g is r-concave on every portion of E_0 whose diameter does not exceed δ. Hence by Theorem (6.1) $g_{r,ap}$ and $(g_{r-1})_{ap}$ exists finitely and are equal almost everywhere in E_0. Hence $f_{r,ap}$ and $(f_{r-1})'_{ap}$ exists and are equal almost everywhere in E_0. Since ε is arbitrary, $f_{r,ap}$ and $(f_{r-1})'_{ap}$ exists and are equal almost everywhere in E. Hence by Theorem (7.18) $\lim\limits_{u\to 0} ap \dfrac{\Delta_r(f,x,u)}{u^r}$ exists and equal $f_{r,ap}(x)$ for almost all x in E. Since $\bar{f}_r < \infty$ on E, by Lemma (7.16) $\limsup\limits_{u\to 0} \dfrac{\Delta_r(f,x,u)}{u^r} < \infty$ holds for almost all x in E. Also since f_{r-1} exists finitely $\lim\limits_{u\to 0} \left|\dfrac{\Delta_{r-1}(f,x,u)}{u^{r-1}}\right|$ exists finitely for all $x \in E$. Hence applying Lemma (7.1), the proof is complete.

(7.24) COROLLARY. *The set $\{x\colon f_r(x) = \pm\infty\}$ is of measure zero.*

We remark that Sargent [11] proved analogous results of Theorem (7.22) and Corollary (7.24) for Cesàro derivatives.

References

[1] P. S. Bullen, *A criterion for n-convexity*, Pacific J. Math. 36 (1971), pp. 81–98.
[2] — and S. N. Mukhopadhyay, *Peano derivatives and general integrals*, Pacific J. Math. (to appear).
[3] E. Corominas, *Contribution à la théorie de la dérivation d'ordre supérieur*, Bull. Soc. Math. France 81 (1953), pp. 177–222.
[4] A. Denjoy, *Sur l'intégration des coefficients différentiels d'ordre supérieur*, Fund. Math. 25 (1935), pp. 273–326.
[5] — *Leçons sur le calcul des coefficients d'une série trigonometrique*, Paris 1941.
[6] J. Marcinkiewicz, *Sur les séries de Fourier*, Fund. Math. 27 (1937), pp. 38–69.
[7] — and A. Zygmund, *On the differentiability of functions and the summability of trigonometric series*, Fund. Math. 26 (1936), pp. 1–43.
[8] H. W. Oliver, *The exact Peano derivative*, Trans. Amer. Math. Soc. 76 (1954), pp. 444–456.
[9] T. Popoviciu, *Les fonctions convexes*, Paris 1944.
[10] S. Saks, *Theory of the Integral*, Warsaw 1937.
[11] W. L. C. Sargent, *On the Cesàro derivates of a function*, Proc. London Math. Soc. (2) 40 (1936), pp. 235–254.
[12] — *On generalized derivatives and Cesàro–Denjoy integral*, Proc. London Math. Soc. (2) 52 (1951), pp. 365–376.
[13] M. Weiss, *On symmetric derivatives in L^p*, Studia Math. 24 (1964), pp. 89–100.
[14] A. Zygmund, *Trigonometric series*, Cambridge 1968.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRITISH COLUMBIA
Vancouver, Canada

Reçu par la Rédaction le 20. 3. 1973

A note on dimension theory of metric spaces

by

T. Przymusiński (Warszawa)

Abstract. In the second section of this paper we show, using the famous Roy example, that the small inductive dimension ind does not satisfy the finite sum theorem in the class of metric spaces (1). In the third section we give a relatively simple proof of the equivalence of dimensions dim and Ind in the class of metric spaces; on the way we prove some well-known characterizations of these dimensions. In § 1 we consider a natural operation on topological spaces, which is used in § 2.

§ 1. A simple operation on topological spaces.

