Degrees of unsolvability within a regressive isol

by

T. G. McLaughlin (Urbana Ill.)

Abstract. J. C. E. Dekker introduced in [2] the notion of *degree of unsolvability of a regressive isol*; his definition relies on the fact that among all the sets belonging to a regressive isol $\mathfrak{J}$ there is one of smallest degree. We here prove that there exist regressive isol $\mathfrak{J}$ and representatives $A \in \mathfrak{J}$, $B \in \mathfrak{J}$ such that A and B are of incomparable degree; the proof is a version of the standard approach to the Friedberg–Mučnik result concerning existence of incomparable r.e. degrees. With the aid of a theorem of Shoenfield, we establish that, additionally, $\mathfrak{J}$, A, and B can be required to satisfy the condition: A, B are separated by r.e. sets and degree $(A \cup B)$ is r.e. As a corollary, we conclude that not every immune regressive set of r.e. degree is introreducible.

1. Introduction. In [2], Dekker defined the *degree of unsolvability* of a regressive isol $\mathfrak{J}$ as the Turing degree of a retraceable representative of $\mathfrak{J}$. This definition is based on the observations that *every regressive isol has a retraceable representative* ([1]) and that the *Turing degree of any retraceable element of* $\mathfrak{J}$ *is the g.l.b. of the Turing degrees of all elements of* $\mathfrak{J}$ ([2]). For a given regressive isol $\mathfrak{J}$, let $\mathfrak{D}(\mathfrak{J})$ denote the set of all Turing degrees $\boldsymbol{D}$ of sets D belonging to $\mathfrak{J}$. We shall prove the existence of regressive isols $\mathfrak{J}$, of degree $\leqslant \boldsymbol{O}'$, such that $\mathfrak{D}(\mathfrak{J})$ is not a linearly ordered set of degrees. Our proof will be a variation on the usual proof of the Friedberg–Mučnik theorem [1].

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[1] We are indebted to the referee for some astute remarks on exposition. While we have not adopted wholesale his recommendations on format, two features of the present form of our proof of Theorem 2.2 which serve to make it more readable should be credited to him: first, a brief intuitive discussion of our attack on individual requirements precedes the detailed construction; and second, several lemmas of a thoroughly routine and typical nature have been stated with only the barest accompanying indications of proof (since in those cases the somewhat tedious verifications can be supplied easily by any reader experienced in recursion theory).

For the most part, our notation and terminology follow the conventions employed in [1]. Let $\langle\varphi_k^{1,X}\rangle_{k=0}^{\infty}$ be a standard enumeration (e.g., that of [4], § 65) of the class of unary partial recursive operators acting on subsets X of N and taking (unary) partial X-recursive functions as values; here, of course, N denotes the set $\{0, 1, 2, \ldots\}$ of all natural numbers. From here on, by the term *function* (*partial function*) we shall always mean a function whose domain is $= N^n$ for some integer $n \geqslant 1$ ($\subseteq N^n$ for some $n \geqslant 1$) and whose range is $\subseteq N$. By $\varphi_k^{1,X_0,s}$ we mean the set of pairs placed in φ_k^{1,X_0} by the end of the sth computational step, according to some fixed definition of "computational step" which is fully effective as a function of the pair $\langle k, C_{X_0}[s]\rangle$. (By *fully effective*, we mean that the *exact* (finite) membership of $\varphi_k^{1,X,s}$ is computed from $\langle k, C_X[s]\rangle$; here $C_X[s]$ denotes the characteristic function of X restricted to arguments $\leqslant s$. For the sake of definiteness, we remark that our convention regarding characteristic functions is: $C_A(x) = 0$ if $x \in A$ & $C_A(x) = 1$ if $x \notin A$.) We also assume, of course, that $\varphi_k^{1,X} = \bigcup_s \varphi_k^{1,X,s}$ holds for all X and k; and as a matter of convenience, we shall further assume that $(\forall k)(\forall X)(\forall s)[\varphi_k^{1,X,s} \subseteq \varphi_k^{1,X,s+1}]$. It will be assumed that our enumeration $\langle\varphi_k^{1,X}\rangle_{k=0}^{\infty}$ satisfies the condition $(\forall X)[\varphi_0^{1,X} = \emptyset]$. For f a partial function and for any $x \in N$, $\hat{f}(x)$ denotes the "orbit" $\{x, f(x), f(f(x)), \ldots\}$; here, of course, the sequence of iterates halts at any point at which the last term obtained is not in δf. (As in [1], we denote the domain and range of f by δf and ϱf respectively.) The notations $f^n(x)$ and $f^*(x)$ (where $n > 0 \Rightarrow x \in \delta f$ and where $\hat{f}(x)$ is subjected to suitable restrictions) are to be understood as in [1]. Let W_e denote $\delta\varphi_e$, where $\langle\varphi_e\rangle_{e=0}^{\infty}$ is a standard recursive enumeration (as in [4]) of the partial recursive functions of one variable; thus $W_e =$ the eth recursively enumerable set (2). Further, let W_e^s denote $\{x \mid x$ is in $\delta\varphi_e$ after s steps of computation$\}$; here again, we refer to some fixed, fully effective definition of "step of computation", this time in connection with a simultaneous computation of the functions φ_e, $e \in N$. In the statement of Theorem 2.2 of the next section, $\simeq$ denotes (as usual in contexts involving isols) the relation of recursive equivalence; $\boldsymbol{a}|\boldsymbol{b}$, for any two degrees $\boldsymbol{a}$ and $\boldsymbol{b}$, means that $\boldsymbol{a}$ and $\boldsymbol{b}$ are incomparable. The statement that a partial recursive function p *regresses* a set A, A infinite, means that there is a non-repeating sequential ordering $a_0, a_1, a_2, \ldots$ of A such that $A \subseteq \delta p$ & $p(a_0) = a_0$ & $(\forall n)[p(a_{n+1}) = a_n]$. Given that the infinite set A is regressed by the partial recursive function p with respect to the ordering $a_0, a_1, a_2, \ldots$ of A, we denote by $E(A; p)$ the set $\{a_{2k} \mid k \in N\}$ (the "even half" of A relative to p) and by $O(A; p)$ the set $\{a_{2k+1} \mid k \in N\}$ (the "odd half" of A relative to p). We shall conclude this introductory section by explicitly recalling some terminology, regarding limits and uniformity, introduced by Shoenfield in [5]. A sequence $\langle\xi_n\rangle_{n=0}^{\infty}$ of unary functions is said to be *uniform in a degree* $\boldsymbol{a}$ just in case there exists a binary function τ of degree $\leqslant \boldsymbol{a}$ such that $(\forall n)(\forall m)[\xi_n(m) = \tau(n, m)]$. Suppose that $\langle\xi_n\rangle_{n=0}^{\infty}$ is a sequence of unary functions with the property that $\lim_{m\to\infty} \xi_n(m)$ exists for all n and m. A *modulus of convergence* for $\langle\xi_n\rangle_{n=0}^{\infty}$ is a unary function τ such that

