Prime sequences and distributivity in local Noether lattices *

by

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Abstract. We investigate the influence of a prime sequence on the multiplicative sublattice of a local Noether it generates. This sublattice is isomorphic to RL_k. We also investigate some conditions sufficient for a local Noether lattice to be distributive.

1. Introduction. If L is a distributive regular local Noether lattice, K. Bogart [3] showed that L is isomorphic to RL_k, where k is the dimension of L and RL_k is the multiplicative sublattice of the ideal lattice of $F[x_1, \ldots, x_k]$ generated by the principal ideals (x_i), F a field. In this paper we generalize Bogart's result and investigate distributivity in local Noether lattices in general. One distinguishing characteristic of RL_k is that it is generated by a prime sequence (Definition 2.2) of length k. In Theorem 2.10 we show that, in a local Noether lattice, the sub-multiplicative-lattice generated by any prime sequence of length k is isomorphic to RL_k. Theorem 3.1 shows that if (L, M) is a local Noether lattice and each M-primary element distributes, then L is distributive. Theorem 3.2 shows that (L, M) is distributive provided that each element in a set of parameters for L distributes. In Theorem 3.3 we show that, in the regular case, L is distributive if some powers of each three-element subset of L form a distributive triple in L.

2. Prime sequences and RL_k. If R is a commutative Noetherian ring with identity and A and B are ideals of R, then if r is an element of $A+B$, $r = a+b$, for some a in A and b in B. Moreover,

$$(r)+(a) = (r)+(b) = (a)+(b)$$

where ( ) denotes ideal generation. The following theorem gives an appropriate analog for this property in local Noether lattices and is a useful computational tool in these lattices.

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For the remainder of this section, (L, M) is a local Noether lattice unless otherwise specified.

THEOREM 2.1. *Let A be a principal element and B and C be elements of (L, M) such that $A \leqslant B \vee C$. Then there exist principal elements $B' \leqslant B$ and $C' \leqslant C$ in L such that*

$$A \vee B' = A \vee C' = B' \vee C'.$$

Proof. Let $B_1, \ldots, B_n$ be principal elements with join $B \wedge (A \vee C)$. Since $A \leqslant B \vee C$, it follows by modularity that $A \vee C = B_1 \vee \ldots \vee B_n \vee C$. And since $A \vee C$ is principal in L/C, it follows that $A \vee C = B_i \vee C$, for some $i = 1, \ldots, n$. Hence, there exists a principal element $B' \leqslant B$ such that $A \leqslant B' \vee C$ and, consequently, a principal element $C' \leqslant C$ such that $A \leqslant B' \vee C'$.

Now, suppose $A \leqslant B \vee C$ as in the hypothesis of the theorem. Our result holds if $A \leqslant B$ or $A \leqslant C$. So, assume $A \nleqslant B$ and $A \nleqslant C$. Since $A \nleqslant \bigwedge_n (B \vee M^n C) = B$, we choose j so that $A \leqslant B \vee M^j C$ but $A \nleqslant B \vee M^{j+1} C$. Then there exists a principal element $C' \leqslant M^j C$ such that $A \leqslant B \vee C'$. Similarly, since $A \nleqslant C' = \bigwedge (M^n B \vee C')$, there is a principal element $B' \leqslant M^k B$ such that $A \leqslant B' \vee C'$, where $A \leqslant M^k B \vee C'$ but $A \nleqslant M^{k+1} B \vee C'$. Moreover,

$$A \vee B' = B' \vee \big((A \vee B') \wedge C'\big) = B' \vee [(A \vee B') : C'] C'.$$

If $(A \vee B') : C' \leqslant M$, then $A \leqslant B' \vee MC \leqslant B \vee M^{j+1} C$. Hence, $(A \vee B') : C' = I$ and $C' \leqslant A \vee B'$. Therefore, $A \vee B' = B' \vee C'$. Similarly, $A \vee C' = B' \vee C'$. Q.E.D.