PROPOSITION 1 (2). *For every topological space X and a continuous mapping $\varphi\colon X\to I$ of X into the interval $I = [0,1]$ such that $\varphi^{-1}(0) \neq \emptyset \neq \varphi^{-1}(1)$ there exist a topological space $\hat{X}$, a continuous mapping $\hat{f}\colon X\to\hat{X}$ onto $\hat{X}$, and a continuous mapping $\hat{\varphi}\colon \hat{X}\to I$, satisfying the following conditions:*

(i) *the diagram*

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\hat{f}"'] & & I \\ & \hat{X} \arrow[ur, "\hat{\varphi}"'] & \end{tikzcd}$$

is commutative;

(ii) *$\hat{f}(\varphi^{-1}(i)) = q_i$, for $i = 0, 1$, where q_0 and q_1 are distinct points of $\hat{X}$;*

(iii) *the restriction of $\hat{f}$ to the subspace $X\setminus\varphi^{-1}(\{0,1\}) \subset X$ is a homeomorphism onto $\hat{X}\setminus\{q_0, q_1\}$;*

(iv) *the families $\{\hat{\varphi}^{-1}([0,1/n))\}_{n=1}^{\infty}$ and $\{\hat{\varphi}^{-1}((1-1/n,1])\}_{n=1}^{\infty}$ form neighbourhood bases in $\hat{X}$ for points q_0 and q_1 respectively.*

Moreover, the triple $\hat{X}, \hat{f}, \hat{\varphi}$ is uniquely determined, i.e. if a topological space $\tilde{X}$ and continuous mappings $\tilde{f}\colon X\to\tilde{X}$, $\tilde{\varphi}\colon\tilde{X}\to I$ satisfy the counter-

(1) My attention to this problem was called by V. V. Filippov. All undefined notions and symbols are as in [2].

(2) Cf. [6], § 22, IV, Theorem 1.

parts of (i)–(iv), *then there exists a homeomorphism* $h\colon \hat{X}\to\tilde{X}$ *such that the diagram*

$$\begin{tikzcd}
 & & \hat{X} \arrow[dl, "\hat{\varphi}"] \arrow[dd, "h"] \\
X \arrow[urr, "\hat{f}"] \arrow[r, "\varphi"] \arrow[drr, "\tilde{f}"'] & I & \\
 & & \tilde{X} \arrow[ul, "\tilde{\varphi}"']
\end{tikzcd}$$

is commutative.

Proof. Let $\hat{X}=X\setminus\varphi^{-1}(\{0,1\})\cup\{q_0,q_1\}$, where $q_0\neq q_1$ and $q_0, q_1\notin X$; for every $x\in X$ assume

$$\hat{f}(x)=\begin{cases} x, & \text{if} \quad x\in X\setminus\varphi^{-1}(\{0,1\}),\\ q_0, & \text{if} \quad x\in\varphi^{-1}(0),\\ q_1, & \text{if} \quad x\in\varphi^{-1}(1),\end{cases}$$

and for every $y\in\hat{X}$ put $\hat{\varphi}(y)=\varphi(\hat{f}^{-1}(y))$.

Consider $\hat{X}$ with the topology generated by the base

$$\mathfrak{B}=\{U\colon U \text{ is open in } X\setminus\varphi^{-1}(\{0,1\})\}\cup \\ \cup\{\hat{\varphi}^{-1}([0,1/n))\}_{n=1}^{\infty}\cup\{\hat{\varphi}^{-1}((1-1/n,1])\}_{n=1}^{\infty}.$$

It is easy to verify that the triple $\hat{X}, \hat{f}, \hat{\varphi}$ satisfies conditions (i)–(iv). To prove that $\hat{X}, \hat{f}, \hat{\varphi}$ are uniquely determined it suffices to put

$$h(y)=\tilde{f}(\hat{f}^{-1}(y)) \quad \text{for} \quad y\in\hat{X}. \ \blacksquare$$

Remark 1. Let X be a completely regular space and $\varphi\colon X\to I$ a continuous mapping. Assume that $\bar{\varphi}\colon \alpha X\to I$ is a continuous extension of φ onto a compactification αX of X. If we identify compact subsets $\bar{\varphi}^{-1}(0)$ and $\bar{\varphi}^{-1}(1)$ of $Y=X\cup\bar{\varphi}^{-1}(\{0,1\})\subset\alpha X$ to points, then the obtained quotient space is homeomorphic to $\hat{X}$ (cf. Examples 1 and 2). $\blacksquare$