$$(\forall n)(\forall m)(\forall q)\big[(m \geqslant \tau(n) \,\&\, q \geqslant \tau(n)) \Rightarrow \xi_n(m) = \xi_n(q)\big].$$

(2) For convenience, we identify φ_e with $\varphi_e^{\emptyset}$.

2. Isolated regressive sets whose even and odd halves are recursively equivalent but Turing incomparable.

2.1. Lemma ([5]). *Let A be a set of natural numbers whose characteristic function C_A satisfies the condition*

(i) $(\forall x)[C_A(x) = \lim_{n\to\infty} \xi_n(x)]$, *where $\langle\xi_n\rangle_{n=0}^{\infty}$ is a sequence of functions uniform in the degree $\boldsymbol{O}$ of the empty set; suppose further that*

(ii) *the sequence $\langle\xi_n\rangle_{n=0}^{\infty}$ of (i) admits a modulus of convergence $\varkappa$ such that the degree $\boldsymbol{\kappa}$ of $\varkappa$ is $\leqslant$ the degree $\boldsymbol{A}$ of A.*

Then A contains a recursively enumerable set.

(As noted in [5], property (i) suffices for the conclusion that A has degree $\leqslant \boldsymbol{O}'$. This observation, together with its converse, has become one of the most frequently exploited technicalities in the theory of relative recursivity.)

2.2. Theorem. *There exist an immune set A of natural numbers (A infinite) and a partial recursive function p which regresses A, such that the following assertions hold:*

(1) $(\forall x)[C_A(x) = \lim_{k\to\infty} \xi_k(x)]$, *where $\langle\xi_k\rangle_{k=0}^{\infty}$ is uniform in $\boldsymbol{O}$ and admits a modulus of convergence recursive in A;*

(2) $E(A; p) \simeq O(A; p)$;

(3) $\boldsymbol{E}(A; p) | \boldsymbol{O}(A; p)$.

Proof. In view of the Friedberg–Mučnik theorem, since all r.e. sets are regressive, the result would be immediate if the requirement of immunity were omitted; however, in the present paper our interest is concentrated on *isols*, i.e., on recursive equivalence types of immune sets. Accordingly, our construction must insure that A is not recursively enumerable. This demand is very easily met. Moreover, to make A immune it in fact *suffices* to insure that A is not r.e., since all non-r.e. regressive sets are immune. To facilitate our description of the construction of A (and, simultaneously, of functions p and q such that p regresses A and q witnesses $E(A; p) \simeq O(A; p)$), we shall employ two sequences,