DEFINITION 2.2. An (ordered) set of principal elements, $A_1, \ldots, A_n$ in a Noether lattice forms a *prime sequence* if $A_i \neq I$ for $i = 1, \ldots, n$ and if $(A_1 \vee \ldots \vee A_{i-1}) : A_i = A_1 \vee \ldots \vee A_{i-1}$ for each $i = 1, \ldots, n$. (We set $A_0 = 0$.)

Our first objective is to remove the parenthesized "ordered" in the definition in the semi-local case.

THEOREM 2.3. *Let L' be a semi-local Noether lattice with Jacobson radical, $\mathfrak{M}$. If $A_1, \ldots, A_n$ is a prime sequence in L' such that $A_i \leqslant \mathfrak{M}$ for $i = 1, \ldots, n$, then any permutation of the A_i's is also a prime sequence.*

Proof. It suffices to establish the case $n = 2$. Hence, assume $0 : A_1 = 0$ and $A_1 : A_2 = A_1$. Since

$$(A_2 : A_1) A_1 = \big((A_2 : A_1) A_1\big) \wedge A_2 = \Big(\big((A_2 : A_1) A_1\big) : A_2\Big) A_2$$
$$\leqslant (A_1 : A_2) A_2 \leqslant A_1 A_2,$$

it follows that $A_2 : A_1 \leqslant A_2$. On the other hand, $0 : A_2 \leqslant A_1 : A_2 = A_1$, so $0 : A_2 = (0 : A_2) \wedge A_1 = \big((0 : A_2) : A_1\big) A_1 = \big((0 : A_1) : A_2\big) A_1 = (0 : A_2) A_1$, so $0 : A_2 = 0$, by the Intersection Theorem. Q.E.D.

In particular, any permutation of a prime sequence in (L, M) is a prime sequence. We use this result in

LEMMA 2.4. *Let $A_1, \ldots, A_n$ be a prime sequence in (L, M), C a principal element in L, and $A = A_1 \vee \ldots \vee A_n$. Then for all $m \geqslant 1$,*

(1) $A^m : A_n = A^{m-1}$, *and*

(2) $A : C = A$ *implies* $A^m : C = A^m$.

Proof. Let $A_1 : C = A_1$ and E be a principal element such that $E \leqslant A_1^m : C \leqslant A_1 : C = A_1$. Choose t so that $E \leqslant A_1^t$ but $E \nleqslant A_1^{t+1}$ and suppose $t < m$. Then there exists a principal element F such that $E = FA_1^t$ and $FA_1^t C \leqslant A_1^m$. Hence, $FC \leqslant A_1^{m-1}$, since $0 : A_1 = 0$. So $F \leqslant A_1$ by induction on m, and $E = FA_1^t \leqslant A^{t+1}$, which contradicts our choice of t. Therefore, $E \leqslant A_1^m$ and $A_1^m : C = A_1^m$, for all $m \geqslant 1$. Hence, (1) and (2) hold for all m, if $n = 1$.

Now, assume (1) and (2) hold for all m, if $n \leqslant s$.

Let $A_1, \ldots, A_{s+1}$ be a prime sequence. Set $B = A_1 \vee \ldots \vee A_s$ and $A = B \vee A_{s+1}$. Assume $m \geqslant 2$.

If $CA_{s+1} \leqslant A^m = B^m \vee A^{m-1} A_{s+1}$, where C is a principal element, then (Theorem 2.1)

$$CA_{s+1} \vee D = CA_{s+1} \vee EA_{s+1} = D \vee EA_{s+1},$$

for some principal elements $D \leqslant B^m$ and $E \leqslant A^{m-1}$. Hence, $D \leqslant (C \vee E) A_{s+1}$, and $D = FA_{s+1}$, for some principal $F \leqslant C \vee E$. Consequently, $FA_{s+1} = D \leqslant B^m$, and $F \leqslant (B^m : A_{s+1}) = B^m$, by the inductive hypothesis. Therefore $CA_{s+1} \leqslant (E \vee F) A_{s+1}$, and $C \leqslant F \vee E \leqslant B^m \vee A^{m-1} = A^{m-1}$. Hence $A^m : A_{s+1} = A^{m-1}$, for all m.

