A theory of proper shape for locally compact metric spaces

by

B. J. Ball and **R. B. Sher** (Athens, Georgia)

Abstract. The notion of the topological shape of a compactum was introduced by K. Borsuk in 1968; it may be considered a generalization of homotopy type in the sense that (1) any two compact metric spaces of the same homotopy type have the same shape and (2) any two compact ANR's which have the same shape are of the same homotopy type. Of the several extensions of Borsuk's shape theory which have been suggested, all retain the applicable versions of (1) and (2) and hence generalize the notion of homotopy type. For noncompact spaces, however, a more geometric approach might be to generalize *proper* homotopy type instead, and one way of doing this, for locally compact metric spaces, is given here. This notion, called "proper shape", agrees with Borsuk's definition of shape in the case of compacta and satisfies (1) and (2) with "compact" replaced by "locally compact" and "homotopy type" by "proper homotopy type".

1. Introduction. The notion of topological shape was first introduced by K. Borsuk ([1], [4]) for compact metric spaces. The concept has been extended to arbitrary metrizable spaces by Borsuk ([5], [6]) and by R. H. Fox [12], to compact Hausdorff spaces by S. Mardešić and J. Segal ([19], [20]), to arbitrary topological spaces by Mardešić [17], and to Hausdorff spaces by L. R. Rubin and T. J. Sanders [21]. While these extensions do not all agree on the overlap of their domains ([15], [22]), all coincide with Borsuk's original notion in the case of compacta and all share with the original theory the property of generalizing homotopy types, in the sense that any two spaces (in the class considered) which are of the same homotopy type have the same shape.

In the case of compacta, spaces of the same homotopy type have a certain geometric similarity, and compacta of the same shape have a corresponding (global) geometric similarity. For non-compact spaces, however, homotopy type does not seem to distinguish adequately between spaces with essentially different geometric properties. The homotopy type of a point, for example, includes such geometrically diverse spaces as a closed interval, a line, a ray, the plane, all Euclidean spaces and half-spaces, the Hilbert cube and Hilbert space, etc., and thus all these spaces have the same shape in each of the extensions mentioned above (except, of course, that of Mardešić–Segal, which does not apply to non-compact spaces).

The notion of *proper* homotopy type seems to give a more appropriate geometric classification for non-compact spaces which are locally nice (e.g., locally compact ANR's) than does the notion of homotopy type, and this suggests that in extending Borsuk's shape theory for compacta to apply to non-compact spaces, it might be more appropriate to retain the property of generalizing *proper* homotopy types rather than that of generalizing homotopy types. One way of doing this is given here, for locally compact metrizable spaces; on the class of all such spaces, we define an equivalence relation, "having the same proper shape", which satisfies the following conditions:

(1) two compact metrizable spaces have the same proper shape if and only if they have the same shape in Borsuk's original sense,

(2) any two locally compact metrizable spaces of the same proper homotopy type have the same proper shape, and

(3) two locally compact ANR's have the same proper shape if and only if they are of the same proper homotopy type.

It follows easily from our definition that (within the class of locally compact metrizable spaces) a compact space cannot have the same proper shape as a non-compact space, nor can a separable space have the same proper shape as a non-separable one. We show also that if X and Y are separable, locally compact metrizable spaces which have the same proper shape, then their one-point compactifications have the same shape (in Borsuk's original sense), and so do their Freudenthal (endpoint) compactifications FX and FY in case these are metrizable. In particular, the number of "ends" of a connected manifold or other suitable space is a proper shape invariant. A (one-direction) analog of Chapman's characterization [8] of shapes of compacta is also given, as are several results on sums and partitions of spaces having the same proper shape.

2. Definitions and conventions.

Although many of the concepts discussed below are applicable to more general spaces, for convenience we assume at the outset that *all spaces considered are metrizable.*

A map $f\colon P\to Q$ is said to be *proper* if $f^{-1}(C)$ is compact for every compact subset C of Q. It is clear that the composition of two proper maps is proper. It will be useful to note the following additional elementary facts.

(1) If $f\colon P\to Q$ is a proper map and P' is a *closed* subset of P, then $f|P'\colon P'\to Q$ is a proper map.

(2) If $f\colon P\to Q$ is a proper map, Q is a *closed* subspace of Q' and $j\colon Q\to Q'$ is the inclusion map, then $jf\colon P\to Q'$ is proper.

(3) If $f\colon P\to Q$ is a proper map, Q' is a *closed* subset of Q containing $f(P)$, and $f'\colon P\to Q'$ agrees with f, then f' is a proper map. Note also that a proper map $f\colon P\to Q$ is necessarily a *closed map*; i.e., if K is a closed subset of P, then $f(K)$ is closed in Q.

Two maps $f, g\colon P\to Q$ are said to be *properly homotopic*, denoted by $f\underset{p}{\simeq} g$, if there exists a proper map $\varphi\colon P\times I\to Q$ such that $\varphi(x,0)=f(x)$ and $\varphi(x,1)=g(x)$ for each $x\in P$. (It is sometimes convenient to observe that $f\underset{p}{\simeq} g$ if and only if there exists a proper map $\psi\colon P\times I\to Q\times I$ with $\psi(x,0)=\big(f(x),0\big)$ and $\psi(x,1)=\big(f(x),1\big)$ for each $x\in P$. The map ψ can be required to be *level-preserving*; i.e., for each $t\in I$, $\psi(x,t)\in Q\times\{t\}$.) It is a well known and easily verifiable fact that $\underset{p}{\simeq}$ is a compositive equivalence relation. Two spaces X, Y belong to the same *proper homotopy type* if there exist proper maps $f\colon X\to Y$, $g\colon Y\to X$ such that $gf\underset{p}{\simeq} i_X$ and $fg\underset{p}{\simeq} i_Y$, where $i_X\colon X\to X$ and $i_Y\colon Y\to Y$ are the respective identity maps; if only the relation $fg\underset{p}{\simeq} i_Y$ is assumed, then X is said to *properly homotopically dominate* Y.

If Q_0, Q_1 and Z are subsets of a space Q, then maps $f_0\colon P\to Q_0$, $f_1\colon P\to Q_1$ are said to be *properly homotopic in* Z if there is a proper map $\varphi\colon P\times I\to Z$ with $\varphi(x,0)=f(x)$ and $\varphi(x,1)=g(x)$ for each $x\in P$; this relation will be denoted by "$f_0\underset{p}{\simeq} f_1$ in Z".

If $Y\subset Q$, then maps $f, g\colon P\to Y$ are said to be *weakly properly homotopic in* Q if for every neighborhood V of Y in Q, $f\underset{p}{\simeq} g$ in V.

By a *directed set* we understand a non-empty set Λ together with a transitive binary relation $\geqslant$ on Λ such that for each λ_1,λ_2 in Λ, there is a $\lambda\in\Lambda$ with $\lambda\geqslant\lambda_1$, $\lambda\geqslant\lambda_2$. It is evident that the cartesian product $\Lambda\times\Delta$ of two directed sets is itself a directed set with respect to the relation $(\lambda,\delta)\geqslant(\lambda_0,\delta_0)\Leftrightarrow\lambda\geqslant\lambda_0,\ \delta\geqslant\delta_0$. This directing relation will always be intended whenever we refer to the product of two directed sets.

A *net of maps of P into Q* is a family $\underline{f}=\{f_\lambda|\ \lambda\in\Lambda\}$ of maps $f_\lambda\colon P\to Q$, indexed by a directed set Λ.

3. Separable local compacta.

Throughout this section, we let $K=H-\{\omega\}$, where H is the Hilbert cube and ω is a point of H. Since H is homogeneous [16], the choice of ω is immaterial.

3.1. Lemma. *Every separable locally compact metrizable space X can be embedded as a closed subset of K.*

Proof. If X is compact, it can be embedded as a proper subset X' of H; if $\omega'\in H-X'$, then $H-\{\omega'\}$ is homeomorphic to K and contains X' as a closed subset.

If X is not compact, its one-point compactification $X \cup \{p\}$ can be embedded as a subset $X' \cup \{p'\}$ of H, and then $H - \{p'\}$ is homeomorphic to K and contains X' as a closed subset.

In view of Lemma 3.1 and the fact (to be established later) that proper shapes are topologically invariant, it is sufficient here to consider only the closed subsets of K. Our development in this case will closely parallel that given by Borsuk in [1] for compacta. The only essential differences in the definitions are the replacement of the sequence $\{f_k|\ k = 1, 2, \ldots\}$ of maps $f_k\colon H \to H$ by a net $\{f_\lambda|\ \lambda \in \Lambda\}$ of maps $f_\lambda\colon K \to K$, and the addition of the requirement (automatically satisfied in the compact case) that all homotopies involved be proper homotopies on closed neighborhoods. Many of the basic results will have proofs nearly identical with those which apply in the case of compacta. (Indeed, if one omits the requirement that the homotopies in the definitions be proper, a notion of shape is obtained which generalizes homotopy type, can be readily adapted to apply to arbitrary metrizable spaces, and requires virtually no changes in the proofs of all basic properties; this notion of shape differs from that given by Borsuk in [5], but may well coincide with the notion developed by Fox [12].)

Suppose X and Y are closed subsets of K. If $\underline{f} = \{f_\lambda|\ \lambda \in \Lambda\}$ is a net of maps of K into K, then the ordered triple $(\underline{f}, X, Y)$ will be called a *proper fundamental net from X to Y (in K)* provided that for every closed neighborhood V of Y, there exist a closed neighborhood U of X and an index $\lambda_0 \in \Lambda$ such that for all indices $\lambda \geqslant \lambda_0$,

$$f_\lambda|U \underset{p}{\simeq} f_{\lambda_0}|U \quad \text{in } V.$$

Two proper fundamental nets $(\underline{f}, X, Y)$ and $(\underline{g}, X, Y)$, where $\underline{f} = \{f_\lambda|\ \lambda \in \Lambda\}$ and $\underline{g} = \{g_\delta|\ \delta \in \Delta\}$, are said to be *properly homotopic*, denoted by $(\underline{f}, X, Y) \underset{p}{\simeq} (\underline{g}, X, Y)$, if for every closed neighborhood V of Y, there exist a closed neighborhood U of X and indices $\lambda_0 \in \Lambda$, $\delta_0 \in \Delta$ such that for all $\lambda \geqslant \lambda_0$, $\delta \geqslant \delta_0$,

$$f_\lambda|U \underset{p}{\simeq} g_\delta|U \quad \text{in } V.$$

If $\underline{f} = \{f_\lambda|\ \lambda \in \Lambda\}$ and $\underline{g} = \{g_\delta|\ \delta \in \Delta\}$ are nets of maps of K into K, then clearly $\underline{gf} = \{g_\delta f_\lambda|\ (\lambda, \delta) \in \Lambda \times \Delta\}$ is a net of maps of K into K. Moreover, it is easy to verify that if X, Y and Z are closed subsets of K such that $(\underline{f}, X, Y)$ and $(\underline{g}, Y, Z)$ are proper fundamental nets from X to Y and from Y to Z, respectively, then $(\underline{gf}, X, Z)$ is a proper fundamental net from X to Z; $(\underline{gf}, X, Z)$ is called the *composition* of the proper fundamental nets $(\underline{f}, X, Y)$ and $(\underline{g}, Y, Z)$.

For directed sets Λ, Δ, Σ, we will not distinguish between the products $(\Lambda \times \Delta) \times \Sigma$ and $\Lambda \times (\Delta \times \Sigma)$, identifying each with $\Lambda \times \Delta \times \Sigma$. Hence composition of proper fundamental nets, when defined, is associative.

It can be shown, by virtually the same argument as in the compact case ([1], p. 232, Th. 6.4) that if $(\underline{f}, X, Y) \underset{p}{\simeq} (\underline{f}', X, Y)$ and $(\underline{g}, Y, Z) \underset{p}{\simeq} (\underline{g}', Y, Z)$, then $(\underline{gf}, X, Z) \underset{p}{\simeq} (\underline{g'f'}, X, Z)$.

If $f_0\colon K \to K$ and Λ_0 is the degenerate directed set $\{0\}$, then $\underline{f} = \{f_\lambda|\ \lambda \in \Lambda_0\} = \{f_0\}$ is a *degenerate net*, consisting of the single function f_0. The degenerate net $\underline{i} = \{i_K\}$ where $i_K\colon K \to K$ is the identity map, is called the *identity net* (*on* K). It is clear that for every closed subset X of K, the triple $(\underline{i}, X, X)$ is a proper fundamental net from X to X.

We will use the phrase "$\underline{f}$ is a proper fundamental net from X to Y", or the notation $\underline{f}\colon X \to Y$, to indicate that $(\underline{f}, X, Y)$ is a proper fundamental net. When confusion is unlikely, we will use the notation $\underline{f} \underset{p}{\simeq} \underline{g}$ rather than the more cumbersome $(\underline{f}, X, Y) \underset{p}{\simeq} (\underline{g}, X, Y)$. In addition, we use $\underline{i}_X$ to denote the proper fundamental net $(\underline{i}, X, X)$.

As in the case of fundamental sequences, the set of all proper fundamental nets from X to Y is divided into equivalence classes by the relation of proper homotopy; the equivalence class containing a given proper fundamental net $\underline{f}\colon X \to Y$ will be denoted by $[\underline{f}]$ and will be called a *proper fundamental class*. If we define the composition $[\underline{g}][\underline{f}]$ of two proper fundamental classes $[\underline{f}]$ and $[\underline{g}]$, where $\underline{f}\colon X \to Y$ and $\underline{g}\colon Y \to Z$, to be the proper fundamental class $[\underline{gf}]$, then — almost exactly as in the case of compacta — it is easy to verify that a category is obtained whose objects are the closed subsets of K and whose morphisms are the proper fundamental classes. This category, or variants of it, will be the *proper shape category*. In particular, two closed subsets X, Y of K will be said to be *properly fundamentally equivalent* if X and Y are equivalent objects in this category; i.e., if there exist proper fundamental nets $\underline{f}\colon X \to Y$ and $\underline{g}\colon Y \to X$ such that $\underline{gf} \underset{p}{\simeq} \underline{i}_X$ and $\underline{fg} \underset{p}{\simeq} \underline{i}_Y$; if only the condition $\underline{fg} \underset{p}{\simeq} \underline{i}_Y$ is assumed, then X is said to *properly fundamentally dominate* Y. These relations will be denoted by $X \underset{pF}{\simeq} Y$ and $X \underset{pF}{\geqslant} Y$, respectively. It is easy to see that the relation $\underset{pF}{\geqslant}$ is transitive and that $\underset{pF}{\simeq}$ is an equivalence relation on the class of closed subsets of K (cf. [2], p. 25, Th. 5.2). We will show that proper fundamental equivalence of closed subsets of K is topologically invariant, and will then define proper shapes for separable, locally compact metrizable spaces by saying that two such spaces have the same proper shape if and only if they have embeddings as closed subsets of K which are properly fundamentally equivalent. First we establish some preliminary results, most of which have analogs in the compact case.