Remark 2. In general the mapping $\hat{f}\colon X\to\hat{X}$ is not quotient and it need not be a homeomorphism when it is one-to-one. It is quotient if and only if the families $\{\varphi^{-1}([0,1/n))\}_{n=1}^{\infty}$ and $\{\varphi^{-1}((1-1/n,1])\}_{n=1}^{\infty}$ form neighbourhood bases for the sets $\varphi^{-1}(0)$ and $\varphi^{-1}(1)$ respectively. It follows from [7], that if X is normal and $\hat{f}$ is quotient, then the sets $\mathrm{Fr}\,\varphi^{-1}(i)$ are countably compact for $i=0,1$. $\blacksquare$

Definition. A class $\mathfrak{K}$ of topological spaces will be called *invariant under the* $\wedge$*-operation* if for every $X\in\mathfrak{K}$ and a continuous mapping $\varphi\colon X\to I$ such that $\varphi^{-1}(0)\neq\emptyset\neq\varphi^{-1}(1)$, the space $\hat{X}$ belongs to $\mathfrak{K}$.

Owing to the fact that $\hat{f}$ in general differs from the natural quotient map, many classes of topological spaces are invariant under the $\wedge$-operation; below we list some of them.

Proposition 2. *Each of the following classes is invariant under the* $\wedge$*-operation;*

(a) *metrizable spaces;*
(b) *paracompact spaces;*
(c) *perfectly normal spaces;*
(d) *totally normal spaces;*
(e) T_i*-spaces, for* $i=0,1,2,3,3\frac{1}{2},4$;
(f) *spaces complete in the sense of Čech;*
(g) *spaces of weight not greater than* $\mathfrak{m}\geqslant\aleph_0$;
(h) *spaces of character not greater than* $\mathfrak{m}\geqslant\aleph_0$;

Proof. As a sample, we shall prove invariance of the class (a); proofs for all remaining classes are not harder.

Let $\mathfrak{B}$ be a σ-locally finite base in a metrizable space X. Define

$$\hat{\mathfrak{B}}_n=\{\hat{f}(B)\setminus\hat{\varphi}^{-1}([0,1/n]\cup[1-1/n,1])\colon B\in\mathfrak{B}\}\cup \\ \cup\{\hat{\varphi}^{-1}([0,1/n)),\hat{\varphi}^{-1}((1-1/n,1])\}.$$

One can easily check that $\hat{\mathfrak{B}}=\bigcup_{n=1}^{\infty}\hat{\mathfrak{B}}_n$ is a σ-locally finite base in $\hat{X}$. As $\hat{X}$ is obviously regular it is a metrizable space. $\blacksquare$

§ 2. The sum theorem for the small inductive dimension.

Definition. A dimension function D defined on a class $\mathfrak{K}$ of topological spaces *satisfies the finite* (σ*-locally finite*) *sum theorem in* $\mathfrak{K}$, if for every $X\in\mathfrak{K}$ and a finite (σ-locally finite) closed covering $\mathfrak{F}$ of X such that $D(F)\leqslant n$, for every $F\in\mathfrak{F}$, we have $D(X)\leqslant n$.

It is well known that in the class of metric spaces the dimensions dim and Ind coincide and satisfy the σ-locally finite sum theorem (cf. § 3). We shall show that the small inductive dimension ind does not satisfy the finite sum theorem in the class of metric spaces.

Theorem 1. *Let* $\mathfrak{K}$ *be a class of normal spaces invariant under the* $\wedge$*-operation and hereditary with respect to closed subspaces.*

If the small inductive dimension ind *satisfies the finite sum theorem in* $\mathfrak{K}$, *then dimensions* ind *and* Ind *coincide in* $\mathfrak{K}$.