Now, let C and D be principal elements such that $A : C = A$ and $CD \leqslant A^m$. Then $CD/A_{s+1} \leqslant B^m/A_{s+1}$ in L/A_{s+1}, so $D/A_{s+1} \leqslant (B/A_{s+1})^m$, since B/A_{s+1} is the join of a prime sequence of length s. Hence $D \leqslant B^m \vee A_{s+1}$. Then $D \vee E = D \vee FA_{s+1} = E \vee FA_{s+1}$, for some principal elements $E \leqslant B^m$ and F. Therefore $CFA_{s+1} \leqslant CD \vee CE \leqslant A^m$, so that $CF \leqslant A^{m-1}$ by the above. By induction on m, it now follows that $F \leqslant A^{m-1}$, and hence that $D \leqslant E \vee FA_{s+1} \leqslant B^m \vee A^{m-1} A_{s+1} \leqslant A^m$. Therefore $A : C = A$ implies $A^m : C = A^m$, for all m. Q.E.D.

We define a Macaulay local lattice to be a local Noether lattice which has a prime sequence of length equal to its altitude. We note that if the lattice satisfies the union condition on prime elements [8], the length of a maximal prime sequence is an invariant for the lattice. Using lattice theoretic interpretations for the discussion in [11, II, p. 397] we remark:

THEOREM 2.5. *Let (L, M) be a Macaulay local lattice of altitude d satisfying the union condition on prime elements, and let $A_1, \ldots, A_s$ be principal elements in L such that the altitude of $L/(A_1 \vee \ldots \vee A_s)$ is $d-s$. Then $A_1, \ldots, A_s$ is a prime sequence in L and every prime divisor of $A_1 \vee \ldots \vee A_s$ has height s and depth $d-s$.*

Now, let $A_1, \ldots, A_n$ be a prime sequence in (L, M). Let $RL(A_1, \ldots, A_n) = RL(A_i)$ be the multiplicative sublattice of L generated by the collection of finite joins of products of the A_i's. Our objective is to show that $RL(A_i)$ is a distributive sublattice of L isomorphic to RL_k [see 3].

By Lemma 2.4, since $(A_i : A_j) = A_i$ for $i \neq j$, $(A_i^t : A_j) = A_i^t$ for all positive integers t, whenever $i \neq j$. More generally,

LEMMA 2.7. *Let J be a join of products of $A_2, \ldots, A_n$. Then $(J : A_1) = J$.*

Proof. Assume that J is the join of products of $A_2, \ldots, A_n$. By renumbering, if necessary, we may assume that A_n actually appears in one of the products.

Write $J = K \vee A_n B$, when K is the join of products of $A_2, \ldots, A_{n-1}$, and B is the join of products of $A_2, \ldots, A_n$. We induct on the sum of the degrees of the products which form J.

Assume $XA_1 \leqslant J$. Then in L/A_n, $(X/A_n)(A_1/A_n) \leqslant K/A_n$, where K is the join of products of $A_2/A_n, \ldots, A_{n-1}/A_n$. Since the sum of the degrees of the products which form K/A_n is smaller than the sum of the degrees of the products which form J, we have that $X/A_n \leqslant K/A_n$, and hence that $X \leqslant K \vee A_n$ in L. It follows that

$$X \vee K = K \vee \big((X \vee K) \wedge A_n\big) = K \vee \big((X \vee K) : A_n\big) A_n ,$$

and hence that

$$\big((X \vee K) : A_n\big) A_n A_1 \leqslant XA_1 \vee KA_1 \leqslant K \vee A_n B .$$