$\langle \Lambda_k \rangle_{k=0}^{\infty}$ and $\langle \Sigma_k \rangle_{k=0}^{\infty}$, of "movable markers"; Λ_k will be used to keep track of our attacks on the operator $\varphi_k^{1,X}$ at the argument $X = E(A; p)$, and Σ_k will have a similar role to play with respect to $\varphi_k^{1,X}$ at $X = O(A; p)$. The priority ordering imposed on the markers (and so on the pairs $\langle \varphi_k^{1,X}, X \rangle$, $X = O(A; p)$ or $E(A; p)$) is the usual "interlace" ordering $\Lambda_0 \prec \Sigma_0 \prec \Lambda_1 \prec \Sigma_1 \prec \ldots \prec \Lambda_{n+1} \prec \Sigma_{n+1} \prec \ldots$, where $\Lambda_i \prec \Sigma_i \prec \Lambda_{i+1}$ signifies that Λ_i *has higher priority than* Σ_i which in turn *has higher priority than* Λ_{i+1}. A number k will be called *free*, at (a given point during) a given Stage s of the construction, provided no number $j \geqslant k$ has borne a marker or a tag or been paired with another number at any prior point in the construction. We shall so arrange the construction that at the end of any given Stage s, the Λ-markers and Σ-markers which are attached to numbers as we await the beginning of Stage $s+1$ are precisely the members of the set $\{\Lambda_k |\ k \leqslant \beta(s)\} \cup \{\Sigma_k |\ k \leqslant \beta(s)\}$, where β is a recursive function of s; in describing the $(s+1)$-st Stage, we understand $\beta(s)$ to have been calculated at the end of Stage s, and we then conclude Stage $s+1$ by calculating $\beta(s+1)$. If $t \leqslant \beta(s)$, then λ_t^s, σ_t^s respectively denote the positions of Λ_t and Σ_t at the end of Stage s. We now give a short informal discussion of the way in which we shall attack the individual requirements involved in satisfying condition (3) and in insuring the non-r.e. character of A. (Of course, the construction as a whole proceeds by way of ever-expanding finite approximations to the graphs of p and q, with all individual requirements treated simultaneously via the priority ordering.) To get A non-r.e., we attempt to associate with each r.e. set W_e a number n_e such that n_e will be banished permanently from A in case we discover that $n_e \in W_e$. Such banishment is easily accomplished, since we are allowed unlimited branchings in the construction of a regressive set. To insure that the eth partial recursive operator does not reduce $E(A; p)$ to $O(A; p)$, we attempt to associate with e a number m_e so that if at some point in the construction our current approximation to the characteristic function of $E(A; p)$ threatens (when plugged into the eth operator) to assert correctly that $m_e \notin O(A; p)$, then m_e is placed in A *with odd p-height*; simultaneously, we strive to freeze all information about $E(A; p)$ used in an essential way in the offending computation. In putting m_e into A, we shall in general be obliged to (permanently) exclude certain other numbers w from A. (Similar measures prevail against reductions from $O(A; p)$ to $E(A; p)$.) Since the requirements in question are all purely local, finitely many attempts will suffice for each requirement (provided only that we adhere to the pre-specified order of priorities). Finally, to insure that we can also verify property (1) for A we need only be a little careful about specifying the branching which occurs when a number n_e or w is kicked out of A.

The formal stage-by-stage procedure is as follows.

Stage 0. Attach Λ_0 to 2 and Σ_0 to 3. Set $p^{(0)} = \{\langle 0, 0\rangle, \langle 1, 0\rangle, \langle 2, 1\rangle, \langle 3, 2\rangle\}$ and $q^{(0)} = \{\langle 0, 1\rangle, \langle 1, 0\rangle, \langle 2, 3\rangle, \langle 3, 2\rangle\}$. Let 4 be *paired with* 2, and 5 with 3. Attach a tag $\square 0$ to 6 and a tag $\triangle 0$ to 7. Let 8 be paired with 6, and 9 with 7. Set $r^{(0)} = \emptyset = u^{(0)}$; then go to Stage 1. (Note that $\beta(0) = 0$.)

Stage $s+1$. Our procedure here divides into three steps, the first of which is designed to insure that A is not recursively enumerable.

Step A. There are two cases.

Case A1. $(\exists t)(\exists m)[t \leqslant \beta(s)\ \&\ (m = \lambda_t^s$ or $m = \sigma_t^s)\ \&\ m \in W_t^s\ \&\ t$ does not currently bear an ε-tag].

In this case, let t_0 be the least such t and let m_0 be the corresponding m of smallest $p^{(s)}$-height. Give t_0 an ε-tag. Detach all Λ_i and Σ_i such that $t_0 \leqslant i \leqslant \beta(s)$. Let k_0 be the number paired with $\lambda_{t_0}^s$ and let j_0 be the number paired with $\sigma_{t_0}^s$. Attach Λ_{t_0} to k_0 and Σ_{t_0} to j_0. Define $r^{(s+1)} = p^{(s)} \cup \{\langle j_0, k_0\rangle, \langle k_0, p^{(s)}(m_0)\rangle\}$. (When our description of the construction is complete, it will be clear that $\lambda_{t_0}^s$ is paired with a number $k_0 > \lambda_{t_0}^s$, that $\sigma_{t_0}^s$ is paired with a number $j_0 > \sigma_{t_0}^s$ where $j_0 \neq k_0\ \&\ \{j_0, k_0\} \cap (\delta p^{(s)} \cup \delta q^{(s)}) = \emptyset$, that k_0 and j_0 can be effectively calculated from $\lambda_{t_0}^s$ and $\sigma_{t_0}^s$, that one of $\lambda_{t_0}^s, \sigma_{t_0}^s$ is the $p^{(s)}$-image of the other, and that $\widehat{p^{(s)}}(x)$ terminates at 0 for each x of the form λ_t^s or σ_t^s, $t \leqslant \beta(s)$.) Next, define $u^{(s+1)} = q^{(s)} \cup \{\langle k_0, j_0\rangle, \langle j_0, k_0\rangle\}$. Remove all currently attached tags of the forms $\square j$, $\triangle j$, where $j \geqslant t_0$. Let n_0 be the smallest number currently free. Give n_0 a tag $\square t_0$, and give n_0+1 a tag $\triangle t_0$. Pair n_0+i+2 with n_0+i, $i = 0$ or 1.

