THEOREM. *Let* $|\mathfrak{B}| = \lambda$, $\mathfrak{B}$ *satisfy* σ-*cc*, *and* $\lambda^{\underline{\sigma}} < \varkappa$. *Moreover, suppose that for each* **ZF**-*formula* Φ *with parameters from* $\check{V}$ *we have* $\|\Phi\| \in \{0, 1\}$ (*i.e.* [D] *from* § 1). *Then* $\mathrm{E}(\varkappa, \varkappa)$ *implies* $\|\mathrm{E}(\check{\varkappa}, \check{\varkappa})\| = 1$ *in* $V^{(\mathfrak{B})}$.

Proof. Since $|\mathfrak{B}| = \lambda$, $\mathfrak{B}$ satisfies σ-cc, $\lambda^{\underline{\sigma}} < \varkappa$ and $\mathrm{E}(\varkappa, \varkappa)$ are assumed, by the Main Lemma we have $\mathrm{E}(\mathfrak{B}, \varkappa)$. Next, obviously $\sigma \leqslant \varkappa$; thus $\mathfrak{B}$ satisfies also $\varkappa$-cc. Consequently all the assumptions of Theorem 1.9.2, are fulfilled. Thus, by 1.9.2, we have $\|\mathrm{E}(\check{\varkappa}, \check{\varkappa})\| = 1$. Q.E.D.

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Examples of disks in E^3/G which cannot be approximated by P-liftable disks*

by

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Abstract. In *Conditions under which disks are P-liftable* the author defined a set $X \subset E^3/G$ to be P-liftable if there exists a set $X' \subset E^3$ such that X and X' are homeomorphic and X is the image of X' under the natural projection mapping P. It was proved that in certain decomposition spaces, each disk $D \subset E^3/G$ can be approximated by P-liftable disks, i.e., for any $\varepsilon > 0$ there exists a P-liftable disk D_ε that is ε-homeomorphic to D. In this paper we give examples of decomposition spaces each containing a disk D that cannot be approximated by P-liftable disks.

Analogous to the problem of the existence of an approximating P-liftable disk is a question posed by Armentrout for 2-spheres when G is a pointlike decomposition. This question is answered in the negative.

An example is given of a pair of decomposition spaces that are "equivalent" in the terminology of Armentrout, Lininger, and Meyer, but differ in the property of containing P-liftable approximating disks.

A construction called a *knit Cantor set of nondegenerate elements* is defined. A newly defined property entitled *equi-locally connected* is not possessed by every point of a knit Cantor set of nondegenerate elements. Hypothesizing this property for the points in the nondegenerate elements, questions are formulated concerning the existence of P-liftable approximating disks.

Key words and phrases. Lift of a space, P-lift, topology of E^3, decomposition space, monotone decomposition, Cantor set of nondegenerate elements, equi-LC^n, equi-locally connected.

1. Introduction.

In *Conditions under which disks are P-liftable* [16] the author defined a set $X \subset E^3/G$ to be *P-liftable* if there exists a set $X' \subset E^3$ such that X and X' are homeomorphic and X is the image of X' under the natural projection mapping P. The set X' is said to be the *P-lift* of X. Note that this generalizes the lifting concept (McAuley [9]) in which the projection mapping is a homeomorphism on the set that is called the lift. For spaces which (1) are definable by 3-cells, or (2) in which G has a countable number of nondegenerate elements and E^3/G is homeo-

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morphic to E^3, it was proved that for each disk $D \subset E^3/G$ and each $\varepsilon > 0$ there exists a P-liftable disk D_ε that is ε-homeomorphic to D.