Proof. Assume that dimensions ind and Ind do not coincide in the class $\mathfrak{K}$. Let n be the smallest integer such that for some X in $\mathfrak{K}$ we have $\operatorname{ind}X=n<\operatorname{Ind}X$. There exist two disjoint non-empty closed subsets $P_0, P_1\subset X$ such that for every neighbourhood U of P_0 satisfying $U\cap P_1=\emptyset$, we have $\operatorname{Ind}\operatorname{Fr}U\geqslant n$. As $\mathfrak{K}$ is hereditary with respect to closed subspaces, by the definition of n we have also $\operatorname{ind}\operatorname{Fr}U\geqslant n$. Let $\varphi\colon X\to I$ be a continuous mapping of X into I such that $\varphi^{-1}(i)\supset P_i$ for $i=0,1$. Take a triple $\hat{X}, \hat{f}, \hat{\varphi}$ and points q_0, q_1 satisfying conditions (i)–(iv) of Proposition 1 and for $i=0,1$ put $F_i=\hat{\varphi}^{-1}(C_i)$, where C_0 and C_1 are closed subsets of I defined by

$$C_0 = \{0\} \cup \bigcup_{n=1}^{\infty}\left[\frac{1}{2n+1}, \frac{1}{2n}\right] \cup \bigcup_{n=1}^{\infty}\left[1-\frac{1}{2n}, 1-\frac{1}{2n+1}\right] \cup \{1\},$$

$$C_1 = \{0\} \cup \bigcup_{n=1}^{\infty}\left[\frac{1}{2n+2}, \frac{1}{2n+1}\right] \cup \bigcup_{n=1}^{\infty}\left[1-\frac{1}{2n+1}, 1-\frac{1}{2n+2}\right] \cup \{1\}.$$

One can easily check that F_i are closed, $\hat{X} = F_0 \cup F_1$ and $\operatorname{ind} F_i \leqslant n$ for $i = 0, 1$. On the other hand our assumptions imply that $\hat{X} \in \mathfrak{R}$ and that for every neighbourhood V of q_0 not containing q_0, we have $\operatorname{ind} \operatorname{Fr} V \geqslant n$. It follows that $\operatorname{ind} \hat{X} \geqslant n+1$, so the dimension ind does not satisfy the finite sum theorem in $\mathfrak{R}$. ■

The next theorem follows from Proposition 2, Theorem 1 and the existence of a (complete) metric space Δ such that $\operatorname{ind} \Delta = 0 < \operatorname{Ind} \Delta = 1$ [9].

THEOREM 2. *The small inductive dimension* ind *does not satisfy the finite sum theorem in the class of (complete) metrizable spaces.* ■

Arguing as in the proof of Theorem 1 we obtain

THEOREM 3. *Let $\mathfrak{R}$ be a class of normal spaces invariant under the $\wedge$-operation.*

The following propositions are equivalent:

(i) *for every $X \in \mathfrak{R}$, we have*: $\operatorname{ind} X = 0 \Leftrightarrow \operatorname{Ind} X = 0 \Leftrightarrow \dim X = 0$;

(ii) *in the class $\mathfrak{R}$ the small inductive dimension* ind *satisfies the finite (σ-locally finite) sum theorem for $n = 0$.* ■

Remark 3. Theorems 1 and 3 can be regarded as two arguments more pointing, that the small inductive dimension ind is not a "nice" dimension function. Indeed, if in a "good" class of spaces ind behaves "properly", then it coincides with the large inductive dimension Ind. Observe that Theorem 2 can be somewhat strengthened: there exists a non-totally disconnected (3) complete metric space, which is the union of two closed subsets zerodimensional in the sense of ind. Obviously, such a space has to be hereditarily disconnected (4). ■

To finish this section we shall give two examples in which the $\wedge$-operation is used in an implicit way (cf. Remark 1). The existence of spaces with properties described in these examples is well known, but our construction is very simple.

(3) A space X is *totally disconnected* if for any two distinct points $x_1, x_2 \in X$ there exists an open and closed set $U \subset X$ such that $x_1 \in U$ and $x_2 \notin U$.