A proper fundamental net $\underline{f} = \{f_\lambda |\ \lambda \in \Lambda\}$ is said to be *generated by* a map $f\colon X \to Y$ if for each $\lambda \in \Lambda$, $f_\lambda(x) = f(x)$ for all $x \in X$. Clearly, if $f\colon X \to Y$ generates a proper fundamental net $\underline{f}\colon X \to Y$, then f is necessarily a proper map; conversely, as shown below, every proper map $f\colon X \to Y$ generates a proper fundamental net $\underline{f}\colon X \to Y$.

3.2. Lemma. *Suppose P and Q are locally compact metrizable spaces, X is a closed subset of P and $f\colon X \to Q$ is a proper map. If $\hat{f}\colon P \to Q$ is an extension of f, there is a closed neighborhood U of X in P such that $\hat{f}|U\colon U \to Q$ is a proper map.*

Proof. Since Q is locally compact and paracompact, there is a locally finite cover $\mathcal{V} = \{V_a |\ a \in A\}$ of Q by open sets with compact closures, and since Q is normal, there is an open cover $\mathcal{W} = \{W_a |\ a \in A\}$ of Q with $\overline{W}_a \subset V_a$ for each a.

Let $Y = f(X)$. Since $f\colon X \to Q$ is a proper map, Y is closed in Q. For each $a \in A$, let $Y_a = Y \cap \overline{W}_a$ and let $X_a = f^{-1}(Y_a)$; since Y_a is compact and f is proper, X_a is compact. Hence, since $\hat{f}(X_a) = Y_a \subset \overline{W}_a \subset V_a$, there is a compact neighborhood U_a of X_a in P such that $\hat{f}(U_a) \subset V_a$. Since $\{V_a |\ a \in A\}$ is locally finite in Q, $\{U_a |\ a \in A\}$ is locally finite in P and it follows that $U = \bigcup_{a \in A} U_a$ is a closed neighborhood of X in P.

If C is a compact subset of Q, then $C \cap V_a = \emptyset$ for all but a finite number of indices a in A, and hence $\hat{f}^{-1}(C)$ intersects at most a finite number of the U_a's. Thus $(\hat{f}|U)^{-1}(C)$ is a closed subset of the union of a finite number of compact sets and hence is compact. Therefore $\hat{f}|U\colon U \to Q$ is a proper map.

3.3. Corollary. *If X and Y are closed subsets of K, then every proper map $f\colon X \to Y$ generates a proper fundamental net $\underline{f}\colon X \to Y$.*

Proof. Let $j\colon Y \to K$ be the inclusion map. Since K is an AR, $jf\colon X \to K$ can be extended to a map $\hat{f}\colon K \to K$. Since jf is a proper map, it follows from Lemma 3.2 that there is a closed neighborhood U of X in K such that $\hat{f}|U\colon U \to K$ is a proper map. The degenerate net $\{\hat{f}\}$ is then a proper fundamental net from X to Y generated by f.

The proof of the next lemma is modeled on the argument given for Lemma 4.2 of [1]; some care is needed to insure that the homotopy of the conclusion is proper.

3.4. Lemma. *Let P be a locally compact metrizable space and Q a locally compact ANR. Suppose X is a closed subset of P and $f, g\colon X \to Q$ are maps which are properly homotopic in a closed subset Z of Q. If $\hat{f}, \hat{g}\colon P \to Q$ are extensions of f and g, respectively, and V is a closed neighborhood of Z in Q, then there is a closed neighborhood U of X in P such that $\hat{f}|U \underset{p}{\simeq} \hat{g}|U$ in V.*

Proof. Let V_1 be a closed neighborhood of Z in Q such that $V_1 \subset \operatorname{Int} V$. By Lemma 3.2, there is a closed neighborhood U_1 of X in P such that $\hat{f}|U_1$ and $\hat{g}|U_1$ are proper maps; since V_1 is a neighborhood of $\hat{f}(X) \cup \hat{g}(X)$ in Q, U_1 may be chosen so that $\hat{f}(U_1) \cup \hat{g}(U_1) \subset V_1$.

Let $\varphi\colon X \times I \to Z$ be a proper map such that

$$\varphi(x, 0) = f(x) \quad \text{and} \quad \varphi(x, 1) = g(x) \quad \text{for every } x \in X.$$

Let $T = (U_1 \times \{0\}) \cup (X \times I) \cup (U_1 \times \{1\})$, and define a map $\psi\colon T \to V_1$ by setting

$$\begin{aligned} \psi(x, t) &= \varphi(x, t) && \text{for} \quad (x, t) \in X \times I, \\ \psi(x, 0) &= \hat{f}(x) && \text{for} \quad (x, 0) \in U_1 \times \{0\}, \text{ and} \\ \psi(x, 1) &= \hat{g}(x) && \text{for} \quad (x, 1) \in U_1 \times \{1\}. \end{aligned}$$

For each compact subset C of V_1, $\psi^{-1}(C)$ is the union of the compact sets $(\hat{f}|U_1)^{-1}(C) \times \{0\}$, $\varphi^{-1}(C)$, and $(\hat{g}|U_1)^{-1}(C) \times \{1\}$. Hence $\psi\colon T \to V_1$ is a proper map.

Let $\psi' = j\psi$, where $j\colon V_1 \to \operatorname{Int} V$ is the inclusion map. Since V_1 is closed in $\operatorname{Int} V$, $\psi'\colon T \to \operatorname{Int} V$ is proper. Since T is a closed subset of $U_1 \times I$ and $\operatorname{Int} V$ is an ANR, there exist an open neighborhood G of T in $U_1 \times I$ and an extension of ψ' to a map $\Psi\colon G \to \operatorname{Int} V$. Since G and V are locally compact, T is a closed subset of G and $\Psi\colon G \to \operatorname{Int} V$ is an extension of the proper map $\psi'\colon T \to \operatorname{Int} V$, by Lemma 3.2 there is a closed neighborhood G_1 of T in G such that $\Psi|G_1\colon G_1 \to \operatorname{Int} V$ is proper.

Let V_2 be a closed neighborhood of V_1 in Q such that $V_2 \subset \operatorname{Int} V$, and let U be a closed neighborhood of X in P such that $U \times I \subset G_1$ and $\Psi(U \times I) \subset V_2$. Let $\Phi = \Psi|U \times I$. Since $U \times I$ is closed in G_1, $\Phi\colon U \times I \to \operatorname{Int} V$ is proper. Let $\Phi'\colon U \times I \to V$ agree with Φ; since $\Phi(U \times I) \subset V_2$ and V_2 is a closed subset of V, Φ' is a proper map. Since $U \subset U_1$, it easily follows that $\Phi'(x, 0) = \hat{f}(x)$ and $\Phi'(x, 1) = \hat{g}(x)$ for every $x \in X$. Hence $\hat{f}|U \underset{p}{\simeq} \hat{g}|U$ in V.

3.5. Lemma. *Suppose X and Y are closed subsets of K and $f, g\colon X \to Y$ are weakly properly homotopic maps. If $\underline{f}, \underline{g}\colon X \to Y$ are proper fundamental nets generated by f and g, respectively, then $\underline{f} \underset{p}{\simeq} \underline{g}$.*

Proof. Let $\underline{f} = \{f_\lambda |\ \lambda \in \Lambda\}$ and $\underline{g} = \{g_\delta |\ \delta \in \Delta\}$, and suppose V is a closed neighborhood of Y (in K). Since $\underline{f}$ and $\underline{g}$ are proper fundamental nets from X to Y, there exist a closed neighborhood U_1 of X and indices

$\lambda_0 \in \Lambda$, $\delta_0 \in \Delta$ such that for $\lambda \geqslant \lambda_0$, $f_\lambda|U_1 \underset{p}{\cong} f_{\lambda_0}|U_1$ in V and for $\delta \geqslant \delta_0$, $g_\delta|U_1 \underset{p}{\cong} g_{\delta_0}|U_1$ in V.

Let $j\colon Y \to K$ be the inclusion map, and let $f' = jf$, $g' = jg$. Let V' be a closed neighborhood of Y such that $V' \subset \operatorname{Int} V$. Since f and g are weakly properly homotopic in K, $f' \underset{p}{\cong} g'$ in V'. Since f_{λ_0} is an extension of f', g_{δ_0} is an extension of g' and V is a closed neighborhood of V', it follows from Lemma 3.4 that there is a closed neighborhood U of X such that

$$f_{\lambda_0}|U \underset{p}{\cong} g_{\delta_0}|U \quad \text{in } V.$$

Clearly U may be chosen so that $U \subset U_1$, and then for $\lambda \geqslant \lambda_0$ and $\delta \geqslant \delta_0$,

$$f_\lambda|U \underset{p}{\cong} f_{\lambda_0}|U \underset{p}{\cong} g_{\delta_0}|U \underset{p}{\cong} g_\delta|U, \quad \text{all in } V,$$

and it follows that $\underline{f} \underset{p}{\cong} \underline{g}$.

3.6. LEMMA. *If X and Y are closed subsets of K and $f, g\colon X \to Y$ generate proper fundamental nets $\underline{f}\colon X \to Y$ and $\underline{g}\colon X \to Y$, respectively, such that $\underline{f} \underset{p}{\cong} \underline{g}$, then f and g are weakly properly homotopic in K.*

3.7. LEMMA. *If Y is an ANR embedded as a closed subset of a locally compact metrizable space P, then there exist a closed neighborhood W of Y in P and a proper map $r\colon W \to Y$ such that $r(y) = y$ for each $y \in Y$; i.e., r is a proper retraction of W to Y. Moreover, for every closed neighborhood V of Y in P with $V \subset W$, there exist a closed neighborhood V' of Y in V and a proper map $\varphi\colon V' \times I \to V$ such that $\varphi(y, 0) = y$ and $\varphi(y, 1) = r(y)$ for each $y \in V'$.*

Proof. Since Y is an ANR, there exist a closed neighborhood W' of Y in P and a retraction r of W' to Y. Since W' and Y are locally compact metric spaces, Y is a closed subset of W' and $r\colon W' \to Y$ is an extension of the identity map $i_Y\colon Y \to Y$, it follows from Lemma 3.4 that there is a closed neighborhood W of Y in W', and hence in P, such that $r|W\colon W \to Y$ is proper.

Suppose V is a closed neighborhood of Y in W. There exist a closed neighborhood V'' of Y in V and a homotopy $\varphi\colon V'' \times I \to V$ such that $\varphi(y, 0) = y$ and $\varphi(y, 1) = r(y)$ for every $y \in V''$, and $\varphi(y, t) = y$ for every $y \in Y$, $t \in I$ (cf. [2], Lemma 3.8 or [1], Lemma 5.2).

Let T denote the closed subset $(V'' \times \{0\}) \cup (Y \times I) \cup (V'' \times \{1\})$ of $V'' \times I$. Since

$$\varphi(y, t) = y \quad \text{for} \quad (y, t) \in (V'' \times \{0\}) \cup (Y \times I)$$

and

$$\varphi(y, 1) = r(y) \quad \text{for} \quad (y, 1) \in V'' \times \{1\},$$

$\varphi|T\colon T \to V$ is a proper map. Hence by Lemma 3.2, there is a closed neighborhood G of T in $V'' \times I$ such that $\varphi|G\colon G \to V$ is proper. Let V' be a closed neighborhood of Y in V'' such that $V' \times I \subset U$. Then $\varphi|V' \times I\colon V' \times I \to V$ is a proper map satisfying the desired conditions.

3.8. COROLLARY. *If X and Y are closed subsets of K, Y is an ANR, and the maps $f, g\colon X \to Y$ are weakly properly homotopic in K, then $f \underset{p}{\cong} g$.*

Proof. By Lemma 3.7, there exist a closed neighborhood W of Y and a proper retraction r of W to Y. If $\varphi\colon X \times I \to W$ is a proper homotopy joining f and g in W, then $r\varphi\colon X \times I \to Y$ is a proper homotopy joining f and g in Y.

3.9. THEOREM. *If X and Y are closed subsets of K and Y is an ANR, then every proper fundamental net from X to Y is properly homotopic to one generated by a map (i.e., every proper fundamental class $[\underline{f}]\colon X \to Y$ is generated by a map $f\colon X \to Y$).*

Theorem 3.9 can be proved by an argument essentially identical to that for Theorem 3.7 of [2], using the above Lemma 3.7 in place of Lemma 3.8 of [2] to guarantee that the homotopies involved are proper.

3.10. THEOREM. *If X and Y are closed subsets of K which are of the same proper homotopy type, then $X \underset{pF}{\cong} Y$.*

Proof. Let $f\colon X \to Y$ and $g\colon Y \to X$ be proper maps such that $gf \underset{p}{\cong} i_X$ and $fg \underset{p}{\cong} i_Y$. By Corollary 3.3, f and g generate proper fundamental nets $\underline{f}\colon X \to Y$ and $\underline{g}\colon Y \to X$, respectively. Since $\underline{gf}$ is generated by gf and $\underline{i_X}$ is generated by i_X, it follows from Lemma 3.5 that $\underline{gf} \underset{p}{\cong} \underline{i_X}$. Similarly, $\underline{fg} \underset{p}{\cong} \underline{i_Y}$, and hence $X \underset{pF}{\cong} Y$.

3.11. Remark. It clearly follows from the argument for Theorem 3.10 that if X properly homotopically dominates Y, then $X \underset{pF}{\geqslant} Y$.