In this paper sections 3 and 4 are each an example of a decomposition space containing a disk D such that for each sufficiently small $\varepsilon > 0$ there is no P-liftable disk D_ε that is ε-homeomorphic to D. The example in section 3 uses a (2,1) toroidal decomposition. The disk in this decomposition space is not an image of a locally connected set in E^3. On the other hand, in the section 4 example the disk D in that decomposition space is the image of a 2-complex in E^3, but the decomposition of E^3 is not as simple as a (2,1) toroidal decomposition. A construction called a knit Cantor set of nondegenerate elements is used to define the decomposition in section 4. The properties of this construction should make it useful for other kinds of counterexamples. The decompositions in both sections 3 and 4 have the following properties:

(1) E^3/G is homeomorphic to E^3,

(2) each $g \in H$ is a tame arc,

(3) H is continuous and closed,

(4) $P(H)$ is a Cantor set,

(5) H is not definable by 3-cells.

In section 2 a lemma is proved that is basic to the examples in sections 3 and 4.

Armentrout [1] has asked a question concerning 2-spheres that is analogous to the problem of the existence of a P-liftable disk D_ε approximating a given disk D. In section 5 this question is answered in the negative by an example in the (2,1) toroidal decomposition space.

Pairs of equivalent decomposition spaces (Armentrout, Lininger, and Meyer [2]) are discussed in section 6. One of each pair is the decomposition space used in the section 3 or 4 example. An equivalent decomposition space with only a countable number of nondegenerate elements exists. A theorem cited above proves that in the countable decomposition every disk can be approximated by P-liftable disks. Thus, we demonstrate that there is a property not shared by equivalent decompositions.

In section 7 a concept entitled equi-locally connected is defined for a collection of closed point sets. It is noted that a knit Cantor set of nondegenerate elements contains a point that does not have this equi-locally connected property. With this property questions are formulated concerning the existence of P-liftable disks that approximate a given disk which is itself the image of a locally connected set.

It is instructive to compare tameness of a disk D with the property that in a neighborhood of D there exists a P-liftable disk D_ε. Since tameness is equivalent to bicollarability, it, too, is a property of the neighborhood of D. Neither of these neighborhood properties implies the other. The disk in the section 3 example can be chosen to be either tame or wild. On the other hand, if the projection map is taken to be the identity map, then any wild or tame disk in the image space is P-liftable.

Notation and terminology. Throughout the paper we denote set closure by Cl, interior by Int, and boundary by Bd. The symbol H denotes the set of nondegenerate elements of a decomposition G, and H^* is the union of the elements of H. We use P to denote the natural projection mapping of E^3 onto E^3/G. The distance between points p and q is denoted by $d(p, q)$.

A sequence $M_1, M_2, M_3, \ldots$ of compact 3-manifolds-with-boundary is a *defining sequence* for a decomposition G of E^3 if and only if for each positive integer i, $M_{i+1} \subset \operatorname{Int} M_i$, and the nondegenerate components of $\bigcap_{i=1}^{\infty} M_i$ are the nondegenerate elements of G.

A decomposition is called *toroidal* if it has a defining sequence $M_1, M_2, M_3, \ldots$ such that every component of M_i is a solid torus. It is called an (m, n) *toroidal decomposition* if it is an iteration of the embedding of m solid tori each of which is shrinkable in the previous stage and whose union essentially wraps n times around (Sher [12]).

For a positive integer n, let M_n be the nth element of a defining sequence for a (2,1) toroidal decomposition. Let T with a subscript of n digits, each of which is a one or two, be a toroidal component of M_n. As is conventional, the first k digits of the subscript indicate the component of M_k in which the given torus lies. Thus, $T_{1221} \subset T_{122}$. When we are concerned with the tori imbedded in a given torus at the next stage or the next few stages, the subscripts agree except in a small number of final digits. A notation introduced by Casler [7] is then convenient. We let na denote a subscript of n digits, and append digits to na. Thus, $T_{na12} \subset T_{na1} \subset T_{na}$.

2. A basic lemma.

The examples in this paper use the (2,1) toroidal construction and modifications of it. In this section we are concerned with the manner in which a disk intersects the tori in various stages of the defining sequence of a (2,1) toroidal decomposition.

