arbitrarily small and two functions g and h such that*

$$f=g+h$$

where $g^{(k)}\in W(E_0)$ and

$$h_r(x)=0 \quad \text{for} \quad x\in E_0,\ r_0-1\leqslant r\leqslant k.$$

Proof. By Lemma (3.5) there exists a perfect set E_0 E such that $\mu(E-E_0)$ is arbitrarily small and two functions $\tilde g$ and $\tilde h$ such that

(3.7) $$f=\tilde g+\tilde h,$$

where $\tilde g^{(k)}\in W(E_0)$ and

(3.8) $$\tilde h_r(x)=0 \quad \text{for} \quad x\in E_0,\ r_0\leqslant r\leqslant k.$$

From (3.8) we conclude, as in (2.9), that

(3.9) $$\Delta_{r_0-1}(\tilde h,x,u)=u^{r_0-1}\tilde h_{r_0-1}(x)+o(u^k), \quad \text{as} \quad u\to 0,\ x\in E_0.$$

So, if $x, x+u\in E_0$ we have by (2.6)

(3.10) $$u^{r_0-1}[\tilde h_{r_0-1}(x+u)-\tilde h_{r_0-1}(x)]=\Delta_{r_0}(\tilde h,x,u)+o(u^k).$$

Now by (3.8) $\tilde h_{r_0}(x)=0$ for $x\in E_0$ and hence

$$\Delta_{r_0}(\tilde h,x,u)=o(u^k), \quad \text{as} \quad u\to 0,\ x\in E_0.$$

Therefore (3.10) reduces to

(3.11) $$u^{r_0-1}[\tilde{h}_{r_0-1}(x+u)-\tilde{h}_{r_0-1}(x)] = o(u^k), \quad \text{as} \quad u\to 0,\ x,\ x+u \in E_0 .$$

Define a function λ in $[a, b]$ such that

(3.12) $$\lambda(x) = \tilde{h}_{r_0-1}(x), \quad x \in E_0$$

and if $(x_i, x_i+\delta_i)$ is a contiguous interval of E_0 and if $0 \leqslant t \leqslant \delta_i$, then

$$\lambda(x_i+t) = \lambda(x_i)+[\lambda(x_i+\delta_i)-\lambda(x_i)]\,\omega\left(\frac{t}{\delta_i}\right),$$

where ω is a polynomial satisfying

$$\omega(0) = 0, \quad \omega(1) = 1,$$

$$\omega'(0) = \omega^{(2)}(0) = \ldots = \omega^{(k-r_0+2)}(0) = 0,$$

$$\omega'(1) = \omega^{(2)}(1) = \ldots = \omega^{(k-r_0+2)}(1) = 0.$$

Then by (3.10) λ is continuous in $[a, b]$. Regarding the function λ we make the following assertion.

(3.13) ASSERTION. *$\lambda^{(k-r_0+1)}$ exists and is continuous in $[a, b]$ and*

$$\lambda^{(r)}(x) = 0 \quad \textit{for} \quad x \in E_0,\ 1 \leqslant r \leqslant k-r_0+1 .$$

We shall prove the assertion by induction. Let $\xi \in E_0$

(a) If $\xi+u \in E_0$, then from (3.12) and (3.11) we have

(3.14) $$\frac{\lambda(\xi+u)-\lambda(\xi)}{u} = \frac{\tilde{h}_{r_0-1}(\xi+u)-\tilde{h}_{r_0-1}(\xi)}{u} \to 0, \quad \text{as} \quad u\to 0 .$$

(b) If ξ is not isolated from the right and $\xi+u \in (x_i, x_i+\delta_i)$, $u > 0$, then from (3.12) and from the definition of λ we have

(3.15) $$\frac{\lambda(\xi+u)-\lambda(\xi)}{u}$$

$$= \frac{h_{r_0-1}(x_i)-\tilde{h}_{r_0-1}(\xi)}{u} + \frac{h_{r_0-1}(x_i)-h_{r_0-1}(x_i)}{u}\left(\frac{\xi+u-x_i}{\delta_i}\right)$$

$$= \frac{h_{r_0-1}(x_i)-h_{r_0-1}(\xi)}{x_i-\xi}\cdot\frac{x_i-\xi}{u}+$$

$$+\frac{h_{r_0-1}(x_i+\delta_i)-h_{r_0-1}(x_i)}{\delta_i}\cdot\frac{\xi+u-x_i}{u}\cdot\frac{\delta_i}{\xi+u-x_i}\,\omega\left(\frac{\xi+u-x_i}{\xi_i}\right).$$

Now as $u\to 0$, $\delta_i\to 0$ and $x_i\to\xi$. Since $\left|\frac{\xi+u-x_i}{u}\right| \leqslant 1$ and $\left|\frac{x_i-\xi}{u}\right| \leqslant 1$, and since $\frac{\omega(T)}{T}\to 0$ as $T\to 0$, we get from (3.15) and (3.11)

(3.16) $$\frac{\lambda(\xi+u)-\lambda(\xi)}{u} \to 0, \quad \text{as} \quad u\to 0+.$$

(c) If ξ is isolated from the right, $\xi = x_i$, $\xi+u \in (x_i, x_i+\delta_i)$, say and hence from the definition of λ

$$\frac{\lambda(\xi+u)-\lambda(\xi)}{u} = \frac{\tilde{h}_{r_0-1}(x_i+\delta_i)-\tilde{h}_{r_0-1}(x_i)}{\delta_i}\cdot\frac{\delta_i}{u}\,\omega\left(\frac{u}{\delta_i}\right).$$

Since $\frac{\omega(T)}{T}\to 0$ as $T\to 0$, this implies

(3.17) $$\frac{\lambda(\xi+u)-\lambda(\xi)}{u} \to 0, \quad \text{as} \quad u\to 0+.$$

Thus from (3.14), (3.16) and (3.17)

$$\lim_{u\to 0+}\frac{\lambda(\xi+u)-\lambda(\xi)}{u} = 0 .$$

Similarly if $u\to 0-$, then this limit exists and equals zero. So, we conclude that

$$\lambda'(x) = 0 \quad \text{for} \quad x \in E_0 .$$

By the condition imposed on the polynomial ω, λ' exists and is continuous on $[a, b]$.