Therefore, by the inductive hypothesis,

$$\big((X \vee K) : A_n\big) A_1 \leqslant (K \vee A_n B) : A_n \leqslant (K : A_n) \vee B \leqslant K \vee B ,$$

and

$$(X \vee K) : A_n \leqslant K \vee B .$$

Hence,

$$X \leqslant K \vee \big((X \vee K) : A_n\big) A_n = K \vee A_n B = J . \qquad \text{Q.E.D.}$$

If P_1, P_2 are products of the A_i, let $GCD(P_1, P_2)$ be the product, Q, of the A_i, of greatest degree such that $P_1 = QP_1'$ and $P_2 = QP_2'$. If no such product exists, we set $GCD(P_1, P_2) = I$. Since for each non-zero product Q, $0 : Q = 0$,

$$(P_1 : P_2) = (QP_1' : QP_2') = (P_1' : P_2') .$$

COROLLARY 2.8. *If $\bigvee \{J_j \mid j = 1, \ldots, t\}$ is a finite join of elements in $RL(A_i)$ and P is a product of the A_i, then $\big((\bigvee_j J_j) : P\big) = \bigvee_j (J_j : P)$.*

Proof. We induct on t and the sum of the degrees of the J_j. Corollary 2.8 holds if some $J_j = I$, or by induction if some $J_j = 0$. If $GCD(J_j, P) = I$ for each j, then by Lemma 2.7, our conclusion holds. So assume $GCD(J_t, P) = Q < I$. Then

$$\begin{aligned}
\big((\bigvee J_j) : P\big) &= \big((\bigvee_{j=1}^{t-1} J_j) \vee QJ_t' : QP'\big) \\
&= \big((\bigvee_{j=1}^{t-1} J_j) \vee QJ_t' : Q\big) : P' \\
&= \Big(\big((\bigvee_{j=1}^{t-1} J_j) : Q\big) \vee J_t'\Big) : P' \\
&= \Big(\big(\bigvee_{j=1}^{t-1} (J_j : Q)\big) \vee J_t'\Big) : P' \\
&= \Big(\big(\bigvee_{j=1}^{t-1} (J_j : Q)\big) : P'\Big) \vee (J_t' : P') \\
&= \big(\bigvee_{j=1}^{t-1} (J_j : P)\big) \vee (J_t' : P') \\
&= \bigvee_{j=1}^{t} (J_j : P) ,
\end{aligned}$$

by induction on t and induction on the sum of the degrees of the J_j. Q.E.D.

COROLLARY 2.9. *$RL(A_i)$ is a distributive sub-Noether lattice of L.*

Proof. Suppose $J = \bigvee J_j \in RL(A_i)$, where the J_j are products of 0, I, and the A_i, and let P be such a product. Then in L,

$$(\bigvee J_j) \wedge P = (\bigvee J_j : P) P = \big(\bigvee (J_j : P)\big) P = \bigvee \big((J_j : P) P\big) = \bigvee (J_j \wedge P) .$$

Since L is modular, it follows that joins of products of the A_i distribute over joins of joins of products of the A_i. Hence, by Corollary 2.8, the collection of joins of products of the A_i, together with 0 and I, is closed under the residuation and meet operations of L and forms a distributive multiplicative-sublattice of L. It is now clear that $RL(A_i)$ is a distributive sub-Noether lattice of L. Q.E.D.

$RL(A_i)$ is clearly a local Noether lattice with maximal element, $A_1 \vee \ldots \vee A_n$. Consequently, from [3, Thm. 5], we have

THEOREM 2.10. *$RL(A_i)$ is a distributive regular local Noether lattice of altitude n, and hence isomorphic to RL_n.*

Proof. Since $A_1, \ldots, A_n$ is a prime sequence in $RL(A_i)$ as well as in L, $RL(A_i)$ is a distributive regular local Noether lattice. Q.E.D.