DEFINITION. A disk D is said to be *meridional* in a solid torus T if $\operatorname{Bd} D$ circles $\operatorname{Bd} T$ once meridionally. A subdisk D of a disk D'_ε is called a *meridional subdisk* of D'_ε in a torus T if and only if D is itself a meridional disk in T and does not properly contain a subdisk that is a meridional disk in T.

Note that a meridional disk D in a solid torus T is not required to satisfy $\operatorname{Int} D \subset T$.

Let $M_1, M_2, M_3, \ldots$ be the defining sequence for a (2,1) toroidal decomposition G_T with nondegenerate elements H_T. Assume that each M_i is polyhedral. Furthermore, suppose that the sequence is so specified that there are two parallel planes R and S that each contain one end point of each nondegenerate element (see Fig. 1).

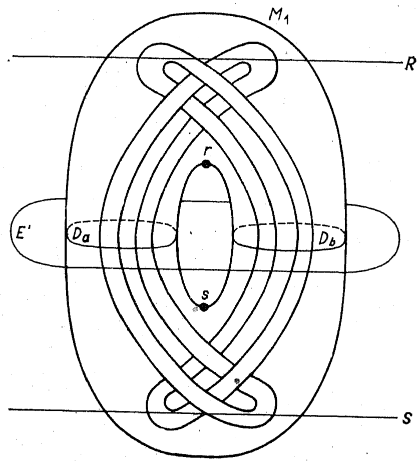

Fig. 1

Lemma 2.1. *In the toroidal decomposition described above, let E' be a disk in E^3 that projects onto a disk in E^3/G_T. Assume that E' is locally polyhedral off H_T^*, and in general position with respect to each* $\mathrm{Bd}\, M_i$. *Suppose that D_a and D_b are disjoint subdisks in E' and that they are meridional disks in the solid torus M_1. Then, for any $g \in H_T$, $g \cap E' \neq \emptyset$.*

The proof of this lemma depends on Lemmas 2.2 and 2.3.

Lemma 2.2. *Suppose that a disk D is locally polyhedral off H_T^*; in general position with respect to* $\mathrm{Bd}\, M_n$ *for every positive integer n; and that for some torus T_{ka} the disk D is meridional in T_{ka}. Then for i either* 1 *or* 2, *the disk D contains two disjoint subdisks d and e that are each meridional in T_{kai}.*

The proof of Lemma 2.2 is a modification of Bing's Theorems 1-4 in [4], in which he uses a (2,2) toroidal decomposition. Details of the modified proof are in [14].

The following definition is used in Lemma 2.3.

Definition. Disjoint sets S_1 and S_2 are said to *link* if there exist simple closed curves $J_1 \subset S_1$ and $J_2 \subset S_2$ such that J_1 and J_2 link by Bing's definition in [5]. Tori T_{kai} and T_{kaj} are said to link in T_{ka} if in the universal covering space of T_{ka} there are linked tori t_{kai} and t_{kaj} that project onto T_{kai} and T_{kaj}, respectively.

Lemma 2.3. *Assume the hypotheses of Lemma 2.2. If for $j \neq i$ the torus T_{kaj} misses D, then it is possible to choose the subdisks d and e of Lemma 2.2 such that T_{kaj} links $D \cup \Delta$, where Δ is either component of $T_{kai} - (d \cup e)$.*

Outline of a proof of Lemma 2.3. (Details are in [14].) Using the linkings of T_{kai} and T_{kaj}, it can be shown that there is a disk Δ such that

(1) $\Delta \subset T_{ka}$,

(2) $\Delta \cap T_{kaj} = \mathrm{Bd}\,\Delta$,

(3) $\mathrm{Bd}\,\Delta$ is a longitudinal simple closed curve in T_{kaj},

(4) $\Delta \cap T_{kai}$ is two disjoint disks Δ_a and Δ_b that are each meridional in T_{kai}, and

(5) Δ misses D. (To get this, it may be necessary to remove from D trivial subdisks that protrude outside T_{ka}.)