From 3.10 and 3.11 we obtain immediately that if X, X', Y, Y' are closed subsets of K with X homeomorphic to X' and Y homeomorphic to Y', then $X \underset{pF}{\cong} Y$ if and only if $X' \underset{pF}{\cong} Y'$ and $X \underset{pF}{\geqslant} Y$ if and only if $X' \underset{pF}{\geqslant} Y'$. Thus we may make the following definition:

Two separable, locally compact metrizable spaces are said to have the *same proper shape*, $\operatorname{Sh}_p X = \operatorname{Sh}_p Y$, if and only if there exist closed subsets X', Y' of K with X homeomorphic to X', Y homeomorphic to Y' and $X' \underset{pF}{\cong} Y'$.

Similarly, we say that $\mathrm{Sh}_p X \geqslant \mathrm{Sh}_p Y$ if there exist closed subsets X', Y' of K homeomorphic to X and Y, respectively, such that $X' \underset{pF}{\geqslant} Y'$.

3.12. THEOREM. *If X and Y are locally compact separable* ANR's, *then $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$ if and only if X and Y are of the same proper homotopy type, and $\mathrm{Sh}_p X \geqslant \mathrm{Sh}_p Y$ if and only if X properly homotopically dominates Y.*

Proof. Suppose $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$ and assume X and Y are embedded as closed subsets of K. Then $X \underset{pF}{\cong} Y$ and hence there exist proper fundamental nets $\underline{f}$: $X \to Y$, $\underline{g}$: $Y \to X$ such that $\underline{g}\underline{f} \underset{p}{\cong} \underline{i}_X$ and $\underline{f}\underline{g} \underset{p}{\cong} \underline{i}_Y$. By Theorem 3.9, there exist proper fundamental nets $\underline{f}'$: $X \to Y$, $\underline{g}'$: $Y \to X$ such that $\underline{f}' \underset{p}{\cong} \underline{f}$ and $\underline{g}' \cong \underline{g}$, and such that $\underline{f}'$ is generated by a map f: $X \to Y$ and $\underline{g}'$ is generated by a map g: $Y \to X$. Since $\underline{g}'\underline{f}' \underset{p}{\cong} \underline{g}\underline{f} \underset{p}{\cong} \underline{i}_X$ and $\underline{g}'\underline{f}'$ is generated by gf and $\underline{i}_X$ is generated by i_X, it follows by Lemma 3.6 that gf is weakly properly homotopic to i_X in K; since X is an ANR, this implies, by Lemma 3.8, that $gf \underset{p}{\cong} i_X$. Similarly, $fg \underset{p}{\cong} i_Y$ and it follows that X and Y are of the same proper homotopy type. The proof that proper shape domination implies proper homotopy domination is implicit in the above argument, and of course the converses follow from Theorem 3.10 and Remark 3.11.

Remark. If X and Y are closed subsets of K and $\underline{f} = \{f_\lambda|\ \lambda \in \Lambda\}$ is a proper fundamental net from X to Y, then X is compact if Y is; to see this, it is only necessary to consider a compact neighborhood V of Y and observe that for some $\lambda_0 \in \Lambda$, $f_{\lambda_0}(X) \subset V$ and $f_{\lambda_0}|X$: $X \to K$ is a proper map, whence $X = (f_{\lambda_0}|X)^{-1}(V)$ is necessarily compact. In particular, no compact space can have the same proper shape as a non-compact space, nor can any compact space properly fundamentally dominate or be dominated by a non-compact space.

3.13. LEMMA. *If X and Y are compact subsets of K and $\underline{f} = \{f_k|\, k = 1, 2, \ldots\}$ is a fundamental sequence (in the sense of* [2]) *from X to Y in (K, K), then $\underline{f}$ is also a proper fundamental net from X to Y. Moreover, any two homotopic fundamental sequences from X to Y are properly homotopic.*

Proof. Suppose V is a closed neighborhood of Y and let V_1 be a compact neighborhood of Y with $V_1 \subset V$. Since $\underline{f}$ is a fundamental sequence from X to Y, there exist a closed neighborhood U of X and a positive integer k_0 such that for $k \geqslant k_0$, $f_k|U \cong f_{k_0}|U$ in V_1. Let U_1 be a compact neighborhood of X with $U_1 \subset U$. Then for $k \geqslant k_0$, $f_k|U_1 \cong f_{k_0}|U_1$ in V_1, and since U_1 and V_1 are compact, $f_k|U_1 \underset{p}{\cong} f_{k_0}|U_1$ in V_1. The proof of the second part of the theorem is analogous.

3.14. LEMMA. *If X and Y are compact subsets of K, then every proper fundamental net from X to Y is properly homotopic to a fundamental sequence from X to Y.*

Proof. Suppose $\underline{f} = \{f_\lambda|\ \lambda \in \Lambda\}$ is a fundamental net from X to Y. Since Y is compact, there is a cofinal sequence $\{V_i\}_{i=1}^{\infty}$ of closed neighborhoods of Y in K. It is easy to obtain a sequence $\{U_i\}_{i=1}^{\infty}$ of closed neighborhoods of X and an increasing sequence $\{\lambda_i\}$ of elements of Λ such that for each i,

$$f_\lambda|U_i \underset{p}{\cong} f_{\lambda_i}|U_i \quad \text{in } V_i, \quad \text{for all indices } \lambda \geqslant \lambda_i.$$

If $\underline{f}' = \{f_{\lambda_i}|\ i = 1, 2, \ldots\}$, it readily follows that $\underline{f}'$ is a proper fundamental net (and hence a fundamental sequence) from X to Y, and that $\underline{f}' \underset{p}{\cong} \underline{f}$.

3.15. THEOREM. *If X and Y are compact subsets of K, then $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$ if and only if $\mathrm{Sh}\, X = \mathrm{Sh}\, Y$.*

Proof. If $\mathrm{Sh}\, X = \mathrm{Sh}\, Y$, there exist fundamental sequences $\underline{f}$: $X \to Y$ in (K, K) and $\underline{g}$: $Y \to X$ in (K, K) such that $\underline{g}\underline{f} \cong \underline{i}_X$ and $\underline{f}\underline{g} \cong \underline{i}_Y$. By Lemma 3.13, $\underline{f}$ and $\underline{g}$ are also proper fundamental nets from X to Y and from Y to X, respectively, and $\underline{g}\underline{f} \underset{p}{\cong} \underline{i}_X$, $\underline{f}\underline{g} \underset{p}{\cong} \underline{i}_Y$. Hence $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$.

Conversely, suppose $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$ and let $\underline{f}$: $X \to Y$ and $\underline{g}$: $Y \to X$ be fundamental nets such that $\underline{g}\underline{f} \underset{p}{\cong} \underline{i}_X$ and $\underline{f}\underline{g} \underset{p}{\cong} \underline{i}_Y$. By Lemma 3.14, there exist fundamental sequences $\underline{f}'$: $X \to Y$ and $\underline{g}'$: $Y \to X$ such that $\underline{f} \underset{p}{\cong} \underline{f}'$ and $\underline{g} \underset{p}{\cong} \underline{g}'$. Then $\underline{g}'\underline{f}' \cong \underline{g}\underline{f} \cong \underline{i}_X$ and $\underline{f}'\underline{g}' \cong \underline{f}\underline{g} \cong \underline{i}_Y$, so $\mathrm{Sh}\, X = \mathrm{Sh}\, Y$.

4. One-point and Freudenthal compactifications. Throughout this section we shall continue to restrict our attention, unless otherwise stated, to separable, locally compact metrizable spaces. If X is such a space, we shall let $CX = X \cup \{\infty\}$ denote the one-point compactification of X. It is well known that CX is metrizable (e.g., [10], p. 247). If Y is another such space and f: $X \to Y$ is a proper map, then f has a unique extension to a map of pairs Cf: $(CX, \{\infty\}) \to (CY, \{\infty\})$.

Recall that if M and N are topological spaces, f: $M \to N$ and g: $M \to N$ are maps, and $M_0 \subset M$, then f and g are *homotopic rel M_0*, denoted $f \cong g \operatorname{rel} M_0$, if there exists a homotopy φ: $M \times I \to N$ joining f and g such that for $m \in M_0$ and $t \in I$, $\varphi(m, t) = f(m) = g(m)$.

The following facts, which are incorporated into a lemma for convenience, are immediate.

4.1. LEMMA. (a) *If X is a locally compact, separable metrizable space, then* $Ci_X = i_{CX}$.

(b) *If $f\colon X\to Y$ and $g\colon Y\to Z$ are proper maps between locally compact, separable metrizable spaces, then* $C(gf) = (Cg)(Cf)$.

(c) *If $f\colon X\to Y$ and $g\colon X\to Y$ are proper maps between locally compact, separable metrizable spaces and $f \underset{p}{\simeq} g$, then $Cf \simeq Cg$ rel $\{\infty\}$.*

The interval $(0,1)$ has two ends while $[0,1)$ has one. The geometric appeal of this statement is evident, and is made precise by the well known theory of ends due to Freudenthal ([13], [14]). This theory is of fundamental importance in geometric topology and has appeared in a variety of contexts; in this section we shall show, among other things, that the number of ends of a locally compact, connected, separable metrizable space is a proper shape invariant.

In [14], Freudenthal defines, for sufficiently nice metrizable spaces X, the set of *ends* of X and the compactification of X by its endpoint set; these sets will here be denoted by EX and $FX = X\cup EX$, respectively. The necessary conditions on X are (1) X is separable, (2) X is semi-compact (i.e., each point of X has arbitrarily small neighborhoods with compact boundary), and (3) QX, the space of quasi-components of X, is compact. The space QX is defined in a natural way (see [14, Section 5]), and we simply remark here that the compactness of QX is equivalent to the condition that every decreasing sequence of nonempty open-closed subsets of X has nonempty intersection. Condition (3) is not required in order to define FX, but is necessary in order that FX be metrizable.

While we shall omit here the precise definition of the set EX, we shall make use of the fact, proved in [14], that FX is characterized, among compactifications of X, by the following properties:

(a) EX is 0-dimensional, and

(b) no open neighborhood of a point $e\in EX$ is separated by EX into two sets each of which is open in X and each of which has e as a limit point.

Perhaps it should be remarked that in many recent investigations, [13] has been used as a reference for the theory of ends. Actually, [13] applies only to locally connected spaces and is not adequate for the more general case considered here. Of course, the construction of [14] reduces to that of [13] in the locally connected case.

4.2 LEMMA. *Suppose X and Y are locally compact, separable metrizable spaces, that QX and QY are compact, and that $f\colon X\to Y$ is a proper map. Then f has a unique extension to a map of pairs $Ff\colon (FX, EX)\to(FY, EY)$.*

Proof. Suppose $\{x_i\}_{i=1}^{\infty}$ is a sequence of points of X converging to $e\in EX$. Then $\{f(x_i)\}_{i=1}^{\infty}$ has an accumulation point $e'\in FY$ and, since f is proper, $e'\in EY$. Suppose now that $e''\neq e'$ is also an accumulation point of $\{f(x_i)\}_{i=1}^{\infty}$. Since e' and e'' are distinct endpoints of Y, there exist disjoint open neighborhoods U_1 and U_2 of e' and e'', respectively, in FY, such that the boundaries of U_1 and U_2 in FY are compact subsets of Y. Then, since f is proper, $f^{-1}(U_1\cup U_2) = V_0$ is an open set in X whose boundary in X is compact. It follows that

$$V = V_0\cup\{z\in EX\mid z \text{ is a limit point of } V_0\}$$

is an open neighborhood in FX of e. But $V-EX$ is the union of two disjoint open subsets of X, namely $f^{-1}(U_1)$ and $f^{-1}(U_2)$, each having e as a limit point. This is a contradiction, so $\{f(x_i)\}_{i=1}^{\infty}$ has precisely one accumulation point, e', in FY and, by the same argument, e' is independent of the choice of the sequence $\{x_i\}_{i=1}^{\infty}$. Thus we may define Ff by letting $Ff(e) = e'$. It is easy to see that the resulting function is continuous and carries the pair (FX, EX) into the pair (FY, EY).

4.3. LEMMA. (a) *If X is a locally compact, separable metrizable space and QX is compact, then* $Fi_X = i_{FX}$.

(b) *If $f\colon X\to Y$ and $g\colon Y\to Z$ are proper maps between locally compact, separable metrizable spaces, where QX, QY, and QZ are compact, then* $F(gf) = (Fg)(Ff)$.

(c) *If $f\colon X\to Y$ and $g\colon X\to Y$ are proper maps between locally compact, separable metrizable spaces, where QX and QY are compact, and $f\underset{p}{\simeq} g$, then $Ff\simeq Fg$ rel EX.*

Proof. Parts (a) and (b) are immediate. Suppose $\varphi\colon X\times I\to Y$ is a proper homotopy joining f and g. It will be shown that for each $t\in I$ and $e\in EX$, $Ff(e) = F\varphi_t(e)$. It easily follows that if $\Phi\colon FX\times I\to FY$ is defined by $\Phi(z, t) = F\varphi_t(z)$ for each $z\in FX$ and $t\in I$, then Φ is a homotopy from Ff to Fg rel EX. For simplicity, we may assume $t = 1$.

Supposing now that for some $e\in EX$, $Ff(e)\neq Fg(e)$, let U be an open neighborhood in FY of $Ff(e)$ such that $Fg(e)\notin \mathrm{Cl}\,U$ and the boundary of U in FY, $\mathrm{Bd}\,U$, is a compact subset of Y. Now let $\{x_i\}_{i=1}^{\infty}$ be a sequence of points of X converging to e such that if $i = 1, 2, \ldots$, then $f(x_i)\in U$ and $g(x_i)\notin U$. It follows that if $i = 1, 2, \ldots$, there exists $t_i\in I$ such that $\varphi(x_i, t_i)\in \mathrm{Bd}\,U$, but this contradicts the fact that φ is proper.

In order to state the results of this section in the strongest possible form, we digress momentarily to give strengthened versions of the notions of *fundamental sequences between pairs* and *relative shape* [cf. 1]. In the following paragraph then, all pairs are compact pairs in the Hilbert cube H; details are omitted since they follow precisely as in [1] with the obvious modifications.