Finally, detach all tags of the forms $\bigcirc *$ and $\surd$ which are found to be attached to numbers i satisfying $t_0 \leqslant i$. Then proceed to Step B.

Case A2. No such pair $\langle t, m\rangle$ exists. In this case, go immediately to Step B.

Step B. If Case A1 held at Step A, let σ_s^* denote the position of Σ_{t_0} at the conclusion of Step A, t_0 as in our description of Case A1. Otherwise, let σ_s^* denote $\sigma_{\beta(s)}^s$. Let E_s^* denote $\{n |\ n \in \widehat{r^{(s+1)}}(\sigma_s^*)\ \&\ r^{(s+1)*}(n)$ is even$\}$; and let O_s^* denote $\{n |\ n \in \widehat{r^{(s+1)}}(\sigma_s^*)\ \&\ r^{(s+1)*}(n)$ is odd$\}$. Again, there are two cases.

Case B1. $(\exists k)(\exists m)[m > 0\ \&\ k \leqslant k_0 =$ the largest index among all indices of markers still attached after Step A & $([\varphi_k^{1,E_s^*,s}(m)$ is defined and $=1\ \&\ m \notin \delta r^{(s+1)}\ \&\ m$ bears a tag of the form $\square k\ \&\ k$ does not currently bear a tag of the form $\bigcirc *]$ or $[\varphi_k^{1,O_s^*,s}(m)$ is defined and $=1\ \&\ m \notin \delta r^{(s+1)}\ \&\ m$ bears a tag of the form $\triangle k\ \&\ k$ does not currently bear a tag of the form $\surd])]$. (When our description of the construction is complete, it will be clear that tags are assigned during the various stages in a fully effective way, so that we are able to tell effectively whether Case B1 holds and, if so, exactly which of the alternatives within its statement are true.)

Let k^* be the least such k, and let m^* be the largest such corresponding number m. There are two subcases, corresponding to the two halves of the incomparability condition.

Subcase B1 (i). $\varphi_{k^*}^{1,E_s^*,s}(m^*)$ is defined and $=1$ & m^* bears a tag of the form $\square k^*$ & $m^* \notin \delta r^{(s+1)}$ & k^* does not currently bear a tag of the form $\bigcirc *$. In this event, we proceed as follows. First, detach all markers Λ_i and Σ_i such that $k^* \leqslant i \leqslant k_0$. Give k^* a tag of the form $\bigcirc *$, and attach Λ_{k^*} to m^* and Σ_{k^*} to n^*, where n^* is the least number currently free. Pair n^*+1 with n^*. Define $\check{r}^{(s+1)} = r^{(s+1)} \cup \{\langle m^*, n^*\rangle, \langle n^*, \sigma_s^*\rangle\}$ and $\check{u}^{(s+1)} = u^{(s+1)} \cup \{\langle m^*, n^*\rangle, \langle n^*, m^*\rangle\}$. (When our description of the construction is done, it will be seen readily that m^* cannot have been in $\delta u^{(s+1)}$.)

Let q^* be the (new) smallest free number, and attach a tag $\triangle k^*$ to q^*. Pair q^*+1 with q^*. Remove $\triangle k^*$ from any number $w \neq q^*$ such that w currently bears $\triangle k^*$. Remove *all* tags of the form $\square w$ or $\triangle w$ for $w > k^*$; and remove all tags $\bigcirc *$ from numbers $k > k^*$ and all tags $\surd$ from numbers $k \geqslant k^*$. Then go to Step C.

Subcase B1 (ii). Subcase B1 (i) does not hold. Then, since Case B1 is in force, we have that $\varphi_{k^*}^{1,O_s^*,s}(m^*)$ is defined and $=1$ & $m^* \notin \delta r^{(s+1)}$ & k^* does not currently bear a tag of the form $\surd$ & m^* bears a tag of the form $\triangle k^*$. Here again, we detach all markers Λ_i and Σ_i such that $k^* \leqslant i \leqslant k_0$. Give k^* a tag of the form $\surd$. Attach Λ_{k^*} to n^* and Σ_{k^*} to m^*, where n^* is as in Subcase B1 (i). Pair n^*+1 with n^*. Define $\check{r}^{(s+1)} = r^{(s+1)} \cup \{\langle n^*, m^*\rangle, \langle m^*, \sigma_s^*\rangle\}$; and define $\check{u}^{(s+1)}$ exactly as in Subcase B1 (i). Let q^* be chosen as in Subcase B1 (i). Give q^* a tag of the form $\square k^*$, and remove $\square k^*$ from any *other* numbers to which it is currently attached. Pair q^*+1 with q^*. Remove *all* tags of the form $\square w$ or $\triangle w$ for $w > k^*$. Remove all tags of the form $\bigcirc *$ or $\surd$ from numbers $k > k^*$. Then go to Step C.