Let the components of $T_{kai} - (\Delta_a \cup \Delta_b)$ be V and W. Then $V \cup \Delta$ is a set that links $\mathrm{Bd}\,D$. It follows that D contains in V a subdisk that is meridional in T_{kai}. Call this subdisk d in the conclusion of the lemma. Similarly, there is in W a subdisk that is meridional in T_{kai}. It can be shown that these subdisks d and e satisfy the conclusion of the lemma.

Outline of a proof of Lemma 2.1. (Details are in [14].) The proof is indirect. At each stage a minimal set of meridional subdisks satisfying Lemmas 2.2 and 2.3 is chosen. Because the existence of this minimal set will lead to a contradiction, we can ignore any other meridional subdisks.

Assume that there is some $g_1 \in H_T$ that misses E'. Then, for some subscript nai, there are tori T_{na} and T_{nai} containing g_1 and such that E' contains a meridional disk in T_{na}, but does not contain a meridional disk in T_{nai}. It can be shown that because E' contains no meridional disks in T_{nai} there must be four meridional disks D_1, D_2, D_3, and D_4 in T_{naj}, where $j \neq i$. At the next stage at least two of these must have pairs of meridional disks in the same toroidal component. Suppose that D_1 and D_2 have pairs D_{11}, D_{12} and D_{21}, D_{22} of meridional subdisks in T_{najj}.

The hypothesis that $P(E')$ is a disk implies that no nondegenerate element intersects both D_1 and D_2. Hence, there is some stage at which subdisks of D_1 do not intersect both toroidal components. Without loss of generality, we can assume this happens in T_{naij} (see Fig. 2). Lemma 2.3 applied to D_1, T_{naj}, and T_{nai} can be used to show that one subdisk of D_1, say D_{12}, contains a disk δ satisfying the following:

(1) Each point of $\mathrm{Bd}\,\delta$ lies farther from Plane $S(R)$ than the point $r(s)$, where r and s are points in Bd M_1 shown in Figure 1.

(2) $\mathrm{Int}\,\delta$ contains a pair of meridional disks at the next stage.

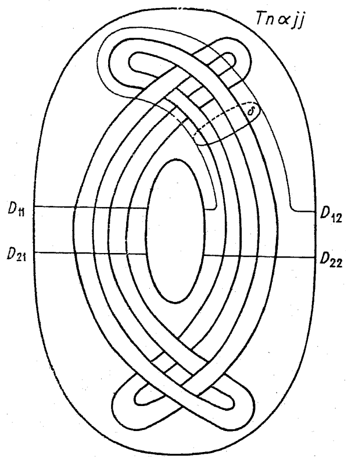

Fig. 2

This argument can be used at each succesive stage. By a careful choice of which subdisks one follows, it can be shown that there is an infinite sequence of nested disks δ_i such that $\delta_{i+1} \subset \mathrm{Int}\,\delta_i$ and each point of $\mathrm{Bd}\,\delta_i$ lies farther from plane S than point r if i is even and farther from plane R than point s if i is odd. This contradicts the hypothesis that E' is a disk and proves the lemma. ■

§ 3. An example of a disk in E^3/G that can not be approximated by P-liftable disks. The decomposition G_T of E^3 used in this example is the (2,1) toroidal decomposition. Let M_1 denote the first stage of a defining sequence for G_T. In Figure 3 the torus M_1 is shown with a set K. The set K is a punctured disk resulting from the removal of four open subdisks from the interior of a disk. The sets K and Bd M_1 intersect in four disjoint meridional simple closed curves, which we denote c_i for $i = 1, 2, 3$, and 4. The punctured disk K misses Int M_1. Since $\mathrm{Cl}\,H^*$ misses $K \cup \mathrm{Bd}\,M_1$,

there is an open neighborhood N of $K \cup \mathrm{Bd}\,M_1$ on which the projection map P is a homeomorphism. Hence, $P(\mathrm{Bd}\,M_1)$ is a torus.