Thus Assertion (3.13) is true when $k = r_0$. So, we suppose that $r_0 < k$ and prove that if for m, $1 \leqslant m < \tau = k-r_0+1$, $\lambda^{(m)}$ exists and is continuous in $[a, b]$ and $\lambda^{(r)}(x) = 0$, for $x \in E_0$ and for $1 \leqslant r \leqslant m$, then $\lambda^{(m+1)}$ exists and is continuous in $[a, b]$ and $\lambda^{(m+1)}(x) = 0$ for $x \in E_0$, and this will prove (3.13) by induction.

To this end we suppose that $1 \leqslant m < \tau$, $\lambda^{(m)}$ exists and is continuous in $[a, b]$ and $\lambda^{(m)}(x) = 0$ for $x \in E_0$. Let $\xi \in E_0$. If $\xi+u \in E_0$, then

$$\frac{\lambda^{(m)}(\xi+u)-\lambda^{(m)}(x)}{u} = 0 .$$

If $\xi+u\in(x_i,x_i+\delta)$, $u>0$, and ξ is a limit point of E_0 from the right, then by (3.11)

$$(3.18)\quad \frac{\lambda^{(m)}(\xi+u)-\lambda^{(m)}(\xi)}{u}$$

$$=\frac{\lambda^{(m)}(\xi+u)}{u}$$

$$=\frac{\lambda(x_i+\delta_i)-\lambda(x_i)}{u}\cdot\frac{1}{\delta_i^m}\cdot\omega^{(m)}\left(\frac{\xi+u-x_i}{\delta_i}\right)$$

$$=\frac{\tilde h_{r_0-1}(x_i+\delta_i)-\tilde h_{r_0-1}(x_i)}{\delta_i^{m+1}}\cdot\frac{\xi+u-x_i}{u}\cdot\frac{\delta_i}{\xi+u-x_i}\,\omega^{(m)}\left(\frac{\xi+u-x_i}{\delta_i}\right)$$

$$=\frac{o(\delta_i^k)}{\delta_i^{r_0+m}}\cdot\frac{\xi+u-x_i}{u}\cdot\frac{\delta_i}{\xi+u-x_i}\,\omega^{(m)}\left(\frac{\xi+u-x_i}{\delta_i}\right).$$

Now $\dfrac{\omega^{(m)}(T)}{T}\to 0$ as $T\to 0$ and $\left|\dfrac{\xi+u-x_i}{u}\right|\leqslant 1$. Also since $r_0+m\leqslant k$, $\dfrac{o(\delta_i^k)}{\delta_i^{r_0+m}}\to 0$, as $\delta_i\to 0$. Noting that $\delta_i\to 0$ and $x_i\to\xi$ as $u\to 0$, we conclude from these that

$$\frac{\lambda^{(m)}(\xi+u)-\lambda^{(m)}(\xi)}{u}\to 0\,,\quad \text{as}\quad u\to 0+.$$

If $\xi+u\in(x_i,x_i+\delta_i)$, $u>0$, and ξ is isolated from the right, then $\xi=x_i$ and as above

$$\frac{\lambda^{(m)}(\xi+u)-\lambda^{(m)}(\xi)}{u}$$

$$=\frac{\tilde h_{r_0-1}(x_i+\delta_i)-\tilde h_{r_0-1}(x_i)}{\delta_i^{m+1}}\cdot\frac{\xi+u-x_i}{u}\cdot\frac{\delta_i}{\xi+u-x_i}\,\omega^{(m)}\left(\frac{\xi+u-x_i}{\delta_i}\right)$$

and hence

$$\frac{\lambda^{(m)}(\xi+u)-\lambda^{(m)}(\xi)}{u}\to 0\,,\quad \text{as}\quad u\to 0+.$$

Considering the left-hand limit in a similar manner we see that $\lambda^{(m+1)}(x)=0$ for $x\in E_0$. The existence of $\lambda^{(m+1)}$ on $[a,b]-E_0$ and its continuity on $[a,b]$ is obvious from the definition of λ. Hence the assertion (3.13) follows.

Let $\tilde\lambda$ be the indefinite integral of $\lambda^{(k-r_0+1)}$ over $[a,b]$ of order k. Then $\tilde\lambda^{(k)}=\lambda^{(k-r_0+1)}$ on $[a,b]$. Also by the definition of λ and $\tilde\lambda$ we have, as in (3.18), using (3.13)

$$\frac{\tilde\lambda^{(k)}(x_i+t)-\tilde\lambda^{(k)}(x_i)}{t}$$

$$=\frac{\lambda^{(k-r_0+1)}(x_i+t)-\lambda^{(k-r_0+1)}(x_i)}{t}$$

$$=\frac{\lambda^{(k-r_0+1)}(x+t)}{t}$$

$$=\frac{\tilde h_{r_0-1}(x_i+\delta_i)-\tilde h_{r_0-1}(x_i)}{\delta_i^{k-r_0+2}}\cdot\frac{\delta_i}{t}\,\omega^{(k-r_0+1)}\left(\frac{t}{\delta_i}\right).$$

