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The undecidability of the existence of a non-separable normal Moore space satisfying the countable chain condition

by

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Abstract. It is shown that Martin's Axiom plus the negation of the continuum hypothesis implies the existence of a non-separable normal Moore space satisfying the countable chain condition. The consistency and independence of the existence of such spaces follows.

In [22], M. E. Rudin constructed a non-separable Moore space (1) satisfying the contable chain condition (2). The importance of her example lay in showing how far removed Moore spaces can be from metrizable spaces. Any such example cannot be locally metrizable, cannot have a dense metrizable subspace, and cannot be completed (3), indeed cannot be densely embedded in a Moore space satisfying the Baire category theorem [1, Theorem 3.31]. In [17], Pixley and Roy constructed a much simpler example, which in addition is metacompact(4). In this note we construct a subspace of their space with the same properties, which, moreover, is normal, if Martin's Axiom [16], [31] plus the negation of the continuum hypothesis is assumed. These assumptions are consistent with the usual axioms of set theory, e.g. Zermelo–Fraenkel, including the Axiom of Choice [25]. Some such assumption is necessary, since in [26], the second author established the consistency of the assumption that every countable chain condition normal Moore space is metrizable, and hence separable.

(1) A Moore space is a regular Hausdorff space X having a sequence of open covers $\{G_n\}_{n<\omega}$ such that for each $x \in X$ and U open containing x, there is an n such that $\bigcup \{g \in G_n\colon x \in g\} \subset U$.

(2) I.e. every collection of disjoint open sets is countable.

(3) There are several different notions of completeness and completability for Moore spaces. The reader is referred to [1] for details.

(4) The Proceedings of the 1971 Prague Topological Symposium have just reached the second author, who is probably responsible for A. V. Arhangel'skiĭ's incorrect discussion [5] of the Pixley–Roy example. No special set-theoretic assumptions are needed to construct their (completely regular) space.

We obtain the following results:

THEOREM. *Martin's Axiom plus* $2^{\aleph_0} > \aleph_1$ *imply the existence of a non-separable metacompact normal Moore space satisfying the countable chain condition.*

COROLLARY. *Each of the following propositions is independent of the axioms of set theory:*

(i) *There is a normal, countable chain condition, non-metrizable metacompact Moore space.*

(ii) *There is a normal countable chain condition non-separable Moore space.*

(iii) *There is a normal, countable chain condition Moore space with no dense metrizable subspace.*

(iv) *There is a normal, countable chain condition p-space* (*in the sense of* [4]) *with a point-countable base which is not metrizable.*

(v) *There is a normal, countable chain condition, non-Lindelöf space with a point-countable base.*

One of the more important consequences of Martin's Axiom plus $2^{\aleph_0} > \aleph_1$ is that *every set S of reals of power $\aleph_1$ is a Q-set* [21], *i.e., in the subspace topology on S, every subset of S is an F_σ*. This follows immediately from results of Rothberger [21] (the non-existence of Hausdorff's Ω-limits [11] implies every set of reals of power $\aleph_1$ is a Q-set) and Booth [7] (Martin's Axiom plus $2^{\aleph_0} > \aleph_1$ implies there are no Ω-limits). The hypothesis of the existence of an uncountable Q-set was used by Bing [6] to construct a separable normal non-metrizable Moore space, and by the first author to construct a Lindelöf space X such that X^2 is normal but not paracompact [19]. The points of the Pixley–Roy space are all finite sets of reals; we shall use finite subsets of an uncountable Q-set. The proof that our space is a countable chain condition, non-separable metacompact Moore space differs inessentially from Pixley–Roy, but since their paper is not universally available, we sketch the details. The proof that our space is normal is somewhat similar to the proof that the space of [19] is normal.