(4) A space X is *hereditarily disconnected* if every subspace of X containing more than one point is disconnected. Every T_1 space zero-dimensional in the sense of ind is totally disconnected and every totally disconnected space is hereditarily disconnected.

EXAMPLE 1. A connected separable metric space Z which is the union of two closed totally disconnected subsets and becomes totally disconnected after removing of a point $q \in Z$.

Let X be the subset of the Hilbert space consisting of all points with rational coordinates. It is known ([4]; see also [2], Example 6.2.2) that no open and closed non-empty subset of X is contained in $K = \{x = \{x_n\} \in X: \sum_{n=1}^{\infty} x_n^2 < 1\}$. Assume that X is embedded into the *Hilbert cube* $I^{\aleph_0}$ and consider the subspace $Y = X \cup \overline{X \setminus K}$ of $I^{\aleph_0}$. Let Z be the space obtained from Y by identification of $\overline{X \setminus K}$ to a point, $f\colon Y \to Y/\overline{X \setminus K} = Z$ be the quotient mapping and $q = f(\overline{X \setminus K})$. One can easily check that Z has the desired properties (cf. Remark 1 and the proof of Theorem 1).

EXAMPLE 2. A normal space X, such that $\operatorname{ind} X = \operatorname{Ind} X = \dim X = 1$, which is the union of two closed subspaces, zerodimensional in the sense of ind.

Let us consider the Dowker example of a normal space Y such that $\operatorname{ind} Y = 0 < \operatorname{Ind} Y = \dim Y = 1$ ([1]; we use the notation from [2], Example 6.2.3). Take the subspace $Z = Y \cup (X^* \times \{0, 1\})$ of Y^* and identify subsets $X^* \times \{0\}$ and $X^* \times \{1\}$ of Z to points. The obtained quotient space has the desired properties.

§ 3. A new proof of the Katětov–Morita theorem. In this section we present a relatively simple proof of the famous theorem, due to Katětov [5] and Morita [8], stating that in the class of metrizable spaces dimensions dim and Ind coincide. On the way, we obtain the σ-locally finite sum theorem and a few well-known characterizations of these dimensions; all arguments used below are slight modifications of classical proofs. The final form of this section arosed from discussions with Professor R. Engelking. The proof of the implication (ii) ⇒ (iii) in Theorem A is taken from his paper [3].

In the proof of Theorems A and B below, besides of elementary properties of dimensions dim and Ind in the class of *metrizable* spaces, we use only tree following facts:

(1) $\dim X \leqslant 0$ *if and only if* $\operatorname{Ind} X \leqslant 0$ *if and only if X admits a σ-discrete base consisting of open and closed sets.*

(2) *If $\mathfrak{F}$ is a countable closed covering of X such that* $\dim F \leqslant 0$ *for every $F \in \mathfrak{F}$, then* $\dim X \leqslant 0$.

(3) $\dim X \leqslant n$ *if and only if any open covering of X admits a locally finite open refinement of order not greater than n.*

THEOREM A. *For a metric space (X, ϱ) and an integer $n \geqslant 0$ the following conditions are equivalent:*

(i) $\dim X \leqslant n$.

(ii) *There exists a sequence* $\{\mathfrak{U}_i\}_{i=1}^{\infty}$ *of locally finite open coverings of* X *such that for every* $i=1,2,\ldots$ *diameters of elements of* $\mathfrak{U}_i$ *are less than* $1/i$, $\operatorname{ord}\mathfrak{U}_i \leqslant n$ *and closures of elements of* $\mathfrak{U}_{i+1}$ *refine* $\mathfrak{U}_i$.

(iii) *For any two closed subsets* A, B *of* X *with positive distance, there exists an open set* K *such that* $A\subset K\subset X\setminus B$ *and* $\dim\operatorname{Fr}K\leqslant n-1$.

(iv) X *admits a* σ*-discrete base* $\mathfrak{B}$ *such that* $\dim\operatorname{Fr}B\leqslant n-1$ *for every* $B\in\mathfrak{B}$.

(v) $X=\bigcup_{i=0}^{n} Y_i$, *where* $\dim Y_i\leqslant 0$ *for* $i=0,1,\ldots,n$.