Case B2. The hypothesis of Case B1 does not hold. In this event, go directly to Step C.

Step C. Let j_0 denote the largest number j such that Σ_j and Λ_j are attached at the end of Step B. (Clearly, from the fully effective character of our procedure up to this point, j_0 is effectively calculable provided that no non-effective measures are taken in the remainder of Stage $s+1$; but, as the reader will easily note, none are.) Let t_0 be the smallest number currently free. Attach Λ_{j_0+1} to t_0 and Σ_{j_0+1} to t_0+1. Give t_0+2 a tag of the form $\square j_0+1$; and give t_0+3 a tag of the form $\triangle j_0+1$. Pair t_0+i+4 with t_0+i, $0 \leqslant i \leqslant 3$. Let t^* be the position of M at the end of Step B, where M is whichever one of Σ_{j_0}, Λ_{j_0} is attached to the number of greatest $\check{r}$-height at the end of Step B. Define $p^{(s+1)} = \check{r}^{(s+1)} \cup \{\langle t_0+1, t_0\rangle, \langle t_0, t^*\rangle\}$ and $q^{(s+1)} = \check{u}^{(s+1)} \cup \{\langle t_0+1, t_0\rangle, \langle t_0, t_0+1\rangle\}$. Compute $\beta(s+1) = j_0+1$, and proceed to Stage $s+2$.

That completes our description of the construction [3]. Our first observation is that $(\forall s)[p^{(s)} \subseteq p^{(s+1)}]$, whence (from the effectiveness of the construction) we have that $p = \bigcup_s p^{(s)}$ determines p as a partial recursive function. To see this, note that if a number n is placed into $\delta p^{(s+1)}$ during a given stage $s+1$ of the construction, then n is either free just prior to that placement or else is a tagged or paired number *not previously in* $\delta p^{(s+1)}$. But it is plain from our description of Stage $s+1$ that no number can have entered $\delta p^{(s+1)}$ at some earlier point and yet remained free, since as soon as we place a number into $\delta p^{(s+1)}$ we give a marker either to it or to some larger number. It follows that $\bigcup_s p^{(s)}$ is a function; hence if $p = \bigcup_s p^{(s)}$, then p is a partial recursive function. Accordingly, we define p by: $p = \bigcup_s p^{(s)}$. We shall now exhibit a sequence of eight lemmas which, in sum, establish that our construction has the desired effects. The first seven of these lemmas are rather typical of proofs based on the elementary ("finite injury") priority technique; in each case, the lemma in question follows from the statement of the construction (perhaps using some of the preceding lemmas in the sequence) by a more-or-less routine application of Mathematical Induction. We therefore leave the detailed verifications, for the first seven cases, to the reader. Lemma 2.2.H is also based on an induction argument; however, the content of Lemma 2.2.H is a bit unusual (in that, in the great majority of such theorems, property (1) is either not considered or else is superfluous because the set being constructed is itself r.e.) and we shall therefore supply a detailed justification of it.

2.2.A. Lemma. *For each $k \geqslant 0$, there are numbers λ_k, $s(k)$ and numbers σ_k, $t(k)$ such that* (i) *Λ_k is attached to λ_k throughout all stages $s \geqslant s(k)$, and* (ii) *Σ_k is attached to σ_k throughout all stages $t \geqslant t(k)$. (Consequently, $\lambda_k = \lim_{s\to\infty} \lambda_k^s$ & $\sigma_k = \lim_{t\to\infty} \sigma_k^t$. Note, in connection with Lemma 2.2.A, that no marker is ever attached to more than one number at a time).*

2.2.B. Lemma. *If $A \overset{\text{df}}{=} \bigcup_{n=0}^{\infty} \hat{f}(\lambda_n)$, then also $A = \bigcup_{n=0}^{\infty} \hat{f}(\sigma_n)$ and, moreover, A is an infinite set regressed by p.*

[3] There are, of course, alternative ways of framing the construction. One such alternative, noted by the referee, leads to a rather neat alternate verification of Lemma 2.2. H below, i.e., of property (1).

2.2.C. Lemma. *If* $q \overset{\mathrm{df}}{=} \bigcup_{s=0}^{\infty} q^{(s)}$, *then* q *is a* $1-1$ *partial recursive function such that* $E(A;p) \subseteqq \delta q$ & $q\big(E(A;p)\big) = O(A;p)$. (*Here, as in the other lemmas which follow,* A *is the set defined in the statement of Lemma* 2.2.B.)

2.2.D. Lemma. *Once a marker* Σ_m *is in final position* σ_m, *all numbers not in* $\hat{p}(\sigma_m) \cup \hat{p}(\lambda_m)$ *which are subsequently placed in* δp *must be* $> \max\big(\{x|\ \ x \in \hat{p}(\sigma_m) \cup \hat{p}(\lambda_m)\}\big)$.

2.2.E. Lemma. $(\nabla e)[A \neq W_e]$. (The proof of this lemma consists in showing that if there is any "persistent" threat of the equation $A = W_e$ holding, then this threat is eventually countered (permanently) via Step A.)