A *fundamental sequence a from (X, X_0) to (Y, Y_0) rel X_0* is a sequence $\{f_i\}_{i=1}^{\infty}$ of maps $f_i\colon (H, X_0)\to(H, Y_0)$ such that for each neighborhood V

of Y in H, there exist a neighborhood U of X in H and an integer $i_0 > 0$ such that if $i \geqslant i_0$, $f_i|U \simeq f_{i_0}|U \operatorname{rel} X_0$ in V. Two such sequences, $\alpha = \{f_i\}_{i=1}^{\infty}$ and $\beta = \{g_i\}_{i=1}^{\infty}$, are *homotopic rel* X_0 if for each neighborhood V of Y in H, there exist a neighborhood U of X in H and an integer $i_0 > 0$ such that if $i \geqslant i_0$, $f_i|U \simeq g_i|U \operatorname{rel} X_0$ in V. This being the case, we write $\alpha \simeq \beta \operatorname{rel} X_0$. Homotopy rel X_0 is an equivalence relation on the fundamental sequences from (X, X_0) to $(Y, Y_0) \operatorname{rel} X_0$. Compositions are defined as usual, and respect equivalence classes. Letting ι denote the sequence each term of which is the identity map on H, we say that (X, X_0) and (Y, Y_0) are *fundamentally equivalent rel* (X_0, Y_0) if there exist a fundamental sequence α from (X, X_0) to $(Y, Y_0) \operatorname{rel} X_0$ and a fundamental sequence β from (Y, Y_0) to $(X, X_0) \operatorname{rel} Y_0$ such that $\beta\alpha \simeq \iota \operatorname{rel} X_0$ and $\alpha\beta \simeq \iota \operatorname{rel} Y_0$. For this, we write

$$(X, X_0) \underset{F}{\simeq} (Y, Y_0) \operatorname{rel} (X_0, Y_0) .$$

As expected, an equivalence relation is obtained and, if the pairs (X, X_0) and (X', X_0') are homotopically equivalent rel (X_0, X_0'), then $(X, X_0) \underset{F}{\simeq} (X', X_0') \operatorname{rel} (X_0, X_0')$. The latter follows from the facts, to be used subsequently, that a map $(X, X_0) \to (Y, Y_0)$ generates a fundamental sequence from (X, X_0) to $(Y, Y_0) \operatorname{rel} X_0$, that two homotopic such maps generate homotopic fundamental sequences rel X_0, and that the assignment preserves compositions up to homotopy rel X_0 and carries $[i]$ to $[\iota]$.

It follows, in particular, that if $(Z', Z_0') \approx (Z'', Z_0'')$ are compact pairs in H, then $(Z', Z_0') \underset{F}{\simeq} (Z'', Z_0'') \operatorname{rel} (Z_0', Z_0'')$. Hence, if (X, X_0) and (Y, Y_0) are compact metrizable pairs, we may write $\operatorname{Sh}(X, X_0) = \operatorname{Sh}(Y, Y_0) \operatorname{rel} (X_0, Y_0)$ provided there exist homeomorphic copies (X', X_0') and (Y', Y_0') of (X, X_0) and (Y, Y_0), respectively, in H so that $(X', X_0') \underset{F}{\simeq} (Y', Y_0') \operatorname{rel} (X_0', Y_0')$, and a well-defined equivalence relation results. In the obvious way, $\operatorname{Sh}(X, X_0) \geqslant \operatorname{Sh}(Y, Y_0) \operatorname{rel} (X_0, Y_0)$ is also defined.

It should be remarked that if $\operatorname{Sh}(X, X_0) = \operatorname{Sh}(Y, Y_0) \operatorname{rel} (X_0, Y_0)$, then (X, X_0) and (Y, Y_0) have the same shape as pairs in the sense of [1], and therefore [18, Remark 1] in the sense of [19], which (as shown in [18]) differs from that of [1]. The converses are not true, however.

Now, for convenience in stating our results, we shall define several categories and functors.

The category $\mathcal{K}_1$. The objects of $\mathcal{K}_1$, $\operatorname{Ob}(\mathcal{K}_1)$, are the locally compact, separable metrizable spaces. If $X, Y \in \operatorname{Ob}(\mathcal{K}_1)$, then the morphisms from X to Y, $\mathcal{K}_1(X, Y)$, are the proper homotopy classes of proper maps from X to Y, with composition defined in the usual way.

The category $\widetilde{\mathcal{K}}_1$. $\widetilde{\mathcal{K}}_1$ is the full subcategory of $\mathcal{K}_1$ whose objects are those $X \in \operatorname{Ob}(\mathcal{K}_1)$ for which QX is compact.

The category $\mathcal{K}_2$. The objects of $\mathcal{K}_2$ are the compact metrizable pairs. If (X, X_0) and (Y, Y_0) are two such pairs, then $\mathcal{K}_2\big((X, X_0), (Y, Y_0)\big)$ consists of the homotopy classes rel X_0 of maps from (X, X_0) to (Y, Y_0), with composition defined in the usual way.

The category $\mathcal{S}_1$. The objects of $\mathcal{S}_1$ are the locally compact, separable metrizable spaces. To each $X \in \operatorname{Ob}(\mathcal{S}_1)$ assign a closed subset X' of $K = H \times [0, 1)$ so that $X \approx X'$. Then, if $X, Y \in \operatorname{Ob}(\mathcal{S}_1)$, $\mathcal{S}_1(X, Y)$ consists of the set of proper fundamental classes (of proper fundamental nets) from X' to Y' in K. Composition is defined as in Section 3.

Of course, $\mathcal{S}_1(X, Y)$ depends on the choice of X' and Y', but the isomorphism class of $\mathcal{S}_1$ is independent of this choice; this follows from the comments following 3.11. Also by this remark, we shall simply regard each $X \in \operatorname{Ob}(\mathcal{S}_1)$ as *being* a closed subspace of K in the remainder of this section. It is easy to verify that if $X \in \operatorname{Ob}(\mathcal{S}_1)$ and QX is compact, then X' can be chosen so that the closure of X' in $\widetilde{H}_0 = H \times [0, 1]$ can be identified with FX. For technical reasons (e.g., the proof of Theorem 4.5) we shall suppose X' has been chosen in this way.

The category $\widetilde{\mathcal{S}}_1$. $\widetilde{\mathcal{S}}_1$ is the full subcategory of $\mathcal{S}_1$ whose objects are those $X \in \operatorname{Ob}(\mathcal{S}_1)$ for which QX is compact.

The category $\mathcal{S}_2$. The objects of $\mathcal{S}_2$ are the compact metrizable pairs. To each $(X, X_0) \in \operatorname{Ob}(\mathcal{S}_2)$ assign a closed pair (X', X_0') in H so that $(X, X_0) \approx (X', X_0')$. Then, if $(X, X_0), (Y, Y_0) \in \operatorname{Ob}(\mathcal{S}_2)$, $\mathcal{S}_2\big((X, X_0), (Y, Y_0)\big)$ is defined to be the set of homotopy classes rel X_0' of fundamental sequences from (X', X_0') to $(Y', Y_0') \operatorname{rel} X_0'$. As above, the isomorphism class of $\mathcal{S}_2$ is independent of the choice of (X', X_0'), and for simplification we shall usually regard an object of $\mathcal{S}_2$ as being a compact pair in the Hilbert cube.

Perhaps it should be pointed out that $\operatorname{Sh}_p X = \operatorname{Sh}_p Y$ if and only if X and Y are equivalent objects in the category $\mathcal{S}_i$ and $\operatorname{Sh}(X, X_0) = \operatorname{Sh}(Y, Y_0) \operatorname{rel} (X_0, Y_0)$ if and only if (X, X_0) and (Y, Y_0) are equivalent objects in the category $\mathcal{S}_2$.

The functor S_p: $\mathcal{K}_1 \to \mathcal{S}_1$. Define S_p by $S_p(X) = X$ for each $X \in \operatorname{Ob}(\mathcal{K}_1)$ and $S_p([f]) = [\underline{f}]$ for each $f \in \mathcal{K}_1(X, Y)$, where $\underline{f}$ is a proper fundamental net generated by f. (Here, and later, we shall use [ ] for the equivalence class of an object under an equivalence relation. The relation will not usually be explicitly mentioned, but will always be made clear from context.) That S_p is well-defined and a functor follows from the results of Section 3.

The functor $\widetilde{S}_p$: $\widetilde{\mathcal{K}}_1 \to \mathcal{S}_1$. $\widetilde{S}_p$ is the restriction to $\widetilde{\mathcal{K}}_1$ of S_p.

The functor F_{rel}: $\mathcal{H}_2 \to \mathcal{S}_2$. Define F_{rel} on $\mathrm{Ob}(\mathcal{H}_2)$ by $F_{\mathrm{rel}}((X, X_0)) = (X, X_0)$ for each $(X, X_0) \in \mathrm{Ob}(\mathcal{H}_2)$. If $[f] \in \mathcal{H}_2((X, X_0), (Y, Y_0))$, define $F_{\mathrm{rel}}([f])$ to be $[\alpha]$, where α is a fundamental sequence from (X, X_0) to (Y, Y_0) rel (X_0, Y_0) generated by f. By our earlier remarks of this Section, F_{rel} is well-defined and is a functor.

The functor Φ: $\mathcal{H}_1 \to \mathcal{H}_2$. Define Φ by $\Phi(X) = (CX, \{\infty\})$ for each $X \in \mathrm{Ob}(\mathcal{H}_1)$ and $\Phi([f]) = [Cf]$ for each $[f] \in \mathcal{H}_1(X, Y)$. By Lemma 4.1, Φ is well-defined and a functor.

The functor $\tilde{\Phi}$: $\tilde{\mathcal{H}}_1 \to \mathcal{H}_2$. Define $\tilde{\Phi}$ by $\tilde{\Phi}(X) = (FX, EX)$ for each $X \in \mathrm{Ob}(\tilde{\mathcal{H}}_1)$ and $\tilde{\Phi}([f]) = [Ff]$ for each $[f] \in \tilde{\mathcal{H}}_1(X, Y)$. By Lemma 4.3, $\tilde{\Phi}$ is well-defined and is a functor.

With the above machinery now at hand, we are able to state the main results of this section.

4.4. THEOREM. *There exists a functor* Ψ: $\mathcal{S}_1 \to \mathcal{S}_2$ *such that the following diagram is commutative.*

$$\begin{tikzcd} \mathcal{H}_1 \arrow[r, "\Phi"] \arrow[d, "S_p"'] & \mathcal{H}_2 \arrow[d, "F_{\mathrm{rel}}"] \\ \mathcal{S}_1 \arrow[r, "\Psi"'] & \mathcal{S}_2 \end{tikzcd}$$

4.5. THEOREM. *There exists a functor* $\tilde{\Psi}$: $\tilde{\mathcal{S}}_1 \to \mathcal{S}_2$ *such that the following diagram is commutative.*

$$\begin{tikzcd} \tilde{\mathcal{H}}_1 \arrow[r, "\tilde{\Phi}"] \arrow[d, "\tilde{S}_p"'] & \mathcal{H}_2 \arrow[d, "F_{\mathrm{rel}}"] \\ \tilde{\mathcal{S}}_1 \arrow[r, "\tilde{\Psi}"'] & \mathcal{S}_2 \end{tikzcd}$$

Before proving Theorems 4.4 and 4.5, we pause to note the following corollaries.

4.6. COROLLARY. *If X and Y are locally compact, separable metrizable spaces and* $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$, *then*

$$\mathrm{Sh}(CX, \{\infty\}) = \mathrm{Sh}(CY, \{\infty\})\ \mathit{rel}\ (\{\infty\}, \{\infty\}).$$

The following is a corollary to Corollary 4.6 (see [1], Section 12).

4.7. COROLLARY. *If X and Y are locally compact, separable metrizable spaces and* $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$, *then the pointed compacta* (CX, ∞) *and* (CY, ∞) *have the same pointed shape.*

4.8. COROLLARY. *If X and Y are locally compact, separable metrizable spaces with QX and QY compact and* $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$, *then* $\mathrm{Sh}(FX, EX) = \mathrm{Sh}(FY, EY)\ \mathit{rel}\ (EX, EY)$.

4.9. COROLLARY. *If X and Y are locally compact, separable metrizable spaces with QX and QY compact and* $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$, *then* $EX \approx EY$.

The following is a corollary to Corollary 4.8 (see [1], Section 8).

4.10. COROLLARY. *If X and Y are locally compact, separable metrizable spaces with QX and QY compact and* $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$, *then the pairs* (FX, EX) *and* (FY, EY) *have the same shape.*

There are obvious analogs of each of the above corollaries (except 4.9) with the notion of equivalence replaced by that of domination in each case.

For the definition of Z-set, see [8].

4.11. COROLLARY. *Suppose X and Y are Z-sets in K and* $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$. *Then* $K - X \approx K - Y$.

Proof. It follows from Corollary 4.6 that CX and CY are fundamentally equivalent in CK. Also, CX and CY are Z-sets in CK, so according to Chapman [8], $CK - CX \approx CK - CY$. But $CK - CX = K - X$ and $CK - CY = K - Y$.

The converse of Corollary 4.11 does not hold, as shown by taking X to be a triod less two endpoints and Y to be a "circle with a sticker" less the endpoint of the "sticker". For suppose X and Y are embedded as Z-sets in K. Then CX and CY are Z-sets in CK and $\mathrm{Sh}(CX) = \mathrm{Sh}(CY)$. By [8], $CK - CX \approx CK - CY$, so $K - X \approx K - Y$, but $\mathrm{Sh}_p(X) \neq \mathrm{Sh}_p(Y)$. Another example is afforded by 4.14.

Borsuk has shown [1, p. 236] that among plane continua there are only countably many shapes. By the following, the analog of this result does not hold for proper shapes.