(vi) $\operatorname{Ind}X\leqslant n$.

THEOREM B. *If* $\mathfrak{F}$ *is a countable closed covering of a metric space such that* $\dim F\leqslant n$ *for every* $F\in\mathfrak{F}$, *then* $\dim X\leqslant n$.

Proof of Theorems A and B. Assume that $n=0$ or $n>0$ and both Theorems A and B are proved for $k<n$.

Ad Theorem A: (i) $\Rightarrow$ (ii). We omit the proof of this implication which is an easy consequence of (3) (cf. [2], Lemma 7.3.1).

(ii) $\Rightarrow$ (iii). Assume that the distance of A and B is greater than $1/N$ for an integer N and that $\{\mathfrak{U}_i\}_{i=1}^{\infty}$ satisfies (ii).

Put $K_0=A$, $M_0=B$ and $K_i=X\setminus H_i$, $M_i=X\setminus G_i$, where $G_i=\bigcup\{U\in\mathfrak{U}_{N+i}\colon \overline{U}\cap M_{i-1}=\emptyset\}$, $H_i=\bigcup\{U\in\mathfrak{U}_{N+i}\colon \overline{U}\cap M_{i-1}\neq\emptyset\}$, for $i\geqslant 1$. Notice that for $i\geqslant 1$.

(4) *If* $U\in U_{N+i}$ *and* $\overline{U}\cap M_{i-1}\neq\emptyset$, *then* $\overline{U}\cap K_{i-1}=\emptyset$.

The validity of (4) for $i=1$ follows from the definition of N. If $i>1$, $U\in\mathfrak{U}_{N+i}$ and $\overline{U}\cap M_{i-1}\neq\emptyset$, then for $V\in U_{N+i-1}$ that contains $\overline{U}$ we have $V\cap M_{i-1}\neq\emptyset$, so V is not contained in G_{i-1}. Hence $V\subset H_{i-1}$ and $\overline{U}\cap K_{i-1}=\emptyset$.

From local finitness of $\mathfrak{U}_i$, definitions of G_i and H_i, and (4) we infer that $\overline{G}\cap M_{i-1}=\emptyset=\overline{H}_i\cap K_{i-1}$, for $i=1,2,\ldots$ This implies that $K_{i-1}\subset X\setminus\overline{H}_i=\operatorname{Int}K_i$ and $M_{i-1}\subset X\setminus\overline{G}_i=\operatorname{Int}M_i$. We have also $K_i\cap M_i=\emptyset$, as $G_i\cup H_i=X$. The sets $K=\bigcup_{i=0}^{\infty}K_i$ and $M=\bigcup_{i=0}^{\infty}M_i$ are open and disjoint, $A\subset K$ and $B\subset M$.

Let $L=X\setminus(K\cup M)=\bigcap_{i=1}^{\infty}(G_i\cap H_i)$. For $i=1,2,\ldots$ the family $\mathfrak{W}_i=\{U\cap L\colon U\in\mathfrak{U}_{N+i}$ and $\overline{U}\cap M_{i-1}\neq\emptyset\}$ is an open covering of L and $\operatorname{ord}\mathfrak{W}_i\leqslant n-1$, because every $x\in L\subset G_i\cap H_i$ is contained also in a $U\in\mathfrak{U}_{N+i}$ such, that $\overline{U}\cap M_{i-1}=\emptyset$. If $U\in\mathfrak{U}_{N+i+1}$ and $\overline{U}\cap M_i\neq\emptyset$, then for $V\in\mathfrak{U}_{N+i}$ such that $\overline{U}\subset V$ we have $V\cap M_i\neq\emptyset$, so V is not contained in G_i. It follows that $V\cap M_{i-1}\neq\emptyset$, i.e. that $V\cap L\in\mathfrak{W}_i$ and consequently closures of elements of $\mathfrak{W}_{i+1}$ refine $\mathfrak{W}_i$. Obviously, diameters of elements of $\mathfrak{W}_i$ are less than $1/N+i\leqslant 1/i$.