2.2.F. Lemma. $(\nabla s)[\sigma_s^*$ *is of odd* p*-height*$]$ & $(\nabla s)(\nabla k)\Big[\sigma_k^s \in \widehat{r^{(s+1)}}(\sigma_s^*) - \{\sigma_s^*\} \Rightarrow \min\Big(\widehat{r^{(s+1)}}(\sigma_s^*) - \big(\widehat{r^{(s+1)}}(\sigma_k^s) \cup \widehat{r^{(s+1)}}(\lambda_k^s)\big)\Big) > \max\big(\widehat{r^{(s+1)}}(\sigma_k^s)\big)\Big]$. (This lemma is a purely technical fact useful in establishing 2.2.G.)

2.2.G. Lemma. *Neither of* $E(A;p)$, $O(A;p)$ *is Turing reducible to the other; thus,* $\boldsymbol{E(A;p)|O(A;p)}$. (*A* key ingredient in the proof of Lemma 2.2.G, of course, in addition to Step B, is the continuity of partial recursive operators. The treatment of the two halves of the incomparability relation are different, essentially because movements of Σ_k induce movements of Λ_k even though $\Lambda_k \prec \Sigma_k$; the difference, however, is very slight.)

2.2.H. Lemma. *The characteristic function* $C_A(x)$ *of* A *is the limit,* $\lim_{s\to\infty} g(s,x)$, *of a recursive function* $g(s,x)$ *such that if* $\xi_s(x) \overset{\mathrm{df}}{=} g(s,x)$ *for each fixed* s, *then* $\langle \xi_s(x)\rangle_{s=0}^{\infty}$ *admits a modulus of convergence recursive in the degree of* A.

Proof of Lemma 2.2.H. We shall argue that there is a function $\tau_A(x)$, recursive in A, such that

$$\text{(I)} \qquad (\nabla x)[x \in A \Leftrightarrow x \in \widehat{p^{(\tau_A(x))}}(\sigma_{\beta(\tau_A(x))}^{\tau_A(x)})]\ \&\ \widehat{p^{(t)}}(\sigma_{\beta(t)}^{t}) \cap \{y|\ \ y \leqslant x\} = \widehat{p^{(\tau_A(x))}}(\sigma_{\beta(\tau_A(x))}^{\tau_A(x)}) \cap \{y|\ \ y \leqslant x\}$$

holds for all $t \geqslant \tau_A(x)$.

If this is so, then we can define $g(s,x) = 0$ or 1 according as $x \in \widehat{p^{(s)}}(\sigma_{\beta(s)}^{s})$ or $x \notin \widehat{p^{(s)}}(\sigma_{\beta(s)}^{s})$; the function g, so defined, will clearly have the required properties. To verify (I), we first show that with the help of c_A (= the characteristic function of A) we can inductively compute

the terms of the sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$ To begin with, we know that $\sigma_0 = 3$. Suppose that $\sigma_0, \ldots, \sigma_k$ are already known. Run the construction out to the first stage $s+1$ at which (a) σ_i^s, $0 \leqslant i \leqslant k$, are defined and respectively equal to σ_i, $0 \leqslant i \leqslant k$, and (b) σ_{k+1}^s is defined and belongs to A; here, obviously, (b) can be handled with help of c_A. Now, it is clear from the construction that no marker ever returns to a previously abandoned position; hence $\sigma_i^t = \sigma_i^s$ for $0 \leqslant i \leqslant k$ and all $t \geqslant s$. Next we observe that if Σ_{k+1} were to move from σ_{k+1}^s via Step A at some Stage $t \geqslant s+1$, then σ_{k+1}^s would thereby be excluded permanently from membership in A; hence, if Σ_{k+1} moves after Stage s it must do so *for the first time* via Step B. But at the beginning of Stage $s+1$, there are exactly two numbers, say m_0 and m_1, which bear tags of the forms $\square k+1$, $\triangle k+1$; and m_0 and m_1 can be found explicitly by examining the construction up to the beginning of Stage $s+1$. The numbers n_0 and n_1 *paired with* m_0 and m_1 respectively, at the beginning of Stage $s+1$, can also be found effectively. Suppose, for definiteness, that m_0 bears $\square k+1$ and m_1 bears $\triangle k+1$ at the beginning of Stage $s+1$. There are now a number of cases to be considered; we shall trace out the procedure in just one such case, leaving the similar treatment of the remaining cases to the reader. To begin with, it is plain from the construction that *at most one* of $c_A(m_0)$, $c_A(n_0)$, $c_A(m_1)$, $c_A(n_1)$ can be $=0$. If *all* these numbers are $=1$, then, clearly, $\sigma_{k+1}^s = \sigma_{k+1}$. Suppose (for definiteness) that we have, instead, $c_A(m_0) = 0$. This means that the first movement of Σ_{k+1} after Stage s, say at Stage $t_1 > s$, occurs on account of Subcase B1 (i). Then, having (effectively) found t_1, we calculate the numbers r_1, r_2 such that, at the end of Stage t_1, r_1 bears $\triangle k+1$ and r_2 is paired with r_1. Again, exactly one of the equations $c_A(r_1) = 0$, $c_A(r_2) = 0$ can hold. If $c_A(r_1) = c_A(r_2) = 1$, then $\sigma_{k+1}^{t_1} = \sigma_{k+1}$; so let us assume, say, $c_A(r_2) = 0$. This means that Λ_{k+1} and Σ_{k+1} make their next move via Step A, at some stage $t_2 > t_1$. Having calculated t_2, we note that no further movement of Σ_{k+1} *under Step* A will ever occur. Let r_3 bear $\square k+1$ and r_4 bear $\triangle k+1$ at the end of Stage t_2. If $c_A(r_3) = c_A(r_4) = 1$, then $\sigma_{k+1}^{t_2} = \sigma_{k+1}$. Suppose instead that $c_A(r_4) = 0$. This means that there is a first stage $t_3 > t_2$ at which Σ_{k+1} moves, and that the movement of Σ_{k+1} at Stage t_3 occurs via Subcase B1 (ii). Having calculated t_3, let r_5 be the number which bears $\square k+1$ at the end of Stage t_3. Suppose $c_A(r_5) = 1$. Then $\sigma_{k+1}^{t_3} = \sigma_{k+1}$. This completes the analysis of one particular case among all possible cases. Disposing of the various other possibilities in like manner, we conclude (by induction on the index n of the marker Σ_n) that there is a function $\xi(x)$, recursive in A, such that $(\nabla n)[\xi(n) = \sigma_n]$. To obtain our desired function $\tau_A(x)$, we now define:

$$\tau_A(x) = (\mu s)[x \leqslant \beta(s)\ \&\ \sigma_x^s = \sigma_x].$$

It is obvious from the construction of A that $x \leqslant \sigma_x$ holds for all x; hence, in view of Lemma 2.2.D, we have

$$(\forall x)[x \in A \Leftrightarrow x \in \widehat{p^{(\tau_A(x))}}(\sigma_x^{\tau_A(x)}) \Leftrightarrow x \in \widehat{p^{(\tau_A(x))}}(\sigma_{\beta(\tau_A(x))}^{\tau_A(x)})] \;\&\; \widehat{p^{(t)}}(\sigma_{\beta(t)}^{t}) \cap \{y|\ y \leqslant x\}$$

$$= \widehat{p^{(\tau_A(x))}}(\sigma_{\beta(\tau_A(x))}^{\tau_A(x)}) \cap \{y|\ y \leqslant x\} \quad \text{for all } t \geqslant \tau_A(x)\,,$$

and the proof is complete (since $\tau_A(x)$, so defined, is plainly recursive in A).

2.3. Remark. The sets A, $O(A;\, p)$, and $E(A;\, p)$ constructed in our proof of Theorem 2.2 are all *differences of r.e. sets* (4). It clearly suffices to establish this fact for A, and for A it is an easy consequence of the following readily verifiable feature of the construction: for all x, if $(\exists s)[x \in \hat{p}(\sigma_{\beta(s)}^{s}) - \hat{p}(\sigma_{\beta(s+1)}^{s+1})]$ then $x \notin A$.

3. Some consequences of Theorem 2.2. We shall now make a couple of applications of § 2 to the class of those regressive isols having degree $\leqslant O'$.

3.1. Corollary. *There exist degrees D_1 and D_2, with r.e. least upper bound, and a regressive isol $\mathfrak{J}$ such that $\mathfrak{J} \cap D_1 \neq \emptyset$, $\mathfrak{J} \cap D_2 \neq \emptyset$, and $D_1|D_2$.*

Proof. Theorem 2.2 and Lemma 2.1.

3.2. Remark. Let $A \in \mathfrak{J} \cap D_1$ and $B \in \mathfrak{J} \cap D_2$, where D_1, D_2, and $\mathfrak{J}$ are as in Corollary 3.1. Then it is not possible for either A or B to have recursively enumerable complement. This observation follows from [2], Proposition P14, and [1], Proposition 10.

3.3. Corollary. *There exists an immune regressive set A, of recursively enumerable degree, which is not introreducible.* (By contrast, it is well known that all *retraceable* sets are introreducible; for a general study of introreducibility and the associated concept of introenumerability, see [3].)

Proof. Let A and p be as in Theorem 2.2. Since $E(A;\, p)$ and $O(A;\, p)$ are separated by disjoint recursively enumerable sets, we have that $A = E(A;\, p)$ join $O(A;\, p)$. Now, if A were introreducible we would have, for instance, $A \leqslant E(A;\, p)$. But $O(A;\, p) \leqslant A$ since $A = E(A;\, p)$ join $O(A;\, p)$, and we get a contradiction. Therefore A is not introreducible. By Lemma 2.1, A is of r.e. degree.

From Corollary 3.3, since the property of introreducibility is invariant under recursive permutations of N, we at once get:

3.4. Corollary. *There exists an immune regressive set of degree $\leqslant O'$ which is not recursively isomorphic to any retraceable set.* (Dekker showed in [1] that every regressive set is *recursively equivalent* to a retraceable set; and in [1], Proposition 10, it was established that any *co-r.e.* regressive set is *recursively isomorphic* to a (co-r.e.) retraceable set.)

(4) This was brought to our attention by the referee.

3.5. Remark. It is possible to prove, by direct construction, a considerably stronger version of Corollary 3.4: *there is an immune regressive set of degree $\leqslant O'$ which is not a finite union of recursive isomorphs of retraceable sets.* We plan to publish the latter result as a portion of a future article.