Since $\dfrac{\omega^{(k-r_0+1)}(T)}{T}\to 0$ as $T\to 0$, from this we conclude

$$\tilde\lambda^{(k)}(x_i+t)-\tilde\lambda^{(k)}(x_i)=o(t)\,,\quad \text{as}\quad t\to 0\,.$$

Similarly

$$\tilde\lambda^{(k)}(x_i+\delta_i)-\tilde\lambda^{(k)}(x_i+t)=o(\delta_i-t)\,,\quad \text{as}\quad t\to\delta_i\,.$$

Hence $\tilde\lambda^{(k)}\in W(E_0)$, by (3.2). Also by (3.13)

$$\tilde\lambda^{(k)}(x)=0\quad\text{for}\quad x\in E_0\,.$$

Set

$$g=\tilde g+\tilde\lambda\,,\qquad h=\tilde h-\tilde\lambda\,.$$

Then by (3.7)

$$f=g+h\,.$$

Also since $\tilde g^{(k)}\in W(E_0)$ and $\tilde\lambda^{(k)}\in W(E_0)$, $g^{(k)}\in W(E_0)$. Finally if $x\in E_0$ and $r_0-1\leqslant r\leqslant k$, then by (3.8), (3.12) and (3.13)

$$h_r(x)=\tilde h_r(x)-\tilde\lambda_r(x)=0\,.$$

This completes the proof of Lemma (3.6).

Proof of Theorem (3.1). The proof of Theorem (3.1) now follows from Lemma (3.3), (3.5) and (3.6) by applying induction on r.

4. The Peano derivatives and the generalized derivatives. Denjoy [5] and Corominas [3] proved that if the derivative $D_rf(x_0)$ exists finitely at a point x_0, then the Peano derivative $f_r(x_0)$ also exists finitely (and hence all the previous derivatives $f_n(x_0)$, $0\leqslant n\leqslant r$, also exist) and is equal to $D_rf(x_0)$; the converse is also proved by them for finite $D_rf(x_0)$ or $f_n(x_0)$.

Recently it is shown in [2] that these remain valid if one considers one sided finite derivatives $D_r^+ f(x_0)$ or $f_r^+(x_0)$. Here we prove that these results are true for all the derivatives $\bar{f}_r^+(x_0)$, $\bar{D}_r^+ f(x_0)$, etc. and even when they are infinite.

(4.1) LEMMA. *If f is continuous at $x_0 \in [a, b]$, then*

$$\lim_{x_r \to x_0} \dots \lim_{x_1 \to x_0} V_r(f;\ x_1, x_2, \dots, x_{r+1}) = \frac{1}{(x_{r+1}-x_0)^r}\left\{f(x_{r+1}) - \sum_{k=0}^{r-1} \frac{(x_{r+1}-x_0)^k}{k!} f_k(x_0)\right\}$$

provided $f_{r-1}(x_0)$ exists finitely.

Proof. The lemma is true for $r = 1$. Suppose that this is true for $r = n$ and we prove it to be true for $r = n+1$. Since $r = n+1$, we may suppose, by hypothesis that $f_n(x_0)$ exists finitely. Also by supposition,

$$\text{(4.2)} \quad \lim_{x_{n+1} \to x_0} \dots \lim_{x_1 \to x_0} V_n(f;\ x_1, \dots, x_{n+1}) = \lim_{x_{n+1} \to x_0} \left[\frac{1}{(x_{n+1}-x_0)^n}\left\{f(x_{n+1}) - \sum_{k=0}^{n-1} \frac{(x_{n+1}-x_0)^k}{k!} f_k(x_0)\right\}\right] = \frac{1}{n!} f_n(x_0)$$

and

$$\text{(4.3)} \quad \lim_{x_{n+1} \to x_0} \dots \lim_{x_2 \to x_0} V_n(f;\ x_2, \dots, x_{n+2}) = \frac{1}{(x_{n+2}-x_0)^n}\left\{f(x_{n+2}) - \sum_{k=0}^{n-1} \frac{(x_{n+2}-x_0)^k}{k!} f_k(x_0)\right\}.$$

Hence from (4.2) and (4.3) we have, by applying the recurrence formula (2.2)

$$\lim_{x_{n+1} \to x_0} \dots \lim_{x_1 \to x_0} V_{n+1}(f;\ x_1, x_2, \dots, x_{n+2}) = \frac{\dfrac{1}{(x_{n+2}-x_0)^n}\left\{f(x_{n+2}) - \displaystyle\sum_{k=0}^{n-1} \frac{(x_{n+2}-x_0)^k}{k!} f_k(x_0)\right\} - \dfrac{1}{n!} f_n(x_0)}{x_{n+2}-x_0}$$

$$= \frac{1}{(x_{n+2}-x_0)^{n+1}}\left\{f(x_{n+2}) - \sum_{k=0}^{n} \frac{(x_{n+2}-x_0)^k}{k!} f_k(x_0)\right\}.$$

Thus the lemma is true for $r = n+1$ and the proof follows by induction.