If $n=0$, then $\operatorname{Fr}K\subset L=\emptyset$ and if $n>0$, then the inductive assumption implies that $\dim\operatorname{Fr}K\leqslant n-1$.

(iii) $\Rightarrow$ (iv). Let $\mathfrak{B}$ be a σ-discrete base in X. For every $V\in\mathfrak{B}$ define $A(V,i)=\{x\in V\colon d(x,X\setminus V)\geqslant 1/i\}$ and take an open set $K(V,i)$ such that $A(V,i)\subset K(V,i)\subset V$ and $\dim\operatorname{Fr}K(V,i)\leqslant n-1$. It is easy to verify that $\mathfrak{B}=\{K(V,i)\colon V\in\mathfrak{B},\ i=1,2,\ldots\}$ satisfies condition (iv).

(iv) $\Rightarrow$ (v). Assume (iv) and put $Z=\bigcup\{\operatorname{Fr}B\colon B\in\mathfrak{B}\}$ and $Y_n=X\setminus Z$. As Y_n admits a σ-discrete base consisting of open and closed sets, we infer from (1) that $\dim Y_n\leqslant 0$. If $n=0$, then $Z=\emptyset$ and if $n>0$, then Z is the union of a countable family $\mathfrak{F}$ of closed sets such that $\dim F\leqslant n-1$ for every $F\in\mathfrak{F}$. The inductive assumption gives $\dim Z\leqslant n-1$ and consequently $Z=\bigcup_{i=0}^{n-1}Y_i$, where $\dim Y_i\leqslant 0$ for $i=0,1,\ldots,n-1$.

(v) $\Rightarrow$ (vi). This can be proved by a standard argument (cf. [2], Lemma 7.3.2).

(vi) $\Rightarrow$ (i). As (vi) and the inductive assumption obviously imply (iii), it suffices to prove that (v) $\Rightarrow$ (i).

Let $\mathfrak{U}$ be a finite open covering of $X=\bigcup_{i=0}^{n}Y_i$, where $\dim Y_i\leqslant 0$ for $i=0,1,\ldots,n$. For every i we can find a finite open covering $\mathfrak{V}_i=\{V_k^i\}_{k=1}^{m_i}$ of Y_i consisting of disjoint sets and refining $\mathfrak{U}$. As X is hereditarily normal, for every $i=0,1,\ldots,n$, there exists a family $\mathfrak{G}_i=\{G_k^i\}_{k=1}^{m_i}$ of disjoint open subsets of X refining $\mathfrak{U}$ and such that $V_k^i\subset G_k^i$ for $k=1,2,\ldots,m_i$. It is clear that the finite open covering $\mathfrak{G}=\bigcup_{i=0}^{n}\mathfrak{G}_i$ of X refines $\mathfrak{U}$ and that $\operatorname{ord}\mathfrak{G}\leqslant n$.

Ad Theorem B: Let $\mathfrak{F}=\{F_m\}_{m=1}^{\infty}$ and put $X_m=F_m\setminus\bigcup_{j<m}F_j$. By (iv) we have $\dim X_m\leqslant n$ and every X_m is an F_σ-set in X. From (v) it follows that $X_m=\bigcup_{i=0}^{n}Y_i^m$, where $\dim Y_i^m\leqslant 0$. For every $i=0,1,\ldots,n$ the intersection $Y_i\cap X_m=Y_i^m$ is an F_σ-set in $Y_i=\bigcup_{m=1}^{\infty}Y_i^m$, so — by (2) — $\dim Y_i\leqslant 0$. We infer from (v) that $\dim X\leqslant n$. ∎

Let us notice that from Theorem B we can easily derive.

THEOREM C. *The dimension* dim *satisfies the* σ*-locally finite sum theorem in the class of metric spaces.*

Proof. By Theorem B it suffices to prove that if $\mathfrak{F}$ is a locally finite closed covering of a metric space X such that $\dim F\leqslant n$ for every $F\in\mathfrak{F}$, then $X=\bigcup_{m=1}^{\infty}C_m$, where C_m are closed and $\dim C_m\leqslant n$.