4. Some additional results and open questions.

4.1. The standard generalization of Theorem 2.2 would consist in effectively breaking (a suitable) A up into $\aleph_0$ mutually (r.e.)-separated pieces forming a (sequentially) Turing-independent collection of recursively equivalent regressive subsets of A. However, our technique for insuring $E(A;\, p) \simeq O(A;\, p)$ in the foregoing proof of Theorem 2.2 is not only simple but excessively rigid: we have not been able to endow it with enough flexibility to get us to the theorem in question. We conjecture, however, that what we have just called the "standard generalization" of Theorem 2.2 is indeed true. On the other hand, we can prove a related though weaker result, namely: there is a sequentially Turing-independent family of $\aleph_0$ immune regressive sets, each of degree $\leqslant O'$, such that the corresponding isols are finite translates of one another. (Hence, there is a regressive isol $\mathfrak{J}$, of degree $\leqslant O'$, having $\aleph_0$ representatives which form a (not necessarily sequentially) Turing-independent family.)

4.2. Using the well-known "permitting" technique of Friedberg and Yates, we can rather easily obtain $\mathfrak{J}$, D_1, D_2 as in Corollary 3.1, with D_1 and D_2 each $\leqslant a$ given non-zero r.e. degree; but, we do not know whether A (of Theorem 2.2) can be made to have arbitrary non-zero r.e. degree.

4.3. Finally, we observe that there are some interesting limitations on the properties possible for D_1 and D_2 as members of the full Turing semilattice. For example, it is clear that since any retraceable element of $\mathfrak{J}$ has minimum degree among all members of $\mathfrak{J}$, D_1 and D_2 cannot constitute a minimal pair; nor, of course, can either of D_1, D_2 separately be minimal. We do not discern with any accuracy, however, the range of such limitations. For instance, can D_1 and D_2 belong to the subsemilattice of *r.e.* degrees? (Theorem 2.2 only places the *join of* D_1 and D_2 among the r.e. degrees.) Presumably they can, though all that we have so far been able to extract (from our particular construction) in this regard is that it is possible to require $D_1 \geqslant d \;\&\; D_2 \geqslant c \;\&\; d|c$ for some pair d, c of r.e. degrees. (Remark 2.3, unfortunately, is of no help in this respect, since, as has been shown by A. H. Lachlan, there exist differences of r.e. sets which are not of r.e. degree.) As to the *lack* of complexity which $\mathfrak{D}(\mathfrak{J})$ may exhibit, it suffices to remark that if A is any *retraceable* set of degree $\geqslant O'$, then every set recursively equivalent to A has the same degree as A.

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On h-regular graded algebras

by

A. Tyc (Toruń)

Abstract. Let R be a commutative ring with identity. In the paper the well-known notion of a regular sequence in R (or an R-sequence) is generalized as follows: A sequence $u_1, \ldots, u_n$, $u_i \in R$, is called an h-regular sequence in R if $(u_1, \ldots, u_n) \neq R$ and $(u_1, \ldots, u_{k-1}) : (u_k) = (u_1, \ldots, u_{k-1}, u_k^{h_k-1})$ $(k = 1, \ldots, n)$, where $h_i = h(u_i)$ is the minimum of integers $n > 0$ such that $u_i^n = 0$ (if there is no such an integer $h(u_i) = \infty$ and $u_i^\infty = 0$). A local Noetherian ring R is said to be h-*regular* if its unique maximal ideal is generated by an h-regular sequence. It is shown that any commutative graded R-algebra $A = \bigoplus_{i=0}^{\infty} A_i$ with the ideal $I = \bigoplus_{i>0} A_i$ generated by an h-regular set is of the form $\bigoplus_i R[X]/(X^{h_i})$ for some $h_i \in N \cup \{\infty\}$ (N is the set of positive integers). Moreover, the Tate resolution of such algebras is found provided R is an h-regular local Noetherian ring.

Introduction. Let R be a commutative local Noetherian ring with the unique maximal ideal m. Recall that a sequence $u_1, \ldots, u_n$, $u_k \in m$, is called an R-*sequence* if $(u_1, \ldots, u_{k-1}) : (u_k) = (u_1, \ldots, u_{k-1})$ for $k = 1, \ldots, n$. In [1] T. Józefiak adapts this definition for commutative graded R-algebras. Namely, if $A = \bigoplus A_i$ is such an algebra, then a sequence $u_1, \ldots, u_n$ of homogeneous element from the ideal $I = m \oplus (\bigoplus_{i>0} A_i) \subset A$ is said to be *normal* (or *regular*) (¹) in A provided

$$(u_1, \ldots, u_{k-1}) : (u_k) = \begin{cases} (u_1, \ldots, u_{k-1}) & \text{if } \deg u_k \text{ is even,} \\ (u_1, \ldots, u_k) & \text{if } \deg u_k \text{ is odd} \end{cases}$$

for $k = 1, \ldots, n$ (we assume $x^2 = 2$ for any homogeneous element $x \in A$ of odd degree). In this paper the notion of a regular sequence in A is

(¹) The term "regular sequence" instead of "normal sequence" is used in Józefiak's next paper [3]. We prefer the term "regular sequence" also.