(4.4) THEOREM. *If $f_r(x_0)$ exists finitely, then*

$$\bar{f}_{r+1}^+(x_0) = \bar{D}_{r+1}^+ f(x_0)\,, \quad \textit{etc.}$$

and if $D_r f(x_0)$ exists finitely, then

$$\bar{D}_{r+1}^+ f((x_0) = \bar{f}_{r+1}^+(x_0)\,, \quad \textit{etc.}$$

Proof. If $r = 0$, this is obvious. So, assume $r > 0$. Since $f_r(x_0)$ exists, f is continuous at x_0 and hence by (2.1) we have

$$\lim_{x_1 \to x_0} V_{r+1}(f;\ x_1, x_2, \dots, x_{r+2}) = V_{r+1}(f;\ x_0, x_2, \dots, x_{r+2})\,.$$

Hence by Lemma (4.1)

$$\lim_{x_{r+1} \to x_0} \dots \lim_{x_2 \to x_0} V_{r+1}(f;\ x_0, x_2, \dots, x_{r+2}) = \lim_{x_{r+1} \to x_0} \dots \lim_{x_1 \to x_0} V_{r+1}(f;\ x_1, x_2, \dots, x_{r+2})$$

$$= \frac{1}{x_{r+2}-x_0}\left\{f(x_{r+2}) - \sum_{k=0}^{r} \frac{(x_{r+2}-x_0)^k}{k!} f_k(x_0)\right\}.$$

Multiplying by $(r+1)!$ and letting $x_{r+2} \to x_0+$ we have

$$\bar{D}_{r+1}^+ f(x_0) = \bar{f}_{r+1}^+(x_0)\,, \quad \text{etc.}$$

Conversely, if $D_r f(x_0)$ exists finitely, then by our earlier remarks, $f_r(x_0)$ exists finitely and so by repeating the above arguments the proof is completed.

5. Convex functions.

We shall require the following theorem on convex functions.

(5.1) THEOREM. *If f is r-concave on a set $E \subset [a, b]$ on which f_{r-1} exists, then f_{r-1} is non-increasing on E.*

Proof. Let $x_1 < x_2 < \dots < x_r$ and $y_1 < y_2 < \dots < y_r$ be any two sets of points in E such that $x_i \leqslant y_i$ for $1 \leqslant i \leqslant r$. Then as in [1]

$$\text{(5.2)} \qquad V_{r-1}(f;\ x_1, \dots, x_r) \geqslant V_{r-1}(f;\ y_1, \dots, y_r)\,.$$

Now if x_0 and y_0 be any two points in E such that $x_0 < y_0$, then choosing the points x_i and y_i, $1 \leqslant i \leqslant r$, in the above manner and then taking limit when all $x_i \to x_0$ and all $y_i \to y_0$ we get from (5.2) and Theorem (4.4)

$$f_{r-1}(x_0) \geqslant f_{r-1}(y_0)$$

which completes the proof.

6. Properties of derivatives and convex functions.

(6.1) THEOREM. *If f is r-concave on a measurable set $E \subset [a, b]$ on which f_{r-1} exists finitely, then $f_{r,ap}$ and $(f_{r-1})'_{ap}$ exists finitely and equal each other almost every where in E.*

Proof. Since f_{r-1} exists on E, by Theorem (5.1) f_{r-1} is non-increasing on E. Let $E_1 \subset E$ be such that $\mu(E_1) = \mu(E)$ and a finite derivative $(f_{r-1})'_E$ of f_{r-1} with respect to E exists on E_1. By Theorem (3.1) there exists a perfect set $E_0 \subset E_1$ such that $\mu(E_1 - E_0)$ is arbitrarily small and two functions g and h such that

$$f = g + h, \tag{6.2}$$

where $g^{(r-1)}$ exists and is continuous in $[a, b]$ and if $(x_i, x_i+\delta_i)$ is any contiguous interval of E_0 and $0 \leqslant t \leqslant \delta_i$, then

$$g^{(r-1)}(x_i+t) - g^{(r-1)}(x_i) = o(t), \quad \text{as} \quad t \to 0, \tag{6.3}$$

$$g^{(r-1)}(x_i+\delta_i) - g^{(r-1)}(x_i+t) = o(\delta_i - t), \quad \text{as} \quad t \to \delta_i \tag{6.4}$$

and

$$h_k(x) = 0 \quad \text{for} \quad x \in E_0, \; k = 0, 1, \ldots, r-1. \tag{6.5}$$

Let $\xi \in E_0$ be a point of density of E_0. Then since $g^{(r-1)}$ is continuous,

$$\begin{aligned} g_r(\xi) &= \lim_{u\to 0} \frac{r!}{u^r}\left\{g(\xi+u) - \sum_{k=0}^{r-1} \frac{u^k}{k!} g_k(\xi)\right\} \\ &= \lim_{u\to 0} \frac{g^{(r-1)}(\xi+u) - g^{(r-1)}(\xi)}{u}. \end{aligned} \tag{6.6}$$

Now if $u \to 0$ with $\xi + u \in E_0$, then since $(f_{r-1})'_E(\xi)$ exists, by (6.2) and (6.5) the last limit of (6.6) exists and equals $(f_{r-1})'_E(\xi)$. Suppose that $\xi + u \notin E_0$. Then $\xi + u \in (x_i, x_i+\delta_i)$, where $(x_i, x_i+\delta_i)$ is a contiguous interval of E_0. Supposing $u > 0$, we have from (6.3)

$$\begin{aligned} &\left|\frac{g^{(r-1)}(\xi+u) - g^{(r-1)}(\xi)}{u} - \frac{g^{(r-1)}(x_i) - g^{(r-1)}(\xi)}{x_i - \xi} \cdot \frac{x_i - \xi}{u}\right| \\ &= \left|\frac{g^{(r-1)}(\xi+u) - g^{(r-1)}(x_i)}{u}\right| \\ &= \left|\frac{o(\xi+u-x_i)}{\xi+u-x_i} \cdot \frac{\xi+u-x_i}{u}\right|, \quad \text{as} \quad \xi+u \to x_i \\ &\leqslant \left|\frac{o(\xi+u-x_i)}{\xi+u-x_i}\right|, \quad \text{as} \quad \xi+u \to x_i. \end{aligned} \tag{6.7}$$