Now to construct the space. Let S be any subset of the reals of power $\aleph_1$ with the subspace topology. We may assume that every non-empty open subset of S is uncountable. Let X be the set of all finite subsets of S having at least two members (the latter restriction is for technical convenience). We take as a basis for a topology on X all sets of the form

$$U(x, V) = \{y \in X\colon x \subset y \subset V\}, \quad x \in X,$$

where V is open in S. The topology is well-defined since if $z \in U(x_1, V_1) \cap U(x_2, V_2)$, $z \in U(z, V_1 \cap V_2) \subset U(x_1, V_1) \cap U(x_2, V_2)$. For $\varepsilon > 0$ and $x \in X$, let $B(x, \varepsilon)$ be the union of open segments in S of radius ε, centered at the members of x. Let $U(x, \varepsilon)$ abbreviate $U(x, B(x, \varepsilon))$. Let $\mathcal{U}_n = \{U(x, 1/n)\colon x \in X\}$. Then $\bigcup_{n<\omega} \mathcal{U}_n$ is a basis for X. Moreover, each $\mathcal{U}_n$ is point-finite, since $y \in U(x, 1/n)$ implies $x \subset y$, and y has only finitely many subsets. Thus X has a σ-point-finite base.

X is Hausdorff, for given $x \neq y \in X$, say e.g. $t \in x - y$, take V open in S, $y \subset V$, $t \notin V$. Then $U(y, V) \cap U(x, S) = 0$.

Each basic open set is closed, but we won't use this fact, proved by Pixley–Roy. X is a Moore space, but rather than verifying this directly as in Pixley–Roy, we shall show, assuming Martin's Axiom plus $2^{\aleph_0} > \aleph_1$, that X is perfectly normal. By [8], perfectly normal spaces with σ-point-finite bases are normal metacompact Moore spaces (and conversely).

No non-empty open subset of X is separable, for if not, some $U(p, V)$ would be separable, say D countable dense in $U(p, V)$. There is an $x \in V$ which is in no member of D since V is uncountable. Then $U(p \cup \{x\}, V) \subset U(p, V)$, but $D \cap U(p \cup \{x\}, V) = 0$.

We next verify as in Pixley–Roy that X satisfies the countable chain condition. Suppose not. Without loss of generality, assume there is an uncountable collection of mutually disjoint members of $\bigcup_{n<\omega} \mathcal{U}_n$. By two applications of the pigeonhole principle, it follows that there are positive integers m, n and an uncountable subset Y of X, with $\overline{\overline{y}}$ (the cardinality of y) $= m$, for $y \in Y$, such that the $U(y, 1/n)$ are mutually disjoint. Given any point $x \in X$, say $x = \{x_1, \ldots, x_k\}$, $k \geqslant 2$, $x_i \in S$, we may assume that $x_1 < \ldots < x_k$. Thus x corresponds to a unique point $(x_1, \ldots, x_k) \in S^k$, and X can be considered as a subset (not subspace) of $\bigcup_{k=2}^{\infty} S^k$. In S^m then, some $y \in Y$ is a limit point of Y. We can therefore find a $z \in Y$ such that for each i, $1 \leqslant i \leqslant m$, $|y_i - z_i| < 1/n$. Then $y \cup z \in U(y, 1/n) \cap U(z, 1/n)$, contradiction.

It remains to show X is perfectly normal. We need two lemmas.

LEMMA 1. *A T_1 space X is perfectly normal if and only if for every open subset U of X, there exist open sets V_n, $n < \omega$, such that $U = \bigcup_{n<\omega} V_n$, and $\overline{V}_n \subset U$.*

Proof. Clear.

LEMMA 2. *If every set of reals of power $\aleph_1$ is a Q-set, then every subset of every separable metric space of power $\aleph_1$ is an F_σ.*

Proof. The hypothesis clearly implies $2^{\aleph_0} > \aleph_1$. Any separable metric space S of power $< 2^{\aleph_0}$ is 0-dimensional [15, p. 286] and hence is homeomorphic to a subset of the Cantor set [15, p. 285]. A fortiori, S is homeomorphic to a subset S' of the reals. S' is a Q-set, so every subset of S is an F_σ.