Recall that Bing proved [4] that for the (2,1) toroidal decomposition G_T, the space E^3/G_T is homeomorphic to E^3. Using the facts that $P|N$ is a homeomorphism and that $H^* \subset \mathrm{Int}\,M_1$, it can be shown that $P(M_1)$ is a solid torus bounded by the torus $P(\mathrm{Bd}\,M_1)$. The set $P(K \cap \mathrm{Bd}\,M_1)$ is the four disjoint simple closed curves $P(c_i)$. They are unlinked in E^3/G_T because their preimages are unlinked in Bd M_1. It can be shown that each $P(c_i)$ is meridional in $P(\mathrm{Bd}\,M_1)$.

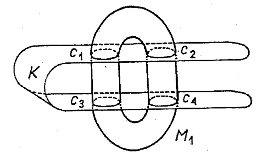

Fig. 3

In the solid torus $P(M_1)$ there must exist four disjoint meridional disks d_1, d_2, d_3, and d_4 that are bounded by $P(c_1)$, $P(c_2)$, $P(c_3)$, and $P(c_4)$, respectively. This implies that $d_1 \cup d_2 \cup d_3 \cup d_4 \cup P(K)$ is a disk. Call it D_T.

The disk $D_T \subset E^3/G$ will be shown to not be ε-homeomorphic to any P-liftable disk D_ε for ε less than a particular value ε_0. We will now specify ε_0. Assume M_1 and K are polyhedral. There is an infinite triangulation of $E^3 - H_T^*$ such that K and Bd M_1 are complexes in it. Let r_i denote the regular neighborhood in this triangulation of the simple closed curve c_i. Choose $\varepsilon_0 > 0$ to be a distance such that for each i, the inverse image of an ε_0-neighborhood of $\mathrm{Bd}\,d_i$ lies in r_i and the one of d_i misses $(\mathrm{Bd}\,M_1) - \bigcup_{i=1}^{4} r_i$. These conditions imply that for any disk D'_ε that projects onto a disk D_ε that is ε-homeomorphic to D_T, it is true that $D'_\varepsilon \cap \mathrm{Bd}\,M_1$ is contained in the regular neighborhood of $K \cap \mathrm{Bd}\,M_1$.

THEOREM 3.1. *For any $\varepsilon < \varepsilon_0$ there is no P-liftable disk D_ε that is ε-homeomorphic to D_T.*

Proof. Suppose that there is such a P-liftable disk D_ε. Let D'_ε be the P-lift disk. Then, by the choice of ε_0, the set $D'_\varepsilon \cap M_1$ contains four meridional subdisks. Denote them by δ_1, δ_2, δ_3, and δ_4. By Lemma 2.1 the set $\delta_1 \cup \delta_2$ must intersect every nondegenerate element. Similarly, $\delta_3 \cup \delta_4$ intersects every nondegenerate element. Therefore, each nondegenerate element intersects more than one of the subdisks δ_1, δ_2, δ_3, and δ_4. This implies that the images of these subdisks are not mutually disjoint subsets in D_ε. This leads to a contradiction of the assumption that D_ε is a disk and proves the theorem. ■

Notice that the disk D_T is not the image of any one of the following:

(1) a 2-complex,

(2) a locally connected set, or

(3) a simply connected set.

4. Another example of a disk in E^3/G that can not be approximated by P-liftable disks. The example we now consider is more easily visualized in E^3 than the last one. In this example the disk chosen in E^3/G will be the image of a simply connected 2-complex.

First, we will define a knit Cantor set of nondegenerate elements. We will then construct a decomposition G_k of E^3 containing two knit Cantor sets of nondegenerate elements. In E^3 a 2-complex will be chosen so that its image in E^3/G_k will be a disk D_k. For any ε less than a particular ε_0, there will be no P-liftable disk that is ε-homeomorphic to D_k.