4.12. COROLLARY. *There exists an uncountable collection of closed connected subsets of the plane no two of which have the same proper shape.*

Proof. Let L denote an arc in the 2-sphere S^2. We identity the plane with $S^2 - L$. There exists an uncountable family $\{Z_\alpha \mid \alpha \in A\}$ of (countable) compact 0-dimensional sets in L so that if $\alpha_1 \neq \alpha_2$, then $Z_{\alpha_1} \not\approx Z_{\alpha_2}$. It is easy to construct, for each $\alpha \in A$, a 1-dimensional connected set $X_\alpha \subset S^2 - L$ so that the pairs $(X_\alpha \cup Z_\alpha, Z_\alpha)$ and (FX_α, EX_α) are homeomorphic. By Corollary 4.9, if $\alpha_1 \neq \alpha_2$, then $\mathrm{Sh}_p(X_{\alpha_1}) \neq \mathrm{Sh}_p(X_{\alpha_2})$.

The following result will be necessary for the proof of Theorems 4.4 and 4.5. We note that its proof is quite similar to the proof of Proposition 4.7 of [12] and of Lemma 4 of [20].

4.13. LEMMA. *Suppose given the following data:*

a) *closed pairs* (X, X_0) *and* (Y, Y_0) *in* H;

b) *closed sets* $A_1 \supset A_2 \supset A_3 \supset \ldots \supset X = \bigcap_{i=1}^{\infty} A_1$, *and closed sets* $B_1 \supset B_2 \supset B_3 \supset \ldots \supset Y = \bigcap_{i=1}^{\infty} B_i$;

c) *maps* $f_i\colon (A_i, X_0) \to (B_i, X_0)$ *such that for* $i = 1, 2, 3, \ldots$, $f_i|A_{i+1} \simeq f_{i+1}$ *rel* X_0 *in* B_i.

Then there exists a unique (up to homotopy rel X_0*) fundamental sequence* $\alpha = \{\hat{f}_i\}_{i=1}^{\infty}$ *from* (X, X_0) *to* (Y, Y_0) *rel* X_0 *such that* $\hat{f}_i|A_i = f_i$.

Proof. Let $H = V_1 \supset V_2 \supset V_3 \supset \ldots$ be open sets in H such that $\bigcap_{i=1}^{\infty} V_i = Y$ and $B_i \subset V_i$ for $i = 1, 2, 3, \ldots$ Let $\hat{f}_1\colon H \to H$ be an extension of f_1 and let $U_1 = H$.

Now suppose, inductively, that $\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n\colon H \to H$ and closed neighborhoods $U_1 \supset U_2 \supset \ldots \supset U_n$ of $A_1 \supset A_2 \supset \ldots \supset A_n$, respectively, have been defined so that

(i) $\hat{f}_i$ is an extension of f_i,

(ii) if $1 \leqslant i \leqslant j \leqslant k \leqslant n$, then $\hat{f}_j|U_i \simeq \hat{f}_k|U_i$ rel X_0 in V_i, and

(iii) $\hat{f}_n(U_n) \subset V_n$.

Now, $f_{n+1} \simeq f_n|A_{n+1}$ rel X_0 in $B_n \subset V_n$. Hence, by (iii) and the Borsuk extension theorem [3, p. 94], we may extend f_{n+1} to a map $g_n\colon U_n \to V_n$ such that $g_n \simeq \hat{f}_n|U_n$ rel X_0 in V_n. Now suppose, inducting downwards, that g_n has been extended to a map $g_m\colon U_m \to V_m$, where $1 < k \leqslant n$, such that if $m \leqslant i \leqslant n$, $g_m|U_i \simeq \hat{f}_n|U_i$ rel X_0 in V_i. By (ii) and the Borsuk extension theorem, we may extend g_m to a map $g_{m-1}\colon U_{m-1} \to V_{m-1}$ such that $g_{m-1} \simeq \hat{f}_n|U_{m-1}$ rel X_0 in V_{m-1}. Continuing the induction we obtain $g_1\colon U_1 \to V_1$ and we let $g_1 = \hat{f}_{n+1}$.

It is clear from the construction that $\{\hat{f}_i\}_{i=1}^{\infty}$ is a fundamental sequence from X to Y rel X_0. To show uniqueness, suppose $\{\bar{f}_i\}_{i=1}^{\infty}$ is a fundamental sequence from X to Y rel X_0 such that for $i = 1, 2, 3, \ldots$, $\bar{f}_i|A_i = f_i = \hat{f}_i|A_i$. Let V be an open neighborhood of Y. Then there exists a neighborhood W of X and a positive integer j_0 such that if $i \geqslant j_0$, $\hat{f}_i|W \simeq \hat{f}_{j_0}|W$ rel X_0 in V and $\bar{f}_i|W \simeq \bar{f}_{j_0}|W$ rel X_0 in V. Let $i_0 \geqslant j_0$ be a positive integer so that $B_{i_0} \subset V$ and $A_{i_0} \subset W$. Let

$$F = W \times \{0\} \cup A_{i_0} \times I \cup W \times \{1\} \subset W \times I\,.$$

Define $h\colon F \to V$ by

$$h(x, t) = \begin{cases} \hat{f}_{i_0}(x) & \text{if } t = 0\,, \\ f_{i_0}(x) & \text{if } x \in A_{i_0}\,, \\ \bar{f}_{i_0}(x) & \text{if } t = 1\,. \end{cases}$$

Then, since V is an ANR, h extends to a map of R into V where R is a neighborhood of F in $W \times I$. Now, let U be a neighborhood of X such that $U \times I \subset R$. Then, if $i \geqslant i_0$,

$$\hat{f}_i|U \simeq \hat{f}_{i_0}|U \simeq \bar{f}_{i_0}|U \simeq \bar{f}_i|U \text{ rel } X_0 \quad \text{in } V.$$

Hence, $\{\hat{f}_i\}_{i=1}^{\infty} \simeq \{\bar{f}_i\}_{i=1}^{\infty}$ rel X_0.

Proof of Theorems 4.4. and 4.5. We explicitly prove Theorem 4.5. The same proof suffices for Theorem 4.4 upon replacement of $\tilde{\Psi}$ by Ψ, $\tilde{H}_0$ by $H_0 = C(H \times [0, 1))$, FX by CX, EX by $\{\infty\}$, etc.

Suppose $X \in \mathrm{Ob}(\tilde{\mathfrak{S}}_1)$. Define $\tilde{\Psi}(X)$ to be the pair (FX, EX). Now suppose $Y \in \mathrm{Ob}(\tilde{\mathfrak{S}}_1)$ and $[\underline{f}] \in \tilde{\mathfrak{S}}_1(X, Y)$. Recall that we are regarding X and Y as closed subspaces of $K = H \times [0, 1)$ with the property that $\mathrm{Cl}\,X$ in $\tilde{H}_0 = H \times [0, 1]$ is FX and $\mathrm{Cl}\,Y$ in $\tilde{H}_0$ is FY. Suppose $\underline{f} = \{f_\lambda |\ \lambda \in \Lambda\}$ is a proper fundamental net from X to Y in K.

Let $B_1' \supset B_2' \supset B_3' \supset \ldots$ be closed neighborhoods of Y in K so that $\bigcap_{i=1}^{\infty} B_i' = Y$ and, if $i = 1, 2, \ldots$, then $FB_i' = \mathrm{Cl}\,B_i'$ in $\tilde{H}_0 = B_i' \cup EY$. This latter condition can be guaranteed by taking each B_i' to be the union of a locally finite countable collection of closed Hilbert cubes in K each of which intersects Y and whose elements form a null-sequence. Denote FB_i' by B_i. Now, since $\underline{f}$ is a proper fundamental net, there exist a closed neighborhood A_1' of X in K and $\lambda_1 \in \Lambda$ such that if $\lambda \geqslant \lambda_1$, $f_\lambda|A_1' \underset{p}{\simeq} f_{\lambda_1}|A_1'$ in B_1'. We may suppose in addition that A_1' lies in the 1-neighborhood of X in K and that $FA_1' = \mathrm{Cl}\,A_1'$ in $\tilde{H}_0 = A_1' \cup EX$.

This begins the inductive construction of a sequence $A_1' \supset A_2' \supset A_3' \supset \ldots$ of closed neighborhoods of X and indices $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots$ such that if $i = 1, 2, \ldots$, then

(i) A_i' lies in the $\frac{1}{i}$-neighborhood of X in K,

(ii) $FA_i' = \mathrm{Cl}\,A_i'$ in $\tilde{H}_0 = A_i' \cup EX$, and

(iii) if $\lambda \geqslant \lambda_i$, then $f_\lambda|A_i' \underset{p}{\simeq} f_{\lambda_i}|A_i'$ in B_i'.

Now, denote FA_i' by A_i and let $f_i\colon (A_i, EX) \to (B_i, EY)$ be the map $F(f_{\lambda_i}|A_i')$. Then, by Lemma 4.13, there exists a unique (up to homotopy rel EX) fundamental sequence $\alpha = \{\hat{f}_i\}_{i=1}^{\infty}$ from (FX, EX) to (FY, EY) rel EX such that if $i = 1, 2, \ldots$, then $\hat{f}_i|A_i = f_i$. We define $\tilde{\Psi}([\underline{f}])$ to be $[\alpha]$.

Of course, we need to show that $[\alpha]$ is independent of the choices made during the construction of α. To this end, suppose $\underline{g}\colon X \to Y$ is a proper fundamental net such that $\underline{f} \underset{p}{\simeq} \underline{g}$, where $\underline{g} = \{g_\delta |\ \delta \in \Delta\}$. Suppose

$D'_1 \supset D'_2 \supset D'_3 \supset \ldots$ are closed neighborhoods of Y in K so that $\bigcap_{i=1}^{\infty} D'_i = Y$, and if $i = 1, 2, \ldots$, then $FD'_i = \mathrm{Cl}\, D'_i$ in $\tilde{H}_0 = D'_i \cup EY$. Suppose $C'_1 \supset C'_2 \supset C'_3 \supset \ldots$ are closed neighborhoods of X and $\delta_1 \leqslant \delta_2 \leqslant \delta_3 \leqslant \ldots$ are indices such that if $i = 1, 2, \ldots$, then

(i) C'_i lies in the $\frac{1}{i}$-neighborhood of X in K,

(ii) $FC'_i = \mathrm{Cl}\, C'_i$ in $\tilde{H}_0 = C'_i \cup EX$, and

(iii) if $\delta \geqslant \delta_i$, then $g_\delta | C'_i \underset{p}{\simeq} g_{\delta_i} | C'_i$ in D'_i.

Let $C_i = FC'_i$, $D_i = FD'_i$, $g_i\colon (C_i, EX) \to (D_i, EY)$ be the map $F(g_{\delta_i}|C'_i)$. Let $\beta = \{\hat{g}_i\}_{i=1}^{\infty}$ be a fundamental sequence from (FX, EX) to $(FY, EY) \operatorname{rel} EX$ such that if $i = 1, 2, \ldots$, then $\hat{g}_i | C_i = g_i$. It is necessary to show that $\alpha \simeq \beta \operatorname{rel} EX$.

Let V be an open neighborhood in $\tilde{H}_0$ of EY. Let k be a positive integer and let $i_0 > k$ be a positive integer so that $B'_{i_0} \cup D'_{i_0} \subset V$. Let $B'_{i_0} \cup D'_{i_0} = V'$. Then, since $\underline{f} \underset{p}{\simeq} \underline{g}$, there exist indices λ_0 and δ_0 and a closed neighborhood W' of X in K such that if $\lambda \geqslant \lambda_0$ and $\delta \geqslant \delta_0$, then $f_\lambda | W' \underset{p}{\simeq} g_\delta | W'$ in V'. We may also suppose that $\lambda_0 \geqslant \lambda_{i_0}$, $\delta_0 \geqslant \delta_{i_0}$, $W' \subset A'_{i_0} \cap C'_{i_0}$, and $FW' = \mathrm{Cl}\, W'$ in $\tilde{H}_0 = W' \cup EX$. Then

$$f_{\lambda_0} | W' \underset{p}{\simeq} f_{\lambda_{i_0}} | W' \quad \text{in } V' \quad (\text{since } \lambda_0 \geqslant \lambda_{i_0},\ W' \subset A'_{i_0},\ B'_{i_0} \subset V'),$$

$$g_{\delta_0} | W' \underset{p}{\simeq} g_{\delta_{i_0}} | W' \quad \text{in } V' \quad (\text{since } \delta_0 \geqslant \delta_{i_0},\ W' \subset C'_{i_0},\ D'_{i_0} \subset V'),$$

$$f_{\lambda_0} | W' \underset{p}{\simeq} g_{\delta_0} | W' \quad \text{in } V' \quad (\text{since } \lambda_0 \geqslant \lambda_0,\ \delta_0 \geqslant \delta_0).$$

Hence, $f_{\lambda_{i_0}} | W' \underset{p}{\simeq} g_{\delta_{i_0}} | W'$ in V'. Let $W = FW'$. It follows that

$$\hat{f}_{i_0} | W \simeq \hat{g}_{i_0} | W \operatorname{rel} EX \quad \text{in } V.$$

Since V is an ANR, it follows that there is an open neighborhood U of W such that $\hat{f}_{i_0} | U \simeq \hat{g}_{i_0} | U \operatorname{rel} EX$ in V. Since i_0 was chosen to be larger than the arbitrary positive integer k, this clearly shows that $\alpha \simeq \beta \operatorname{rel} EX$.

It is easily verified from our construction that $\tilde{\Psi}$ is indeed a functor and that $\tilde{\Psi}\tilde{S}_p = F_{\mathrm{rel}}\tilde{\Phi}$.

We remark that it is possible to identify $\mathcal{K}_1$ with a subcategory of $\mathcal{S}_1$, by restricting the morphisms of $\mathcal{S}_1$ to those generated by maps. A similar remark allows us to identify $\mathcal{K}_2$ with a subcategory of $\mathcal{S}_2$, so that we can think of S_p and F_{rel} as being inclusions and of Φ as being the restriction of Ψ. In this way, the shape categories $\mathcal{S}_1$ and $\mathcal{S}_2$ can be thought of as "enlargements" of the homotopy categories $\mathcal{K}_1$ and $\mathcal{K}_2$. Similar remarks apply, of course, to $\tilde{\mathcal{K}}_1$, $\tilde{\mathcal{S}}_1$, $\tilde{S}_p$, $\tilde{\Phi}$ and $\tilde{\Psi}$.