Now since ξ is a point of density of E_0, $x_i \to \xi$ and $\dfrac{x_i - \xi}{u} \to 1$ as $u \to 0$. Hence taking limit as $u \to 0+$ and noticing that the limit (6.6) exists and equals $(f_{r-1})_E(\xi)$ when the limit is restricted on E_0, we conclude from (6.7) that

$$g_r(\xi) = \lim_{u\to 0+} \frac{g^{(r-1)}(\xi+u) - g^{(r-1)}(\xi)}{u} = (f_{r-1})'_E(\xi). \tag{6.8}$$

Similarly if $u < 0$, then using (6.4) it can be proved that

$$g_r(\xi) = \lim_{u\to 0-} \frac{g^{(r-1)}(\xi+u) - g^{(r-1)}(\xi)}{u} = (f_{r-1})'_E(\xi). \tag{6.9}$$

Thus from (6.6), (6.8), and (6.9) we conclude that g_r exists and equals $(f_{r-1})'_E$ at all points of density of E_0. Also if $\xi \in E_0$ is a point of density of E_0, then from (6.5) and (6.2)

$$\begin{aligned} g_r(\xi) &= \lim_{\substack{u\to 0\\ \xi+u\in E_0}} \frac{r!}{u^r}\left\{g(\xi+u) - \sum_{k=0}^{r-1} \frac{u^k}{k!} g_k(\xi)\right\} \\ &= \lim_{\substack{u\to 0\\ \xi+u\in E_0}} \left\{\frac{r!}{u^r} f(\xi+u) - \sum_{k=0}^{r-1} \frac{u^k}{k!} f_k(\xi)\right\} = f_{r,ap}(\xi). \end{aligned} \tag{6.10}$$

Since ξ is a point of density of $E_0 \subset E$ and since $(f_{r-1})'_E(\xi)$ exists the approximate derivative $(f_{r-1})'_{ap}(\xi)$ exists and equals $(f_{r-1})'_E(\xi)$. So from (6.9) and (6.10) we have

$$f_{r,ap}(\xi) = (f_{r-1})'_{ap}(\xi), \tag{6.11}$$

whenever $\xi \in E_0$ is a point of density of E_0. Thus (6.11) holds almost everywhere in E_0. Since $\mu(E_0) = \mu(E_1)$ and since $\mu(E - E_1)$ can be made arbitrarily small, we conclude that (6.11) holds almost everywhere in E.

7. The derivative $\Delta_r f(x)$. Now we shall establish certain relations between the Peano derivatives and $\Delta_r f(x)$. We begin with a lemma which is an improvement of an interesting result of Marcinkiewicz and Zygmund [7].

(7.1) LEMMA. *If* (i) $\lim_{u\to 0} ap \dfrac{\Delta_k(f, x, u)}{u^k}$ *exists finitely*,

(ii) $\limsup_{u\to 0} \dfrac{\Delta_k(f, x, u)}{u^k} < \infty$, *and*

(iii) $\lim_{u\to 0} \left|\dfrac{\Delta_{k-1}(f, x, u)}{u^{k-1}}\right| < \infty$ *for every x on a measurable set E, then*

$$\lim_{u\to 0+} \sup \frac{\Delta_k(f, x, u)}{u^k} = \lim_{u\to 0} ap \frac{\Delta_k(f, x, u)}{u^k} = \lim_{u\to 0-} \sup \frac{\Delta_k(f, x, u)}{u^k} \tag{7.2}$$

for almost all x in E.

Proof. Let $\eta > 0$ be arbitrary. Then there is $M > 0$, $\delta > 0$ and a subset $E_1 \subset E$ such that

$$\mu(E + E_1) < \eta \tag{7.3}$$

and

$$\frac{\Delta_k(f, x, u)}{u^k} < M, \tag{7.4}$$

$$\left|\frac{\Delta_{k-1}(f, x, u)}{u^{k-1}}\right| < M \tag{7.5}$$

for all $x \in E_1$ and $0 < |u| < \delta$. Let x_0 be a point of density of E_1. We may suppose that $x_0 = 0$ and $\lim\limits_{u\to 0} ap \dfrac{\Delta_k(f, x_0, u)}{u^k} = 0$.

Let G be a subset of E_1 having $x_0 = 0$ as a point of density and

$$\lim_{\substack{u\to 0\\ u\in G}} \frac{\Delta_k(f, 0, u)}{u^k} = \lim_{u\to 0} ap \frac{\Delta_k(f; 0, u)}{u^k} = 0. \tag{7.6}$$

Let $u > 0$. Choose ε arbitrary such that $0 < \varepsilon < 1$. Then if u is sufficiently small, there exists v, $u(1-\varepsilon) \leqslant v \leqslant u$ such that the points

$$\begin{array}{ll} v + \dfrac{u-v}{k} j, & j = 0, 1, \ldots, k-1, \\ iv, & i = 1, 2, \ldots, k, \\ iv + i\dfrac{u-v}{k}, & i = 1, 2, \ldots, k \end{array}$$

all belong to G. Then by (7.4)

$$\sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} f(0) = 0, \tag{7.7}$$

$$\begin{aligned} \Delta_k\left(f, iv, i\frac{u-v}{k}\right) &= \sum_{j=0}^{k} (-1)^{k-1}\binom{k}{j} f\left(iv + i\frac{u-v}{k} j\right) \\ &\leqslant M\left(i\frac{u-v}{k}\right)^k \quad \text{for} \quad i = 1, 2, \ldots, k. \end{aligned} \tag{7.8}$$

Now from (7.7) and (7.8)

$$\begin{aligned} &\sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} \Delta_k\left(f, 0, v + \frac{u-v}{k} j\right), \\ &= \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} \sum_{i=0}^{k} (-1)^{k-i}\binom{k}{i} f\left(iv + i\frac{u-v}{k} j\right) \end{aligned} \tag{7.9}$$

$$\begin{aligned} &= \sum_{i=0}^{k} (-1)^{k-i}\binom{k}{i} \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} f\left(iv + i\frac{u-v}{k} j\right) \\ &= \sum_{i=1}^{k} (-1)^{k-i}\binom{k}{i} \Delta_k\left(f, iv, i\frac{u-v}{k}\right) \\ &= \Sigma^{(+)} + E^{(-)}, \end{aligned}$$

where $E^{(+)}$ denotes the summation over the terms for which $\Delta_k\left(f, iv, i\dfrac{u-v}{k}\right)$ is positive and $E^{(-)}$ denotes the summation over the terms for which $\Delta_k\left(f, iv, i\dfrac{u-v}{k}\right)$ is negative.