Let U be an open subset of X. For every $x \in U$, there exists a $\mu(x)$ such that $U(x, 1/\mu(x)) \subset U$. Define $\varrho(x) = \min_{i \neq j} |x_i - x_j|$. Let

$$A_{n,m} = \{x \in U: \overline{\overline{x}} = n,\ \mu(x) \leqslant m,\ \varrho(x) \geqslant 1/m\}.$$

Then $U = \bigcup_{n=2}^{\infty} \bigcup_{m=1}^{\infty} A_{n,m}$. We shall consider $A_{n,m}$ as a subset of S^n, via the correspondence indicated above. Since S^n is a separable metric space of cardinality $\aleph_1$, by Lemma 2, assuming Martin's Axiom plus $2^{\aleph_0} > \aleph_1$, there exist closed subsets $F_{n,m,k}$ of S^n such that $A_{n,m} = \bigcup_{k=1}^{\infty} F_{n,m,k}$. Let $V_{n,m,k} = \bigcup \{U(x, 1/2m):\ x \in F_{n,m,k}\}$. Clearly $U = \bigcup_{k=1}^{\infty} \bigcup_{n=2}^{\infty} \bigcup_{m=1}^{\infty} V_{n,m,k}$. By Lemma 1, it suffices to show that $\overline{V}_{n,m,k} \subset U$. Let $z = \{z_1, \ldots, z_r\} \in \overline{V}_{n,m,k}$. Then for every positive integer p, there is a $t^{(p)} \in X$ and an $x^{(p)} \in F_{n,m,k}$, such that $t^{(p)} \in U(z, 1/p) \cap U(x^{(p)}, 1/2m)$. Therefore $t^{(p)} \subset B(z, 1/p) \cap B(x^{(p)}, 1/2m)$ and $t^{(p)} \supset z \cup x^{(p)}$, so

(i) $z \cup x^{(p)} \subset B(z, 1/p) \cap B(x^{(p)}, 1/2m)$.

For every p there is an $i_1(p)$ such that $|x_1^{(p)} - z_{i_1(p)}| < 1/p$. Since z is a finite set, there is an i_1 and an infinite set P_1 of positive integers, such that for each $p \in P_1$, $|x_1^{(p)} - z_{i_1}| < 1/p$. For every $p \in P_1$, there is an $i_2(p)$ such that $|x_2^{(p)} - z_{i_2(p)}| < 1/p$. As before, there is an infinite $P_2 \subset P_1$ and an i_2, such that for each $p \in P_2$, $|x_2^{(p)} - z_{i_2}| < 1/p$. Continuing this process, we find an infinite set P_n of positive integers, and positive integers $i_1, \ldots, i_n$, such that for each $p \in P_n$ and for every j, $1 \leqslant j \leqslant n$,

(ii) $|x_j^{(p)} - z_{i_j}| < 1/p$.

Since $\varrho(x^{(p)}) \geqslant 1/m$, from (ii) and the triangle inequality we get $\min_{j \neq j'} |z_{i_j} - z_{i_{j'}}| \geqslant 1/m$ and then $z_{i_1} < \ldots < z_{i_n}$. Let $z' = \{z_{i_1}, \ldots, z_{i_n}\} \subset z$. Then $z' \in F_{n,m,k}$. For if not, there would exist an $\varepsilon > 0$ such that $\prod_{j=1}^{n} B(\{z_{i_j}\}, \varepsilon) \cap F_{n,m,k} = 0$. Take $p \in P_n$ such that $1/p < \varepsilon$. By (ii), for each j we have $x_j^{(p)} \in B(\{z_{i_j}\}, \varepsilon)$, which is impossible.

Since $z' \in F_{n,m,k}$, $U(z', 1/m) \subset U$. Take any $y \in z$ and $p \in P_n$ such that $1/p < 1/2m$. By (i), $y \in B(x^{(p)}, 1/2m)$, so there is a j such that $|y - x_j^{(p)}| < 1/2m$. By (ii), $|x_j^{(p)} - z_{i_j}| < 1/2m$, so $|y - z_{i_j}| < 1/m$. Since y was an arbitrary member of z, it follows that $z \subset B(z', 1/m)$ and hence $z \in U(z', 1/m) \subset U$, completing the proof.