4.14. Example. Let $X = \left\{(x, y) \in E^2 \mid\ 0 < x \leqslant 1,\ \ y = \sin\frac{\pi}{x}\right\} \cup \cup \{(0, y) \mid\ -1 \leqslant y < 1\}$ and let $Y = \{(x, 0) \in E^2 \mid\ x \geqslant 0\}$. Then $\mathrm{Sh}_p Y \not\geqslant \mathrm{Sh}_p X$.

Proof. Let E denote $\{(x, y) \in E^2 \mid\ -2 \leqslant y < 1 + |x|\}$. Then $E \approx E^2_+$. We may regard X, Y and E as closed subspaces of $H \times E = K_0 \approx K$, and we shall henceforth view K_0 and K as having been identified.

Suppose, contrary to the above claim, that there exist proper fundamental nets $\underline{f}\colon X \to Y$ and $\underline{g}\colon Y \to X$ such that $\underline{g}\underline{f} \underset{pF}{\simeq} \underline{i}_X$, where $\underline{f} = \{f_\lambda \mid \lambda \in \Lambda\}$ and $\underline{g} = \{g_\delta \mid\ \delta \in \Delta\}$. We shall show that this supposition brings about a contradiction.

If $i = 1, 2, \ldots$, let $p_i = \left(\frac{2}{4i+1}, 1 - \frac{2}{4i+1}\right) \in E$, and let N_i be a closed circular disk containing p_i and lying in $E - X$. Let $V = H \times [E - \bigcup_{i=1}^{\infty} \mathrm{Int}\, N_i]$. Then V is a closed neighborhood of X in K_0. Hence, there exist $\lambda_0 \in \Lambda$, $\delta_0 \in \Delta$, and a closed neighborhood U of X such that if $\lambda \geqslant \lambda_0$ and $\delta \geqslant \delta_0$, then $g_\delta f_\lambda | U \underset{p}{\simeq} i_U$ in V. Since $\underline{g}$ is a proper fundamental net, there exist a closed neighborhood U_0 of Y and $\delta_1 \in \Delta$ such that $\delta_1 \geqslant \delta_0$ and $g_{\delta_1}(U_0) \subset V$. We may further suppose that U_0 is simply connected. Since $\underline{f}$ is a proper fundamental net, there exists $\lambda_1 \geqslant \lambda_0$ such that $f_{\lambda_1}(X) \subset \mathrm{Int}\, U_0$. Let W be a neighborhood of $(0, -1) \in X$ such that $W \subset U$ and $f_{\lambda_1}(W) \subset \mathrm{Int}\, U_0$. Let k be a positive integer such that the segment A joining $\left(\frac{2}{4k-1}, -1\right)$ and $\left(\frac{2}{4k+3}, -1\right)$ lies in W and let J denote the (unique) simple closed curve lying in $X \cup A$.

Now $J \subset U$, and hence $g_{\delta_1} f_{\lambda_1} | J \underset{p}{\simeq} i_K | J$ in V. But $f_{\lambda_1}(J) \subset U_0$, $g_{\delta_1}(U_0) \subset V$, and U_0 is simply connected. This implies that $g_{\delta_1} f_{\lambda_1} | J$ is nullhomotopic in V. This is a contradiction, since J is essential in V.

We note that Example 4.14 shows that the converse of each of Corollaries 4.6, 4.7, 4.8, 4.9, 4.10, and 4.11 is false.

5. Non-separable spaces. We first generalize the definitions of fundamental nets and related notions by removing the condition that all sets involved be contained in the one space $K = H - \{\omega\}$ used heretofore; this is necessary because, of course, non-separable spaces cannot be embedded in K. The generalizations and terminology are entirely analogous to those used for fundamental sequences in [2] (cf. also [5]).

If X and Y are closed subsets of spaces P and Q, respectively, and $\underline{f} = \{f_\lambda \mid \lambda \in \Lambda\}$ is a net of maps $f_\lambda\colon P \to Q$, then the ordered triple $(\underline{f}, X, Y)$ is called a *(proper) fundamental net from X to Y in (P, Q)* if for every closed neighborhood V of Y in Q, there exist a closed neighborhood U of X in P and an index $\lambda_0 \in \Lambda$ such that for $\lambda \geqslant \lambda_0$, $f_\lambda|U \underset{p}{\cong} f_{\lambda_0}|U$ in V. The definitions of (properly) homotopic fundamental nets from X to Y in (P, Q) and of the composition of a fundamental net from X to Y in (P, Q) with a fundamental net from Y to Z in (Q, R) are the obvious ones; the degenerate fundamental net $(\{i_P\}, X, X)$ is called the *identity fundamental net on X in P*, and will be denoted by $\underline{i}_{X,P}$. We will use the phrase "$\underline{f}$ is a fundamental net from X to Y in (P, Q)", or "$\underline{f}\colon X \to Y$ in (P, Q)", to indicate that $(\underline{f}, X, Y)$ is a proper fundamental net from X to Y in (P, Q).

If X and Y are closed subsets of P and Q, respectively, and there exist fundamental nets $\underline{f}\colon X \to Y$ in (P, Q) and $\underline{g}\colon Y \to X$ in (Q, P) such that $\underline{g}\underline{f} \underset{p}{\cong} \underline{i}_{X,P}$ and $\underline{f}\underline{g} \underset{p}{\cong} \underline{i}_{Y,Q}$, then X and Y are said to be *properly fundamentally equivalent in* (P, Q); if only the relation $\underline{f}\underline{g} \underset{p}{\cong} \underline{i}_{Y,Q}$ is assumed, then X is said to *properly fundamentally dominate Y in (P, Q)*. These relations are denoted by "$X \underset{pF}{\cong} Y$ in (P, Q)" and "$X \underset{pF}{\geqslant} Y$ in (P, Q)", respectively.

It is clear (cf. Lemma 3.2) that if X and Y are closed subsets of locally compact metrizable spaces P and Q, respectively, and Q is an AR, then every proper map $f\colon X \to Y$ generates a proper fundamental net $\underline{f}\colon X \to Y$ in (P, Q). It therefore follows, by an argument identical to that given for Theorem 3.1 of [5], that *if M, N, M', N' are locally compact AR's and X, Y, X', Y' are closed subsets of M, N, M', N', respectively, such that $X \approx X'$ and $Y \approx Y'$, then $X \underset{pF}{\geqslant} Y$ in (M, N) if and only if $X' \underset{pF}{\geqslant} Y'$ in (M', N') and $X \underset{pF}{\cong} Y$ in (M, N) if and only if $X' \underset{pF}{\cong} Y'$ in (M', N').* Since a locally compact AR is necessarily separable, we need a slightly more general result in order to handle arbitrary locally compact metrizable spaces. We first prove a useful theorem on partitions of spaces.

A collection $\{X_\alpha \mid \alpha \in A\}$ of subsets of a space P is said to be *discrete in P* if for each $p \in P$, there is a neighborhood U of p in P such that there is at most one α in A for which $U \cap X_\alpha \neq \emptyset$. By a *partition* of a space P we will mean a collection $\{P_\alpha \mid \alpha \in A\}$ of non-empty subsets of P such that $P = \bigcup_{\alpha \in A} P_\alpha$ and such that $\{P_\alpha \mid \alpha \in A\}$ is discrete in P; note that each P_α is necessarily open and closed in P, and hence P is the *free sum* (topological sum, disjoint union) of the subspaces P_α. We note also that if X is a closed subset of P and $\{X_\alpha \mid \alpha \in A\}$ is a partition of X, then $\{X_\alpha \mid \alpha \in A\}$ is discrete in P.

5.1. LEMMA. *Suppose A and C are metrizable spaces and $f, g\colon A \to C$ are maps which are properly homotopic in a closed subset B of C. If A_1 is a closed subset of A and B_1 is an open and closed subset of B such that $f(A_1) \subset B_1$, then $f|A_1 \underset{p}{\cong} g|A_1$ in B_1.*

Proof. Let $\varphi\colon A \times I \to B$ be a proper map such that for each $a \in A$, $\varphi(a, 0) = f(a)$ and $\varphi(a, 1) = g(a)$. If $a_1 \in A_1$, then $\varphi(\{a_1\} \times I)$ is a connected subset of B containing the point $\varphi(a_1, 0) = f(a_1)$ of B_1, and since B_1 is open and closed in B, it follows that $\varphi(\{a_1\} \times I) \subset B_1$. Hence $\varphi(A_1 \times I) \subset B_1$ and since $A_1 \times I$ is closed in $A \times I$ and B_1 is closed in B, the map $\psi\colon A_1 \times I \to B_1$ defined by $\psi(x, t) = \varphi(x, t)$ for each $x \in A_1$, $t \in I$ is a proper map. Since $\psi(x, 0) = \varphi(x, 0) = f(x)$ and $\psi(x, 1) = \varphi(x, 1) = g(x)$ for each $x \in A_1$, ψ is a proper homotopy joining $f|A_1$ to $g|A_1$ in B_1.

5.2. THEOREM. *Suppose P and Q are locally compact metrizable spaces, X and Y are closed subsets of P and Q, respectively, and $\underline{f}\colon X \to Y$ in (P, Q), $\underline{g}\colon Y \to X$ in (Q, P) are fundamental nets such that $\underline{g}\underline{f} \underset{p}{\cong} \underline{i}_{X,P}$. If $\{X_\alpha \mid \alpha \in A\}$ is a partition of X, then there exists a partition $\{Y_\alpha \mid \alpha \in A\}$ of Y such that for each $\alpha \in A$, $\underline{f}\colon X_\alpha \to Y_\alpha$ in (P, Q), $\underline{g}\colon Y_\alpha \to X_\alpha$ in (Q, P) and $\underline{g}\underline{f} \underset{p}{\cong} \underline{i}_{X_\alpha,P}$. Moreover, if $\underline{f}\underline{g} \underset{p}{\cong} \underline{i}_{Y,Q}$, then also for each α, $\underline{f}\underline{g} \underset{p}{\cong} \underline{i}_{Y_\alpha,Q}$.*

Proof. Let $\underline{f} = \{f_\lambda \mid \lambda \in \Lambda\}$ and $\underline{g} = \{g_\delta \mid \delta \in \Delta\}$.

Since $\{X_\alpha \mid \alpha \in A\}$ is a discrete collection of closed subsets of P, there exists (see [9], p. 308) a discrete collection $\{V_\alpha \mid \alpha \in A\}$ of subsets of P such that for each $\alpha \in A$, V_α is a closed neighborhood of X_α. Let $V = \bigcup_{\alpha \in A} V_\alpha$.

Since $\{V_\alpha \mid \alpha \in A\}$ is discrete in P, it follows that V is a closed neighborhood of X in P. Hence since $\underline{g}$ is a fundamental net from Y to X in (Q, P), there exist a closed neighborhood W of Y in Q and an index δ_1 in Δ such that for all $\delta \geqslant \delta_1$,

$$g_\delta|W \underset{p}{\cong} g_{\delta_1}|W \quad \text{in } V.$$

Since $\underline{f}$ is a fundamental net from X to Y in (P, Q), there exist a closed neighborhood U_1 of X in P and an index $\lambda_1 \in \Lambda$ such that for all $\lambda \geqslant \lambda_1$,

$$f_\lambda|U_1 \underset{p}{\cong} f_{\lambda_1}|U_1 \quad \text{in } W.$$

Since $\underline{g}\underline{f} \underset{p}{\cong} \underline{i}_{X,P}$, there exist a closed neighborhood U of X in P and indices $\lambda_0 \in \Lambda$, $\delta_0 \in \Delta$ such that for all $(\lambda, \delta) \geqslant (\lambda_0, \delta_0)$,

$$g_\delta f_\lambda|U \underset{p}{\cong} i_U \quad \text{in } V;$$

clearly it may also be required that $U \subset U_1 \cap V$ and that $\lambda_0 \geqslant \lambda_1$ and $\delta_0 \geqslant \delta_1$.

For each $\alpha \in A$, let $U_\alpha = U \cap V_\alpha$. Since $U_\alpha \subset V_\alpha$ and V_α is open and closed in V, it follows from Lemma 5.1 that for each $\alpha \in A$ and each $(\lambda, \delta) \geqslant (\lambda_0, \delta_0)$ in $\Lambda \times \Delta$,

$$g_\delta f_\lambda | U_\alpha \underset{p}{\cong} i_{U_\alpha} \quad \text{in } V_\alpha .$$

For each $\alpha \in A$, let $W_\alpha = W \cap g_{\delta_0}^{-1}(V_\alpha)$ and let $Y_\alpha = Y \cap W_\alpha$. Since $\{V_\alpha |\ \alpha \in A\}$ is discrete in P, $\{g_{\delta_0}^{-1}(V_\alpha)|\ \alpha \in A\}$ is discrete in Q; hence also $\{W_\alpha |\ \alpha \in A\}$ is discrete in Q. Since $W = \bigcup_{\alpha \in A} W_\alpha$ and W is a closed neighborhood of Y in Q, it follows that W_α is a closed neighborhood of Y_α in Q, for all $\alpha \in A$. Moreover, since $\delta_0 \geqslant \delta_1$, for all $\delta \geqslant \delta_0$, $g_\delta | W \underset{p}{\cong} g_{\delta_0} | W$ in V and since $g_{\delta_0}(W_\alpha) \subset V_\alpha$ it follows by Lemma 5.1 that $g_\delta | W_\alpha \underset{p}{\cong} g_{\delta_0} | W_\alpha$ in V_α, for each $\alpha \in A$ and all $\delta \geqslant \delta_0$. In particular, then, $g_\delta(W_\alpha) \subset V_\alpha$ for $\alpha \in A$, $\delta \geqslant \delta_0$.