We have from (7.9) and (7.8)

$$\begin{aligned} \Sigma^{(+)}(-1)^{k-i}\binom{k}{i}\Delta_k\left(f, iv, i\frac{u-v}{k}\right) &\leqslant M\left(\frac{u-v}{k}\right)^k \sum_{i=1}^{k}\binom{k}{i} i^k \\ &= MC_1(u-v)^k, \quad \text{where} \quad C_1 = \sum_{i=1}^{k}\binom{k}{i}\left(\frac{i}{k}\right)^k \\ &\leqslant MC_1 u^k \varepsilon^k \end{aligned} \tag{7.10}$$

and from (7.9) and (7.5)

$$\begin{aligned} &\left|\Sigma^{(-)}(-1)^{k-i}\binom{k}{i}\Delta_k\left(f, iv, i\frac{u-v}{k}\right)\right| \\ &\leqslant \Sigma^{(-)}\binom{k}{i}\left|\Delta_{k-1}\left(f;\ iv + i\frac{u-v}{k}, i\frac{u-v}{k}\right) - \Delta_{k-1}\left(f;\ iv, i\frac{u-v}{k}\right)\right| \\ &\leqslant \Sigma^{(-)}\binom{k}{i}\left\{\left|\Delta_{k-1}\left(f;\ iv + i\frac{u-v}{k}, i\frac{u-v}{k}\right)\right| + \left|\Delta_{k-1}\left(f;\ iv, i\frac{u-v}{k}\right)\right|\right\} \\ &\leqslant \Sigma^{(-)}\binom{k}{i} 2M\left(i\frac{u-v}{k}\right)^{k-1} \\ &\leqslant MC_2(u-v)^{k-1}, \quad \text{where} \quad C_2 = 2\sum_{i=1}^{k}\binom{k}{i}\left(\frac{i}{k}\right)^{k-1} \\ &\leqslant MC_2 u^{k-1}\varepsilon^{k-1}. \end{aligned} \tag{7.11}$$

So, by (7.9), (7.10) and (7.11) we have

$$(7.12)\qquad \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\Delta_k\Big(f;\ 0,\ v+\frac{u-v}{k}j\Big)\leqslant MC_1u^k\varepsilon^k+MC_2u^{k-1}\varepsilon^{k-1}.$$

Since the points $v+\frac{u-v}{k}j\in G$, we conclude

$$(7.13)\qquad \Delta_k\Big(f;\ 0,\ v+\frac{u-v}{k}j\Big)=o(u^k),\qquad j=0,1,\dots,k-1.$$

Hence from (7.12) and (7.13)

$$\sum_{j=0}^{k-1}o(u^k)+\Delta_k(f,0,u)\leqslant MC_1u^k\varepsilon^k+MC_2u^{k-1}\varepsilon^{k-1},$$

i.e.

$$\Delta_k(f,0,u)\leqslant o(u^k)+MC_1u^k\varepsilon^k+MC_2u^{k-1}\varepsilon^{k-1}.$$

Since ε is arbitrary, this implies

$$(7.14)\qquad \Delta_k(f;\ 0,\ u)\leqslant o(u^k).$$

So

$$\limsup_{u\to 0+}\frac{\Delta_k(f,0,u)}{u^k}\leqslant 0.$$

Since

$$\limsup_{u\to 0+}\frac{\Delta_k(f,0,u)}{u^k}\geqslant \lim_{u\to 0} ap\,\frac{\Delta_k(f,0,u)}{u^k}=0,$$

we have

$$(7.15)\qquad \limsup_{u\to 0+}\frac{\Delta_k(f,0,u)}{u^k}=0$$

which proves the left hand equality.

If $u<0$, choosing ε, $0\leqslant\varepsilon\leqslant 1$, consider two cases:

(α) Let k be even. Then as in (7.13) we deduce

$$(7.14\alpha)\qquad \Delta_k(f;\ 0,\ u)\leqslant o(|u|^k)$$

and hence

$$\limsup_{u\to 0-}\frac{\Delta_k(f,0,u)}{u^k}=\limsup_{u\to 0-}\frac{\Delta_k(f,0,u)}{|u|^k}\leqslant 0.$$

So, as above

$$(7.15\alpha)\qquad \limsup_{u\to 0-}\frac{\Delta_k(f,0,u)}{u}=0.$$

(β) Let k be odd. Then instead of relations (7.8), (7.10) and (7.11) we have

$$(7.8\beta)\qquad \Delta_k\Big(f;\ iv,\ i\frac{u-v}{k}\Big)\geqslant -M\Big(i\frac{|u-v|}{k}\Big)^k,\qquad i=1,2,\dots,k,$$

$$(7.10\beta)\qquad \Sigma^{(-)}(-1)^{k-i}\binom{k}{i}\Delta_k\Big(f;\ iv,\ i\frac{u-v}{k}\Big)$$
$$\geqslant -M\Big(\frac{|u-v|}{k}\Big)^k\sum_{i=1}^{k}\binom{k}{i}i^k$$
$$=-MC_1|u-v|^k\geqslant -MC_1|u|^k\varepsilon^k$$

and

$$(7.11\beta)\qquad \Big|\Sigma^{(+)}(-1)^{k-i}\binom{k}{i}\Delta_k\Big(f;\ iv,\ i\frac{u-v}{k}\Big)\Big|$$
$$\leqslant \Sigma^{(+)}\binom{k}{i}2M\Big(i\frac{|u-v|}{k}\Big)^k$$
$$\leqslant MC_2|u-v|^{k-1}$$
$$\leqslant MC_2|u|^{k-1}\varepsilon^{k-1}.$$