We now move onto the corollary. Conditions (ii) and (iii) are clearly equivalent. We do not know of any other equivalences.

(i) implies (iv) since completely regular Moore spaces are p-spaces [3] and metacompact Moore spaces have point-countable bases. (iv) implies (v) since Lindelöf p-spaces with point-countable bases are metrizable [18].

The example we constructed earlier proves half of the corollary. The other half follows from the results of [26]. There the second author proves the consistency of the assumption that every normal first countable space is collectionwise normal with respect to discrete collections of $\aleph_1$ points. A sketch of the proof appears in [27]. It follows that a countable chain condition first countable normal space has no uncountable closed discrete subspace. By Jones [14], Moore spaces with the latter property are metrizable. Aquaro [2] showed that a space with no uncountable closed discrete subspace, in which every open cover has a point-countable open refinement, is Lindelöf, which takes care of the final two clauses in the corollary.

Since no open set of our example is separable, it follows, as noted by Pixley and Roy, that any metrizable subspace is nowhere dense, and thus X is not locally metrizable. It also follows that X cannot be completed and indeed, as mentioned earlier is "unbairable". See [1]. Lutzer bears full responsibility for this term. Lest one be tempted to think that if there is a (metacompact) normal non-metrizable Moore space, it must be "bad", we recall that in [28] or [29], the second author constructs, assuming Martin's Axiom plus $2^{\aleph_0} > \aleph_1$, a metacompact, complete, locally metrizable, normal non-metrizable Moore space with a dense metrizable subspace. It is still open whether it is consistent that (metacompact) normal Moore spaces have any "nice" properties. If it could be shown consistent that normal Moore spaces are collectionwise normal with respect to discrete collections of points, the consistency of their having dense metrizable subspaces would follow from [9]. Some progress toward this goal is accomplished in [26], but singular cardinals present as yet unresolved obstacles.

X is a normal Moore space which is metacompact and not screenable [6], and hence is not "strongly star-screenable" [20], answering (at least as a consistency result) a question of Reed [20].

It is ironic that Martin's Axiom plus $2^{\aleph_0} > \aleph_1$ implies the existence of a countable chain condition non-separable Moore space, since these hypotheses also imply Souslin's Conjecture [23] and indeed the separability of a wide variety of countable chain condition spaces [30]. In the same vein, we note that Souslin's Conjecture fails in the model of [26] ($\aleph_2$ subsets of $\aleph_1$ adjoined to L by countable Cohen conditions) in which every countable chain condition normal Moore space is metrizable and hence separable (since Souslin trees exist in L [13] and are not destroyed by countably closed extensions).

It is also surprising that Martin's Axiom plus $2^{\aleph_0} > \aleph_1$ implies any Hausdorff compactification cX of our space X is separable. This follows from [10], since cX satisfies the countable chain condition and has a dense first countable subspace of power less than continuum, namely X.

Observe that for our construction it would suffice to have one uncountable Q-set if the property of having all subsets F_σ were preserved by finite products. We do not know if it is. If so, we are led to conjecture that the existence of a separable normal non-metrizable Moore space is equivalent to the existence of a non-separable normal Moore space satisfying the countable chain condition, in view of the equivalence of the former hypothesis with the existence of an uncountable Q-set [6], [12]. B. Šapirovskiĭ announced in [24]:

THEOREM. *$2^{\aleph_0} < 2^{\aleph_1}$ implies every first countable normal countable chain condition space is collectionwise normal.*

Thus $2^{\aleph_0} < 2^{\aleph_1}$ decides negatively the propositions in the corollary.

Added in proof. Recently W. W. Fleissner proved the consistency of the assumption that every normal first countable space is collectionwise normal with respect to discrete collections of points. It follows from (ii) of our corollary and from [9], that the existence of dense metrizable subsets in normal Moore spaces is independent of the axioms of set theory.

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