3. Conditions for distributivity. It follows from the Artin-Rees Lemma for Noether lattices [7], that if A, B, and C are elements of a Noether lattice, then there is a positive integer, k, such that

$$A\wedge(B\vee C^k)\leqslant(A\wedge B)\vee AC^{n-k}\leqslant(A\wedge B)\vee C^{n-k}\quad \text{for all } n\geqslant k.$$

If A, B, and C form a distributive triple as in [1], we write $(A, B, C)D$. If $(A, B, C)D$ for all B and C in a local Noether lattice L, we say that A distributes over L.

THEOREM 3.1. *If $(Q_1, Q_2, Q_3)D$ for all Q_i which are M-primary elements in (L, M), local, then L is distributive.*

Proof. Let A, B, and C be elements in L and choose k so that

$$(A\vee M^n)\wedge(B\vee M^n)\leqslant\big((A\vee M^n)\wedge B\big)\vee M^{n-k}\leqslant(A\wedge B)\vee M^{n-k}$$

and

$$(A\vee M^n)\wedge(C\vee M^n)\leqslant\big((A\vee M^n)\wedge C\big)\vee M^{n-k}\leqslant(A\wedge C)\vee M^{n-k}$$

for all $n\geqslant k$. Then

$$\begin{aligned}A\wedge(B\vee C)&\leqslant(A\vee M^n)\wedge\big((B\vee M^n)\vee(C\vee M^n)\big)\\&=\big((A\vee M^n)\wedge(B\vee M^n)\big)\vee\big((A\vee M^n)\wedge(C\vee M^n)\big)\\&\leqslant\big((A\wedge B)\vee(A\wedge C)\big)\vee M^{n-k}\end{aligned}$$

for all $n\geqslant k$, since elements joined with M^n are primary for M. Hence,

$$A\wedge(B\vee C)\leqslant\bigwedge_{n\geqslant k}\Big(\big((A\wedge B)\vee(A\wedge C)\big)\vee M^{n-k}\Big)=(A\wedge B)\vee(A\wedge C)$$

by [4, Cor. 3.2, p. 487]. Hence, L is distributive. Q.E.D.

THEOREM 3.2. *Assume (L, M) is a local Noether lattice. If M is the join of principal elements $M_1, \ldots, M_k$ which distribute over L, then L is distributive.*

Proof. If (E, C, D) is a distributive triple in which E is principal, then

$$\big((C\vee D)\colon E\big)E=(C\colon E)E\vee(D\colon E)E\,,\qquad\text{so}\qquad(C\vee D)\colon E=(C\colon E)\vee(D\colon E)\,.$$

Hence, if E and F are principal elements which distribute over L, then EF distributes over L. Also, since L is modular, the join of elements which distribute over L distributes over L.

Since $M_1, \ldots, M_k$ are principal elements which distribute over L, it follows from the above that joins of power products of $M_1, \ldots, M_k$ distribute over L. However, as in [2, prf. of Thm. 5.1], every principal element is a power product of $M_1, \ldots, M_k$, so L is distributive. Q.E.D.

COROLLARY. *Let (L, M) be a local Noether lattice. If M is the join of principal elements $M_1, \ldots, M_k$ such that, for each i, $0\colon M_i=0$ and some power of M_i distributes over L, then L is distributive.*

Proof. Assume M_i^{t+1} distributes over L, $t\geqslant 1$. Let C and D be arbitrary elements of L. Then

$$M_i^{t+1}\wedge(M_iC\vee M_iD)=\big(M_i^t\wedge(M_iC\vee M_iD)\colon M_i\big)M_i=\big(M_i^t\wedge(C\vee D)\big)M_i\,,$$

and

$$\begin{aligned}(M_i^{t+1}\wedge M_iC)\vee(M_i^{t+1}\wedge M_iD)&=\big(M_i^t\wedge(M_iC\colon M_i)\big)M_i\vee\big(M_i^t\wedge(M_iD\colon M_i)\big)M_i\\&=\big((M_i^t\wedge C)\vee(M_i^t\wedge D)\big)M_i\,.\end{aligned}$$

Hence $M_i^t\wedge(C\vee D)=(M_i^t\wedge C)\vee(M_i^t\wedge D)$, and L is distributive, by Theorem 3.2. Q.E.D.