Suppose $x \in U_\alpha$ and $\lambda \in \Lambda$, $\lambda \geqslant \lambda_0$. Since $\lambda_0 \geqslant \lambda_1$ and $U \subset U_1$, $f_\lambda(U) \subset W$, so $f_\lambda(x) \in W$. Since

$$g_{\delta_0} f_\lambda | U_\alpha \underset{p}{\cong} i_{U_\alpha} \quad \text{in } V_\alpha ,$$

$g_{\delta_0}(f_\lambda(x)) \in V_\alpha$ and hence $f_\lambda(x) \in g_{\delta_0}^{-1}(V_\alpha)$. Thus for each $x \in U_\alpha$, $f_\lambda(x) \in W \cap g_{\delta_0}^{-1}(V_\alpha) = W_\alpha$, so $f_\lambda(U_\alpha) \subset W_\alpha$. In particular, $f_\lambda(X_\alpha) \subset W_\alpha$ for each $\alpha \in A$, $\lambda \geqslant \lambda_0$.

Now suppose that for some $\alpha_0 \in A$, $Y_{\alpha_0} = \emptyset$. Let $W' = W - W_{\alpha_0}$. Then W' is a closed neighborhood of Y in Q and since $\underline{f}$ is a fundamental net from X to Y in (P, Q), there exist a closed neighborhood U' of X in P and a $\lambda \geqslant \lambda_0$ such that $f_\lambda(U') \subset W'$. Since $\lambda \geqslant \lambda_0$, $f_\lambda(X_{\alpha_0}) \subset W_{\alpha_0}$, and this is a contradiction since $W_{\alpha_0} \cap W' = \emptyset$ and X_{α_0} is a non-empty subset of U'. Hence for each $\alpha \in A$, $Y_\alpha \neq \emptyset$. It follows that $\{Y_\alpha |\ \alpha \in A\}$ is a partition of Y.

Suppose $\alpha_0 \in A$ and W'_{α_0} is a closed neighborhood of Y_{α_0} in Q, with $W'_{\alpha_0} \subset W_{\alpha_0}$. Let $W' = W'_{\alpha_0} \cup (W - W_{\alpha_0})$. Since $\underline{f}$ is a fundamental net from X to Y in (P, Q), there exist a closed neighborhood U' of X in P and a $\lambda'_0 \in \Lambda$ such that $U' \subset U$, $\lambda'_0 \geqslant \lambda_0$ and for all $\lambda \geqslant \lambda'_0$,

$$f_\lambda | U' \underset{p}{\cong} f_{\lambda'_0} | U' \quad \text{in } W'.$$

Let $U'_{\alpha_0} = U' \cap U_{\alpha_0}$. Since $\lambda'_0 \geqslant \lambda_0$, for all $\lambda \geqslant \lambda'_0$, $f_\lambda(U_{\alpha_0}) \subset W_{\alpha_0}$ and hence $f_\lambda(U'_{\alpha_0}) \subset W_{\alpha_0}$; since also $f_\lambda(U'_{\alpha_0}) \subset W'$, it follows that $f_\lambda(U'_{\alpha_0}) \subset W'_{\alpha_0}$. Hence by Lemma 5.1, for all $\lambda \geqslant \lambda'_0$,

$$f_\lambda | U'_{\alpha_0} \underset{p}{\cong} f_{\lambda'_0} | U'_{\alpha_0} \quad \text{in } W'_{\alpha_0} .$$

Thus $\underline{f}$ is a fundamental net from X_{α_0} to Y_{α_0} in (P, Q).

Similar arguments show that for $\alpha_0 \in A$, $\underline{g}$ is a fundamental net from Y_{α_0} to X_{α_0}, and that $\underline{g}\underline{f} \underset{p}{\cong} \underline{i}_{X_{\alpha_0}, P}$.

In case it is also true that $\underline{f}\underline{g} \underset{p}{\cong} \underline{i}_{Y, Q}$, the above proof may be applied to the partition $\{Y_\alpha |\ \alpha \in A\}$ of Y to obtain a partition $\{X'_\alpha |\ \alpha \in A\}$ of X such that for each $\alpha \in A$, $\underline{g}$: $Y_\alpha \to X'_\alpha$ in (Q, P), $\underline{f}$: $X'_\alpha \to Y_\alpha$ in (P, Q) and $\underline{f}\underline{g} \underset{p}{\cong} \underline{i}_{Y_\alpha, Q}$. (From the fact that $\underline{g}$ is a fundamental net from Y_α to X_α and also from Y_α to X'_α in (Q, P), it easily follows that $X'_\alpha = X_\alpha$.)

5.3. Lemma. *If X and Y are closed subsets of locally compact metrizable spaces P and Q, respectively, and there is a fundamental net $\underline{f} = \{f_\lambda |\ \lambda \in \Lambda\}$ from X to Y in (P, Q), then X is separable if Y is separable.*

Proof. Suppose Y is separable. Then since Q is locally compact, there is a closed neighborhood V of Y in Q which is the union of a countable collection $\{V_i\}_{i=1}^\infty$ of compact sets. Since $\underline{f}$ is a proper fundamental net from X to Y in (P, Q), there is a $\lambda_0 \in \Lambda$ such that $f_{\lambda_0}(X) \subset V$ and $f_{\lambda_0}|X$: $X \to Q$ is a proper map. If $X_i = (f_{\lambda_0}|X)^{-1}(V_i)$, $i = 1, 2, \ldots$, then $X = \bigcup_{i=1}^{\infty} X_i$ and each X_i is compact, so X is separable.

The fact (see, e.g., [10], p. 241, Th. 7.3) that any locally compact metrizable space can be partitioned into separable subspaces suggests the following extension of our definition of proper shape for separable, locally compact metrizable spaces to arbitrary locally compact metrizable spaces.

5.4. Definition. Two locally compact metrizable spaces X, Y will be said to have the *same proper shape*, denoted by $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$, provided there exist partitions $\{X_\alpha |\ \alpha \in A\}$ and $\{Y_\alpha |\ \alpha \in A\}$ of X and Y, respectively, such that for each $\alpha \in A$, X_α and Y_α are separable and $\mathrm{Sh}_p X_\alpha = \mathrm{Sh}_p Y_\alpha$.

In order that this should be a reasonable definition, it is necessary, of course, to show that the relation "X and Y have the same proper shape" is an equivalence relation on the class of all locally compact metrizable spaces. Since this relation is clearly reflexive and symmetric, it is only necessary to show that $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$ and $\mathrm{Sh}_p Y = \mathrm{Sh}_p Z$ imply $\mathrm{Sh}_p X = \mathrm{Sh}_p Z$. This will require several preliminary results.

5.5 Lemma. *If X and Y are locally compact metrizable spaces having the same proper shape, then there exist locally compact ANR's P and Q containing X and Y, respectively, as closed subsets such that $X \underset{pF}{\cong} Y$ in (P, Q).*

Proof. Let $\{X_\alpha |\ \alpha \in A\}$ and $\{Y_\alpha |\ \alpha \in A\}$ be partitions of X and Y, respectively, into separable subspaces such that for each $\alpha \in A$, $\mathrm{Sh}_p X_\alpha = \mathrm{Sh}_p Y_\alpha$. For each $\alpha \in A$, there exist locally compact spaces P_α, Q_α, each homeomorphic to $K = H - \{\omega\}$, containing X_α, Y_α, respectively, as closed subsets, and fundamental nets $\underline{f}^\alpha$: $X_\alpha \to Y_\alpha$ in (P_α, Q_α), $\underline{g}^\alpha$: $Y_\alpha \to X_\alpha$ in (Q_α, P_α) such that $\underline{g}^\alpha\underline{f}^\alpha \underset{p}{\simeq} \underline{i}_{X_\alpha, P_\alpha}$ and $\underline{f}^\alpha\underline{g}^\alpha \underset{p}{\simeq} \underline{i}_{Y_\alpha, Q_\alpha}$. It may be assumed that $P_\alpha \cap P_\beta = \emptyset$ and $Q_\alpha \cap Q_\beta = \emptyset$ for $\alpha \neq \beta$. Let

$$P = \bigoplus_{\alpha \in A} P_\alpha \quad \text{and} \quad Q = \bigoplus_{\alpha \in A} Q_\alpha$$

be the free sums of $\{P_\alpha|\ \alpha \in A\}$ and $\{Q_\alpha|\ \alpha \in A\}$. It follows immediately from the definition of ANR's that each of P and Q is an ANR. For each $\alpha \in A$, let $\underline{f}^\alpha = \{f^\alpha_\lambda|\ \lambda \in \Lambda_\alpha\}$ and $\underline{g}^\alpha = \{g^\alpha_\delta|\ \delta \in \Delta_\alpha\}$. Let $\Lambda = \prod_{\alpha \in A} \Lambda_\alpha$, with $\lambda = \{\lambda_\alpha|\ \alpha \in A\} \geqslant \lambda' = \{\lambda'_\alpha|\ \alpha \in A\}$ if and only if $\lambda_\alpha \geqslant \lambda'_\alpha$ for each $\alpha \in A$; clearly Λ is a directed set with respect to this order relation. Let $\Delta = \prod_{\alpha \in A} \Delta_\alpha$, with the directing relation in Δ defined similarly. For each $\lambda = \{\lambda_\alpha|\ \alpha \in A\} \in \Lambda$, let f_λ: $P \to Q$ be the *combination* ([11], p. 69) of the maps $f^\alpha_{\lambda_\alpha}$: $P_\alpha \to Q_\alpha$; i.e., for each $x \in P$, $f_\lambda(x) = f^\alpha_{\lambda_\alpha}(x)$ if $x \in P_\alpha$. Similarly, for $\delta = \{\delta_\alpha|\ \alpha \in A\}$, let g_δ: $Q \to P$ be the combination of the maps $g^\alpha_{\delta_\alpha}$: $Q_\alpha \to P_\alpha$. It follows easily that $\underline{f} = \{f_\lambda|\ \lambda \in \Lambda\}$ and $\underline{g} = \{g_\delta|\ \delta \in \Delta\}$ are proper fundamental nets from X to Y in (P, Q) and from Y to X in (Q, P) and that $\underline{g}\underline{f} \underset{p}{\simeq} \underline{i}_{X,P}$ and $\underline{f}\underline{g} \underset{p}{\simeq} \underline{i}_{Y,Q}$. Hence $X \underset{pF}{\simeq} Y$ in (P, Q).

Suppose X is a closed subset of P, Y is a closed subset of Q and $\underline{f} = \{f_\lambda|\ \lambda \in \Lambda\}$ is a proper fundamental net from X to Y in (P, Q). If P is a closed subset of M and Q is a closed subset of N, then a proper fundamental net $\hat{\underline{f}}$: $X \to Y$ in (M, N) is said to be an *extension* of $\underline{f}$ if $\hat{\underline{f}} = \{\hat{f}_\lambda|\ \lambda \in \Lambda\}$ and for each $\lambda \in \Lambda$, $\hat{f}_\lambda$: $M \to N$ agrees with f_λ: $P \to Q$ at every point $x \in P$ (cf. [7], p. 56). Although the following results hold in somewhat more general settings, for simplicity we adopt the STANDING HYPOTHESIS, for the next two lemmas, that *X and Y are closed subsets of P and Q, P and Q are locally compact* ANR's, *and M and N are locally compact* AR's *containing P and Q, respectively, as closed subsets.*

5.6. LEMMA. *Every proper fundamental net $\underline{f}$: $X \to Y$ in (P, Q) can be extended to a proper fundamental net $\hat{\underline{f}}$: $X \to Y$ in (M, N).*

Proof. Let $\underline{f} = \{f_\lambda|\ \lambda \in \Lambda\}$. If j: $Q \to N$ is the inclusion map and $j\underline{f} = \{jf_\lambda|\ \lambda \in \Lambda\}$, then $j\underline{f}$ is a proper fundamental net from X to Y in (P, N) and any proper fundamental net from X to Y in (M, N) which is an extension of $j\underline{f}$ is also an extension of $\underline{f}$. Hence it may be assumed that $Q = N$.

By Lemma 3.7, since P is an ANR, there exist a closed neighborhood W of P in M and a proper retraction r: $W \to P$ of W to P. For each $\lambda \in \Lambda$, let $f'_\lambda = f_\lambda r$: $W \to N$. Since M is an AR, each f'_λ can be extended to a map $\hat{f}_\lambda$: $M \to N$. Let $\hat{\underline{f}} = \{\hat{f}_\lambda|\ \lambda \in \Lambda\}$.

Suppose V is a closed neighborhood of Y in Q. Since $\underline{f}$ is a proper fundamental net from X to Y in (P, N), there exist a closed neighborhood U_1 of X in P and a $\lambda_0 \in \Lambda$ such that for $\lambda \geqslant \lambda_0$,

$$f_\lambda|U_1 \underset{p}{\simeq} f_{\lambda_0}|U_1 \quad \text{in } V.$$

Let $U = r^{-1}(U_1)$. Then U is a closed neighborhood of X in W, and hence in M.

Suppose $\lambda \geqslant \lambda_0$ and let φ: $U_1 \times I \to V$ be a proper map such that $\varphi(x, 0) = f_{\lambda_0}(x)$ and $\varphi(x, 1) = f_\lambda(x)$, for each $x \in U_1$. Define ψ: $U \times I \to V$ by setting $\psi(x, t) = \varphi(r(x), t)$ for each $(x, t) \in U \times I$. Since r and φ are proper maps, so is ψ. Moreover, for each $x \in U$, $\psi(x, 0) = \varphi(r(x), 0) = f_{\lambda_0}(r(x)) = f'_{\lambda_0}(x) = \hat{f}_{\lambda_0}(x)$ and, similarly, $\psi(x, 1) = \hat{f}_\lambda(x)$. Hence $\hat{f}_\lambda|U \underset{p}{\simeq} \hat{f}_{\lambda_0}|U$ in V, and it follows that $\hat{\underline{f}}$: $X \to Y$ in (M, N).

5.7 LEMMA. *If $\underline{f}$: $X \to Y$ in (P, Q) and $\underline{g}$: $X \to Y$ in (P, Q) are properly homotopic proper fundamental nets and $\hat{\underline{f}}$: $X \to Y$ in (M, N), $\hat{\underline{g}}$: $X \to Y$ in (M, N) are extensions of $\underline{f}$ and $\underline{g}$, respectively, then $\hat{\underline{f}} \underset{p}{\simeq} \hat{\underline{g}}$.*

Proof. If j: $Q \to N$ is the inclusion map, then $j\underline{f}$ and $j\underline{g}$ (defined as in the proof of Lemma 5.6) are proper fundamental nets from X to Y in (P, N), $j\underline{f} \underset{p}{\simeq} j\underline{g}$, and $\hat{\underline{f}}, \hat{\underline{g}}$ are extensions of $j\underline{f}, j\underline{g}$, respectively. Hence it may be assumed that $Q = N$.