Let h_p and h_q be two line segments that have their endpoints in two parallel planes, A and B, and are perpendicular to them.

DEFINITION. The countably infinite set of arcs

$$\{h_i|\ -\infty < i < \infty\} \cup \{h_p\} \cup \{h_q\}$$

between planes A and B shown in Figure 4 is said to be *knit from the point p to the point q.*

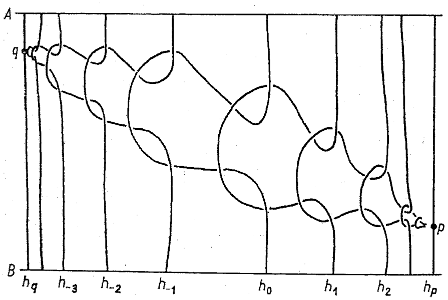

Fig. 4

(Although it is not necessary for this definition that p and q be at different heights, they are so shown in anticipation of a later step in the construction.) Figure 5 illustrates the following generalization of the knitting construction.

DEFINITION. A *Cantor set $\hat{H}$ of arcs is knit from a point p to a point q* if $\hat{H}$ can be realized by the following modification of a countably infinite set of arcs knit from p to q. For each h_i let $N(h_i)$ be a tubular neighbor-

hood of h_i, and let these be such that the members of the set $\{h_p\} \cup \{h_q\} \cup \{N(h_i)|\ -\infty < i < \infty\}$ are pairwise disjoint. In each tubular neighborhood replace the arc h_i by a Cantor set H_i of arcs, each one of which intersects A and B in the same manner as h_i.

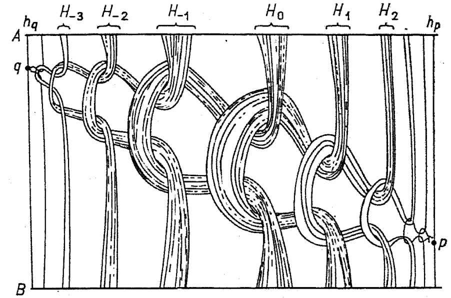

Fig. 5

DEFINITION. Let $N(H)$ be a closed neighborhood containing $\hat{H}$ and such that $\hat{H} \cap \operatorname{Bd} N(H) = h_p \cup h_q$. A set of nondegenerate elements H in E^3 is said to be a *knit Cantor set of nondegenerate elements* if there is an embedding of $\operatorname{Int} N(H)$ into E^3 that takes $[\hat{H} - (h_p \cup h_q)]^*$ into H.

Next we describe a Cantor set of nondegenerate elements which we will modify into a knit Cantor set of nondegenerate elements. The first stage is the two disjoint tori pictured in Figure 6. We decompose each

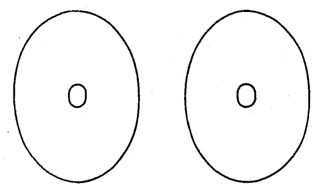

Fig. 6

by a (2,1) toroidal decomposition. Figure 7 indicates the first embedding of pairs of tori whose unions essentially wrap once around. Let P be a plane that cuts each solid torus as indicated in the figure. We can assume that the nondegenerate elements of these two toroidal decompositions intersect P in a standard Cantor set $\mathcal{C}$ of points in a line segment I. Figure 8 shows this plane and some of the points with their usual numerical representation in base 3.