Hence from (7.9), (7.10β), (7.11β) we have

$$(7.12\beta)\qquad \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{j}\Delta_k\Big(f;\ 0,\ v+\frac{u-v}{k}j\Big)\geqslant -MC_1|u|^k\varepsilon^k-MC_2|u|^{k-1}\varepsilon^{k-1}$$

and so, as in (7.14) we get

$$(7.14\beta)\qquad \Delta_k(f;\ 0,\ u)\geqslant o(u^k)$$

showing that

$$\limsup_{u\to 0-}\frac{\Delta_k(f;\ 0,\ u)}{u^k}\leqslant 0.$$

So, as above, we get

$$(7.15\beta)\qquad \limsup_{u\to 0-}\frac{\Delta_k(f;\ 0,\ u)}{u^k}=0.$$

Relations (7.15), (7.15α) and (7.15β) show that (7.2) is true at all points of density of E_1 and hence (7.2) is true almost everywhere in E_1. Since η is arbitrary we conclude from (7.3) that (7.2) is true almost everywhere in E.

(7.16) LEMMA. *If $\bar{f}_r<\infty$ on a measurable set E, then $\limsup_{u\to 0}\frac{\Delta_r(f;x,u)}{u^r}<\infty$ almost everywhere in E.*

Proof. Since $\bar{f}_r < \infty$ on E, by Theorem (4.4) $D_{r-1}f$ exists and $\bar{D}_r f < \infty$ on E and so for any $\varepsilon > 0$ there exists $M_0 > 0$, $\delta_0 > 0$, and a subset $E_0 \subset E$ such that
$$\mu(E - E_0) < \varepsilon$$
and
$$\lim_{x_{r-1}\to x} \ldots \lim_{x_1 \to x} r! V_r(f;\ x, x_1, \ldots, x_r) < M_0$$
for all $x \in E_0$, and all x_r, $0 < |x_r - x| < \delta_0$. Hence there exists $M_1 > M_0$, $\delta_1 > 0$, $\delta_1 < \delta_0$ and $E_1 \subset E_0$ such that
$$\mu(E_0 - E_1) < \varepsilon$$
and
$$\lim_{x_{r-2}\to x} \ldots \lim_{x_1 \to x} r! V_r(f;\ x, x_1, \ldots, x_r) < M_1$$
for all $x \in E_1$, and all x_{r-1}, x_r, $0 < |x_{r-1} - x| < \delta_1$, $0 < |x_r - x| < \delta_0$. So, after a finite number of steps we get $M_{r-1} > M_{r-2} > 0$ and $0 < \delta_{r-1} < \delta_{r-2}$ and $E_{r-1} \subset E_{r-2}$ such that
$$\mu(E_{r-2} - E_{r-1}) < \varepsilon$$
and
$$r! V_r(f;\ x, x_1, \ldots, x_r) < M_{r-1}$$
for all $x \in E_{r-1}$, and all $x_1, x_2, \ldots, x_r$, $0 < |x_i - x| < \delta_{r-i}$, $i = 1, 2, \ldots, r$. Thus if $0 < |u| < \dfrac{\delta_{r-1}}{r}$, then
$$r! V(f;\ x, x+u, \ldots, x+ru) < M_{r-1}\,.$$
Letting $u \to 0$,
$$\limsup_{u\to 0} \frac{\Delta_r(f;\ x, u)}{u^r} \leqslant M_{r-1}$$
for all $x \in E_{r-1}$. Since $\mu(E - E_{r-1}) < r\varepsilon$ and ε is arbitrary, we conclude from (7.17) that
$$\limsup_{u\to 0+} \frac{\Delta_r(f, x, u)}{u^r} < \infty$$
almost everywhere in E.

(7.18) Theorem. *If $f_{r,ap}(x)$ exists finitely, then* $\lim_{u\to 0} ap\, \dfrac{\Delta_r(f, x, u)}{u^r}$ *exists and equals $f_{r,ap}(x)$.*

Proof. We have
$$f_{r,ap}(x) = \lim_{u\to 0} ap\, \frac{r!}{u^r}\left\{f(x+u) - \sum_{i=0}^{r-1} \frac{u^i}{i!} f_i(x)\right\}$$

Hence we can write
$$\text{(7.19)}\qquad \frac{r!}{u^r}\left\{f(x+u) - \sum_{i=0}^{r-1} \frac{u^i}{i!} f_i(x)\right\} = f_{r,ap}(x) + \varepsilon(u)\,,$$
where $\lim_{u\to 0} ap\, \varepsilon(u) = 0$. So from (7.19)
$$\text{(7.20)}\qquad f(x+u) = \sum_{i=0}^{r-1} \frac{u^i}{i!} f_i(x) + \frac{u^r}{r!} f_{r,ap}(x) + \frac{u^r}{r!}\varepsilon(u)\,.$$
Hence from (7.20)
$$\Delta_r(f, x, u) = \sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j} f(x+ju)$$
$$= u^r f_{r,ap}(x) + \frac{u^r}{r!}\sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j}\varepsilon(ju)\,,$$
i.e.
$$\text{(7.21)}\qquad \frac{\Delta_r(f, x, u)}{u^r} = f_{r,ap}(x) + \frac{1}{r!}\sum_{j=0}^{r} (-1)^{r-j}\binom{r}{j}\varepsilon(ju)\,.$$
Now as $u \to 0$ through a set having 0 as a point of density, it can be shown, as in Lemma (7.1), that ju will remain on that set and hence $\lim_{u\to 0} ap\, \varepsilon(ju) = 0$, $j = 0, 1, \ldots, r$ and so from (7.21)
$$\lim_{u\to 0} ap\, \frac{\Delta_r(f;\ x, u)}{u^r} = f_{r,ap}(x)\,.$$