Take an open covering $\mathfrak{U}$ of X, whose elements meet only finite number of members of $\mathfrak{F}$ and a σ-discrete closed refinement $\mathfrak{G} = \bigcup_{m=1}^{\infty} \mathfrak{G}_m$ of $\mathfrak{U}$, where $\mathfrak{G}_m$ are discrete for $m = 1, 2, \ldots$ The sets $C_m = \bigcup \mathfrak{G}_m$ satisfy our assumptions.

Added in proof. Eric van Douwen has independently obtained our Theorem 2. His paper will appear in Indagations Mathematicae.

References

[1] C. H. Dowker, *Local dimension of normal spaces*, Quart. Journ. of Math. Oxford 6 (1955), pp. 101–120.
[2] R. Engelking, *Outline of General Topology*, Amsterdam 1968.
[3] — *Some new proofs in dimension theory*, Ist. di Alta Math. Roma, Symposia Math. (1974).
[4] P. Erdös, *The dimension of the rational points in Hilbert space*, Ann. of Math. 41 (1940), pp. 734–736.
[5] M. Katětov, *On the dimension of non-separable spaces I* (in Russian), Czech. Math. Journ. 2 (1952), pp. 333–368.
[6] K. Kuratowski, *Topology*, vol. I, New York–London–Warszawa 1966.
[7] E. Michael, *A note on closed maps and compact sets*, Israel Journ. of Math. 2 (1964), pp. 173–176.
[8] K. Morita, *Normal families and dimension theory for metric spaces*, Math. Ann. 128 (1954), pp. 350–362.
[9] P. Roy, *Non-equality of dimensions for metric spaces*, Trans. Amer. Math. Soc. 134 (1968), pp. 117–132.

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES, Warsaw

Reçu par la Rédaction le 3. 4. 1973

Further results on the achromatic number

by

Dennis Geller and **Hudson Kronk** (Binghamton, N. Y.)

Abstract. We investigate: (1) the effect on the achromatic number of removing points or lines, (2) exact values for the achromatic numbers of paths and cycles, and (3) general bounds for the achromatic number and similar but tighter bounds for the achromatic numbers of bipartite graphs.

A coloring of a graph (1) G is *complete* if for every two colors i and j there are adjacent points u and v, colored i and j respectively. The achromatic number $\psi = \psi(G)$ is the largest number n such that G has a complete coloring with n colors. The achromatic number was introduced in [4] as the largest order of the complete homomorphisms of G. Later results appeared in [1] and [3]. In this paper we investigate the effect on the achromatic number of removing points and lines from G, find the values $\psi(C_n)$ and $\psi(P_n)$, and develop bounds for the achromatic number of any graph and for the achromatic number of any bigraph.

THEOREM 1. *For any graph G and point $u \in G$,*

$$\psi(G) \geqslant \psi(G-u) \geqslant \psi(G)-1 .$$

Proof. If $\psi(G-u) = n$, then G has a complete n-coloring unless each of the n colors is assigned to some point adjacent to u, in which case G has a complete $(n+1)$-coloring. Thus $\psi(G) \geqslant \psi(G-u)$.

On the other hand, if $\psi(G) = n$, consider the coloring of $G-u$ induced by a complete n-coloring of G, in which u is assigned color i. If this coloring is not complete, there is some color j not adjacent to any point colored i. If all points of $G-u$ which are colored i are recolored j the result is a complete $(n-1)$-coloring, so that $\psi(G-u) \geqslant \psi(G)-1$.

COROLLARY. *If $\psi(G-u) = \psi(G)$ there is a complete $\psi(G)$-coloring of G which induces a complete $\psi(G)$-coloring of $G-u$.*

Proof. Suppose that $\psi(G-u) = \psi(G) = n$. If no complete n-coloring of G induces a complete n-coloring of $G-u$, then in every complete n-coloring of $G-u$ every color appears on some point adjacent to u, in which case $\psi(G) \geqslant 1+\psi(G-u)$ as shown above.

(1) Definitions and notations are those of [2].