In the case of a regular local Noether lattice, we obtain the following generalization:

THEOREM 3.3. *Let (L, M) be a regular local Noether lattice and $M_1, \ldots, M_k$ principal elements with join M. Assume that each of the elements M_i, $i=1, \ldots, k$, has the property that, given $B, C\in L$, there exist natural numbers r, s, t such that (M_i^r, B^s, C^t) is a distributive triple. Then L is distributive.*

Proof. Reduce $M_1, \ldots, M_k$ to a minimal base $M_1, \ldots, M_v$ for M, so that $M_1, \ldots, M_v$ form a regular system of parameters. Let $E\leqslant M$ be any principal element of L.

Let q be least such that E is $\leqslant$ the join of q of the elements $M_1, \ldots, M_v$. We assume that $E\leqslant M_1\vee\ldots\vee M_q$, and that $q>1$. Choose r, s, t so that $\big(M_1^r, E^s, (M_2\vee\ldots\vee M_q)^t\big)$ is a distributive triple. Then

$$\big(M_1^r\vee(M_2\vee\ldots\vee M_q)^t\big)\colon E^s=(M_1^r\colon E^s)\vee\big((M_2\vee\ldots\vee M_q)^t\colon E^s\big).$$

However, by Lemma 2.7, E^s is prime to M_1^r and to $(M_2\vee\ldots\vee M_q)^t$, whereas $E^s\leqslant M_1\vee\ldots\vee M_q$, which is a prime of $M_1^r\vee(M_2\vee\ldots\vee M_q)^t$. Hence $q=1$ and $E\leqslant M_1$. As in the previous theorem it now follows that every principal element of L is a power product of $M_1, \ldots, M_v$, so that $L=RL(M_1, \ldots, M_v)$. Hence, L is distributive, by Theorem 2.10. Q.E.D.

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Remarks on the absolute suspension

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Abstract. There is proved that an n-dimensional compact metric space is n-dimensional sphere whenever each pair of distinct points is a pair of tops of some suspension representation and $n = 1, 2, 3$. This is a positive answer, for $n \leqslant 3$, on de Groot's conjecture.

A *suspension over* Y is a space SY formed from $Y \times [-1, 1]$ by identifying $Y \times \{1\}$ and $Y \times \{-1\}$ to single points, called the *tops* of the suspension (the resulting set being equipped with the quotient topology).

A metrizable compact space will be said to be an *absolute suspension* if for each pair p, q of its distinct points it is a topologically suspension with tops p and q.

If X is the suspension over Y, then for $F \subset Y$, we can assume that F and SF are the subspaces of X.

Professor de Groot at the Prague Symposium 1971 asked whether an absolute suspension is homeomorphic to an n-sphere, whenever it is n-dimensional. We shall show that this conjecture is true in dimensions 1, 2 and 3.

Throughout the paper all the spaces will be assumed to be metric with the finite dimension in the sense of dim.

As was shown by de Groot in [4], Theorem 2, it suffices to show that the absolute suspension is a manifold in order to get the solution even for an arbitrary finite dimension. Thus showing that the absolute suspension in the dimensions 1, 2 and 3 is a manifold, is the most important step in the proof.

LEMMA 1 (Hurewicz; see Kuratowski [2], p. 311). *If* Y *is compact and* $\dim Z = 1$, *then* $\dim(Y \times Z) = \dim Y + 1$.

LEMMA 2. *If* X *is compact and* $X = SY$, *then* Y *is compact.*

Proof. Since $Y \times [-\frac{1}{2}, \frac{1}{2}]$ is a closed subset of compact space X, it is compact. Hence Y is compact.