Let $\underline{f} = \{f_\lambda|\ \lambda \in \Lambda\}$, $\underline{g} = \{g_\delta|\ \delta \in \Delta\}$, $\hat{\underline{f}} = \{\hat{f}_\lambda|\ \lambda \in \Lambda\}$ and $\hat{\underline{g}} = \{\hat{g}_\delta|\ \delta \in \Delta\}$, with $\hat{f}_\lambda$: $M \to N$ an extension of f_λ: $P \to N$ and $\hat{g}_\delta$: $M \to N$ an extension of g_δ: $P \to N$ for all $\lambda \in \Lambda$, $\delta \in \Delta$.

Suppose V is a closed neighborhood of Y in N and let V_1 be a closed neighborhood of Y in N such that $V_1 \subset \operatorname{Int} V$. Since $\underline{f}$ and $\underline{g}$ are properly homotopic proper fundamental nets from X to Y in (P, N), there exist a closed neighborhood W of X in P and indices $\lambda_1 \in \Lambda$, $\delta_1 \in \Delta$ such that for $\lambda \geqslant \lambda_1$, $\delta \geqslant \delta_1$, $f_\lambda|W \underset{p}{\simeq} f_{\lambda_1}|W$ in V, $g_\delta|W \underset{p}{\simeq} g_{\delta_1}|W$ in V and $f_\lambda|W \underset{p}{\simeq} g_\delta|W$ in V_1. Since $\hat{\underline{f}}$ and $\hat{\underline{g}}$ are proper fundamental nets from X to Y in (M, N), there exist a closed neighborhood U_1 of X in M and indices $\lambda_0 \in \Lambda$, $\delta_0 \in \Delta$ such that $U_1 \cap P \subset W$, $\lambda_0 \geqslant \lambda_1$, $\delta_0 \geqslant \delta_1$ and for $\lambda \geqslant \lambda_0$, $\delta \geqslant \delta_0$,

$$\hat{f}_\lambda|U_1 \underset{p}{\simeq} \hat{f}_{\lambda_0}|U_1 \quad \text{in } V_1 \quad \text{and} \quad \hat{g}_\delta|U_1 \underset{p}{\simeq} \hat{g}_{\delta_0}|U_1 \quad \text{in } V_1.$$

Since $\lambda_0 \geqslant \lambda_1$ and $\delta_0 \geqslant \delta_1$, $f_{\lambda_0}|X \underset{p}{\simeq} g_{\delta_0}|X$ in V_1. Hence by Lemma 3.4, since $\hat{f}_{\lambda_0}$ and $\hat{g}_{\delta_0}$ are extensions of $f_{\lambda_0}|X$ and $g_{\delta_0}|X$, respectively, and V is a closed neighborhood of V_1, there is a closed neighborhood U of X in M such that $U \subset U_1$ and $\hat{f}_{\lambda_0}|U \underset{p}{\cong} \hat{g}_{\delta_0}|U$ in V. Then for $\lambda \geqslant \lambda_0$, $\delta \geqslant \delta_0$,

$$f_\lambda|U \underset{p}{\cong} \hat{f}_{\lambda_0}|U \underset{p}{\cong} \hat{g}_{\delta_0}|U \underset{p}{\cong} \hat{g}_\delta|U \quad \text{in } V,$$

so $\underline{\hat{f}} \underset{p}{\cong} \underline{\hat{g}}$.

5.8. LEMMA. *If X and Y are locally compact metrizable spaces having the same proper shape and $\{X_a|\ a \in A\}$ is a partition of X into separable subspaces, then there is a partition $\{Y_a|\ a \in A\}$ of Y into separable subspaces such that for each $a \in A$,* $\mathrm{Sh}_p X_a = \mathrm{Sh}_p Y_a$.

Proof. By Lemma 5.5, there exist locally compact ANR's P and Q containing X and Y, respectively, as closed subsets and proper fundamental nets $\underline{f}\colon X \to Y$ in (P, Q) and $\underline{g}\colon Y \to X$ in (Q, P) such that $\underline{gf} \underset{p}{\cong} \underline{i}_{X,P}$ and $\underline{fg} \underset{p}{\cong} \underline{i}_{Y,Q}$. By Lemma 5.2, there is a partition $\{Y_a|\ a \in A\}$ of Y such that for each $a \in A$, $\underline{f}$ is a proper fundamental net from X_a to Y_a in (P, Q), $\underline{g}$ is a proper fundamental net from Y_a to X_a in (Q, P),

$$\underline{gf} \underset{p}{\cong} \underline{i}_{X_a,P} \quad \text{and} \quad \underline{fg} \underset{p}{\cong} \underline{i}_{Y_a,Q}.$$

Consider a fixed $a \in A$, and let P_a denote the union of all components of P which intersect X_a and let Q_a be the union of all components of Q which intersect Y_a. By hypothesis, X_a is separable and hence by Lemma 5.3, since $\underline{g}\colon Y_a \to X_a$ in (Q, P), Y_a is also separable. Hence there are only a countable number of components of P which intersect X_a and only a countable number of components of Q which intersect Y_a, so P_a and Q_a are separable. Therefore, by Lemma 3.1, there exist locally compact absolute retracts M_a and N_a containing P_a and Q_a, respectively, as closed subsets.

By Lemma 5.6, $\underline{f}$ can be extended to a proper fundamental net $\underline{\hat{f}}\colon X_a \to Y_a$ in (M_a, N_a) and $\underline{g}$ can be extended to a proper fundamental net $\underline{\hat{g}}\colon Y_a \to X_a$ in (N_a, M_a). Then $\underline{\hat{g}\hat{f}}\colon X_a \to X_a$ in (M_a, M_a) is an extension of $\underline{gf}\colon X_a \to X_a$ in (P_a, P_a) and since $\underline{gf} \underset{p}{\cong} \underline{i}_{X_a,P}$ it follows by Lemma 5.7 that $\underline{\hat{g}\hat{f}} \underset{p}{\cong} \underline{i}_{X_a,M_a}$. Similarly, $\underline{\hat{f}\hat{g}} \underset{p}{\cong} \underline{i}_{Y_a,N_a}$ and hence $\mathrm{Sh}_p X_a = \mathrm{Sh}_p Y_a$.

5.9. THEOREM. *If X, Y and Z are locally compact metrizable spaces such that* $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$ *and* $\mathrm{Sh}_p Y = \mathrm{Sh}_p Z$, *then* $\mathrm{Sh}_p X = \mathrm{Sh}_p Z$.

Proof. Let $\{X'_a|\ a \in A\}$ and $\{Y'_a|\ a \in A\}$ be partitions of X and Y, respectively, into separable subspaces such that $\mathrm{Sh}_p X'_a = \mathrm{Sh}_p Y'_a$ for each $a \in A$, and let $\{Y''_\beta|\ \beta \in B\}$ and $\{Z'_\beta|\ \beta \in B\}$ be partitions of Y and Z, respectively, into separable subspaces such that $\mathrm{Sh}_p Y''_\beta = \mathrm{Sh}_p Z'_\beta$ for each $\beta \in B$. Let $G = \{(a, \beta) \in A \times B|\ Y'_a \cap Y''_\beta \neq \emptyset\}$ and for each $\gamma = (a, \beta) \in G$, let $Y_\gamma = Y'_a \cap Y''_\beta$. Then $\{Y_\gamma|\ \gamma \in G\}$ is a partition of Y into separable subspaces and hence by Lemma 5.8, there exist partitions $\{X_\gamma|\ \gamma \in G\}$ and $\{Z_\gamma|\ \gamma \in G\}$ of X and Z, respectively, such that for each $\gamma \in G$,

$$\mathrm{Sh}_p X_\gamma = \mathrm{Sh}_p Y_\gamma \quad \text{and} \quad \mathrm{Sh}_p Z_\gamma = \mathrm{Sh}_p Y_\gamma.$$

Hence for each $\gamma \in G$, $\mathrm{Sh}_p X_\gamma = \mathrm{Sh}_p Z_\gamma$ and therefore $\mathrm{Sh}_p X = \mathrm{Sh}_p Z$.

Thus the relation $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$, as given in Definition 5.4, is an equivalence relation on the class of all locally compact metrizable spaces, as desired. Clearly one can define the relation $\mathrm{Sh}_p X \geqslant \mathrm{Sh}_p Y$ in an analogous fashion and, using appropriate modifications of Lemmas 5.5 and 5.8, show that $\mathrm{Sh}_p X \geqslant \mathrm{Sh}_p Y$ and $\mathrm{Sh}_p Y \geqslant \mathrm{Sh}_p Z$ imply $\mathrm{Sh}_p X \geqslant \mathrm{Sh}_p Z$.

5.10. THEOREM. *If X and Y are locally compact metrizable spaces and X is properly homotopically equivalent to Y, then* $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$.

Proof. Let $f\colon X \to Y$, $g\colon Y \to X$ be maps such that $gf \underset{p}{\cong} i_X$ and $fg \underset{p}{\cong} i_Y$. Let $\{X_a|\ a \in A\}$ be a partition of X into separable subspaces and for each $a \in A$, let $Y_a = g^{-1}(X_a)$. Since X_a is open and closed in X, Y_a is open and closed in Y. Since $gf \underset{p}{\cong} i_X$ and X_a is open and closed in X, $gf(X_a) \subset X_a$ and hence $f(X_a) \subset g^{-1}(X_a) = Y_a$. Hence since $X_a \neq \emptyset$, $Y_a \neq \emptyset$ for each $a \in A$. Thus $\{Y_a|\ a \in A\}$ is a partition of Y. Since X_a is separable and therefore σ-compact and $g|Y_a\colon Y_a \to X$ is a proper map with $g(Y_a) \subset X_a$, it easily follows that $Y_a = g^{-1}(X_a)$ is σ-compact and hence separable. If $f_a\colon X_a \to Y_a$ is defined by $f_a(x) = f(x)$ for $x \in X_a$ and $g_a\colon Y_a \to X_a$ is defined by $g_a(y) = g(y)$ for $y \in Y_a$, then since X_a and Y_a are open and closed in X and Y, respectively, and $gf \underset{p}{\cong} i_X$, $fg \underset{p}{\cong} i_Y$, it follows from Lemma 5.1 that $g_a f_a \underset{p}{\cong} i_{X_a}$ and $f_a g_a \underset{p}{\cong} i_{Y_a}$. Hence, for each $a \in A$, X_a and Y_a are properly homotopically equivalent and therefore by Theorem 3.10, $\mathrm{Sh}_p X_a = \mathrm{Sh}_p Y_a$. Then by Definition 5.4, $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$.

5.11. THEOREM. *If X and Y are locally compact ANR's and* $\mathrm{Sh}_p X = \mathrm{Sh}_p Y$, *then X and Y are properly homotopically equivalent.*

Proof. Let $\{X_a|\ a \in A\}$ and $\{Y_a|\ a \in A\}$ be partitions of X and Y, respectively, into separable subspaces such that for each $a \in A$, $\mathrm{Sh}_p X_a = \mathrm{Sh}_p Y_a$. Since X and Y are ANR's and X_a and Y_a are open subsets of X and Y, respectively, X_a and Y_a are ANR's and hence, by Theorem 3.12, X_a and Y_a are properly homotopically equivalent. For each $a \in A$, let $f_a\colon X_a \to Y_a$ and $g_a\colon Y_a \to X_a$ be maps such that $g_a f_a \underset{p}{\cong} i_{X_a}$ and $f_a g_a \underset{p}{\cong} i_{Y_a}$. Let $f\colon X \to Y$, $g\colon Y \to X$ be the combinations of $\{f_a|\ a \in A\}$ and $\{g_a|\ a \in A\}$, respectively. It readily follows that $gf \underset{p}{\cong} i_X$ and $fg \underset{p}{\cong} i_Y$, and hence X and Y are properly homotopically equivalent.

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On level sets of Darboux functions

by

J. S. Lipiński (Gdańsk)

Abstract. K. M. Garg [2] has found the necessary conditions that a set of real numbers be the set of all points y such that the level sets $f^{-1}(y)$ are single point sets, dense in the selves sets, closed sets, connected sets and perfect sets, where f is a free choice Darboux function. The aim of this paper is to give a proof that all Garg's conditions are not only necessary but also sufficient.

Let f be a real function of a real variable. The set $\{x: f(x) = y\} = f^{-1}(y)$ will be referred to as the level set of f corresponding to the value y. Generally we are not able to draw conclusions as to the properties of functions from the properties of their level sets. E.g., it is well known [5] that all level sets may be closed sets, even one point sets, whereas the function itself is not measurable in the sense of Lebesgue. However, under certain additional stipulations as to the function there follow from an appropriate regularity of sufficiently many level sets strong conclusions about the function itself. E.g. if the function f possesses the Darboux property, and the set of those values y for which the level sets $f^{-1}(y)$ are closed is dense, then f is continuous [3].

Special families of level sets of continuous functions have been discussed in [4], [1] and [2].

Let us denote by 1 the family of all single-point sets on the real axis. Furthermore, let us denote by d the family of all sets dense in themselves, by k the family of all closed sets, by c the family of all connected sets, by p the family of all perfect sets and finally by ∞ the family of all infinite sets. This rather non-typical notation will be adopted here because of the notation adopted in other papers on level sets. If $*$ is a family of sets and f a fixed function, we shall put $Y_*(f) = \{y: f^{-1}(y) \in *\}$. Let us denote, in the usual way, by G_δ the family of all sets of the form $\bigcap_{n=1}^{\infty} G_n$ where G_n are open sets, and by $F_{\sigma\delta}$ the family of all sets of the form $\bigcap_{k=1}^{\infty}\bigcup_{n=1}^{\infty} F_{k,n}$ where $F_{k,n}$ are closed sets. Let F^- denote the family of all sets of the form $A \setminus B$, where A is a closed set and B is a subset of the set of all end-points of components of the complement of A. Every point of B is hence