(7.22) Theorem. *If $\bar{f}_r < \infty$ on a measurable set E, then*
$$\limsup_{u\to 0+} \frac{\Delta_r(f, x, u)}{u^r} = (f_{r-1})'_{ap}(x) = f_{r,ap}(x) = \limsup_{u\to 0-} \frac{\Delta_r(f, x, u)}{u^r}$$
holds for almost all x in E, all the values being finite.

Proof. Since $\bar{f}_r < \infty$ on E, as in Lemma (7.16) for any $\varepsilon > 0$, there exists $M > 0$, $\delta > 0$, and a subset $E_0 \subset E$ such that
$$\mu(E - E_0) < \varepsilon$$
and
$$\text{(7.23)}\qquad r! V_r(f;\ x, x_1, \ldots, x_r) < M$$
for all $x, x_1, \ldots, x_r$ in E_0 with $0 < x_1 - x < x_2 - x < \ldots < x_r - x < \delta$ writing
$$g(x) = f(x) - Mx^r$$
we conclude from (7.23)
$$r!\ V_r(g;\ x, x_1, \ldots, x_r) < 0$$
for all $x, x_1, \ldots, x_r \in E_0$ with $0 < x_1 - x < x_2 - x < \ldots < x_r - x < \delta$. Hence

g is r-concave on every portion of E_0 whose diameter does not exceed δ. Hence by Theorem (6.1) $g_{r,ap}$ and $(g_{r-1})_{ap}$ exists finitely and are equal almost everywhere in E_0. Hence $f_{r,ap}$ and $(f_{r-1})'_{ap}$ exists and are equal almost everywhere in E_0. Since ε is arbitrary, $f_{r,ap}$ and $(f_{r-1})'_{ap}$ exists and are equal almost everywhere in E. Hence by Theorem (7.18) $\lim\limits_{u\to 0} ap \dfrac{\Delta_r(f,x,u)}{u^r}$ exists and equal $f_{r,ap}(x)$ for almost all x in E. Since $\bar{f}_r < \infty$ on E, by Lemma (7.16) $\limsup\limits_{u\to 0} \dfrac{\Delta_r(f,x,u)}{u^r} < \infty$ holds for almost all x in E. Also since f_{r-1} exists finitely $\lim\limits_{u\to 0} \left|\dfrac{\Delta_{r-1}(f,x,u)}{u^{r-1}}\right|$ exists finitely for all $x \in E$. Hence applying Lemma (7.1), the proof is complete.

(7.24) Corollary. *The set $\{x: f_r(x) = \pm\infty\}$ is of measure zero.*

We remark that Sargent [11] proved analogous results of Theorem (7.22) and Corollary (7.24) for Cesàro derivatives.

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A note on dimension theory of metric spaces

by

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Abstract. In the second section of this paper we show, using the famous Roy example, that the small inductive dimension ind does not satisfy the finite sum theorem in the class of metric spaces (1). In the third section we give a relatively simple proof of the equivalence of dimensions dim and Ind in the class of metric spaces; on the way we prove some well-known characterizations of these dimensions. In § 1 we consider a natural operation on topological spaces, which is used in § 2.

§ 1. A simple operation on topological spaces.

Proposition 1 (2). *For every topological space X and a continuous mapping $\varphi\colon X\to I$ of X into the interval $I = [0,1]$ such that $\varphi^{-1}(0) \neq \emptyset \neq \varphi^{-1}(1)$ there exist a topological space $\hat{X}$, a continuous mapping $\hat{f}\colon X\to\hat{X}$ onto $\hat{X}$, and a continuous mapping $\hat{\varphi}\colon \hat{X}\to I$, satisfying the following conditions:*

(i) *the diagram*

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\hat{f}"'] & & I \\ & \hat{X} \arrow[ur, "\hat{\varphi}"'] & \end{tikzcd}$$

is commutative;

(ii) *$\hat{f}(\varphi^{-1}(i)) = q_i$, for $i = 0, 1$, where q_0 and q_1 are distinct points of $\hat{X}$;*

(iii) *the restriction of $\hat{f}$ to the subspace $X\setminus\varphi^{-1}(\{0,1\}) \subset X$ is a homeomorphism onto $\hat{X}\setminus\{q_0, q_1\}$;*

(iv) *the families $\{\hat{\varphi}^{-1}([0,1/n))\}_{n=1}^{\infty}$ and $\{\hat{\varphi}^{-1}((1-1/n,1])\}_{n=1}^{\infty}$ form neighbourhood bases in $\hat{X}$ for points q_0 and q_1 respectively.*

Moreover, the triple $\hat{X}, \hat{f}, \hat{\varphi}$ is uniquely determined, i.e. if a topological space $\tilde{X}$ and continuous mappings $\tilde{f}\colon X\to\tilde{X}$, $\tilde{\varphi}\colon \tilde{X}\to I$ satisfy the counter-

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(1) My attention to this problem was called by V. V. Filippov. All undefined notions and symbols are as in [2].

(2) Cf. [6], § 22, IV, Theorem 1.