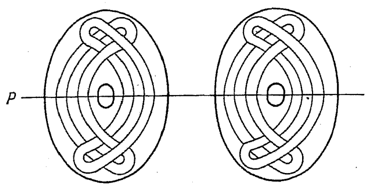

Fig. 7

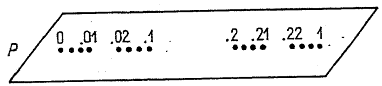

Fig. 8

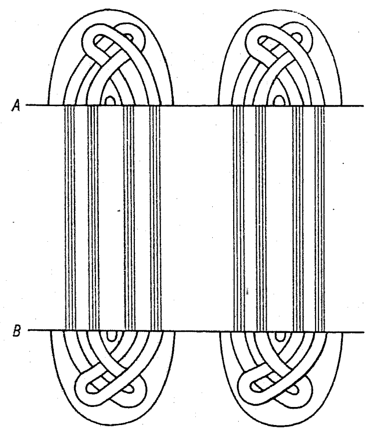

Fig. 9

In Figure 9 we have separated the tori on P. The two copies of P are now labelled planes A and B. Between A and B we indicate the Cantor set of straight line segments connecting copies of $\mathcal{C}$ in A and B. For this Cantor set of segments we substitute the knit Cantor set of Figure 5 in the manner indicated in the following table. The notation $[a, b]$ indi-

$$\ldots \quad H_{-3} \quad H_{-2} \quad H_{-1} \quad H_0 \quad H_1 \quad H_2 \quad \ldots$$

$$\ldots [.0002, .001][.002, .01][.02, .1][.2, .21][.22, .221][.222, .2221] \ldots$$

cates the set of vertical segments containing points in $\mathcal{C}$ in the interval $[a, b] \subset I$, and H_i is a Cantor set of arcs in Figure 5. With these sub-

stitutions we have constructed a knit Cantor set of nondegenerate elements, H. Let g_p and g_q satisfy $g_p^* \supset h_p^*$ and $g_q^* \supset h_q^*$.

Let N be a closed neighborhood of H such that $\mathrm{Bd} N \cap H$ is the elements g_p and g_q. Map N into E^3 in such a way that the following conditions are satisfied. The elements g_p and g_q are identified and the map is an embedding for $N - (g_p \cup g_q)$. In Figure 10, parts of some ele-

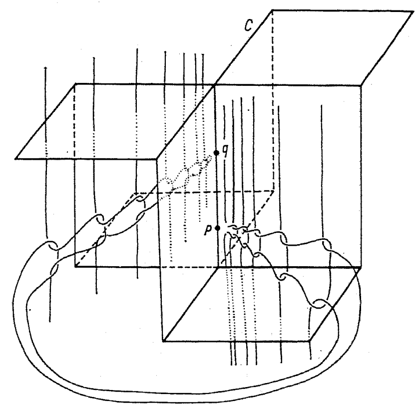

Fig. 10

ments are shown. Notice that p and q are mapped to different points in the identification of the two nondegenerate elements containing them. Also, in Figure 10 is the 2-complex C, which can be specified by:

$$\{(x, y, z) |\ 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1,\ z = 0\},$$
$$\{(x, y, z) |\ -1 \leqslant x \leqslant 0,\ -1 \leqslant y \leqslant 0,\ z = 0\},$$
$$\{(x, y, z) |\ 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 0,\ z = 1\},$$
$$\{(x, y, z) |\ -1 \leqslant x \leqslant 0,\ 0 \leqslant y \leqslant 1,\ z = 1\},$$
$$\{(x, y, z) |\ x = 0,\ -1 \leqslant y \leqslant 1,\ 0 \leqslant z \leqslant 1\}, \text{ and}$$
$$\{(x, y, z) |\ -1 \leqslant x \leqslant 1,\ y = 0,\ 0 \leqslant z \leqslant 1\}.$$

The two identified elements of H are the nondegenerate element g_0, which is the segment of the z-axis: $-1 \leqslant z \leqslant 2$. The points p and q are $z = \frac{1}{3}$ and $z = \frac{2}{3}$. The set $H^* \cap C$ is contained in the line segments g_0,

$$\{(x, y, z) |\ x = y,\ 0 \leqslant x \leqslant 1,\ z = 0\}, \text{ and}$$
$$\{(x, y, z) |\ x = -y,\ -1 \leqslant x \leqslant 0,\ z = 1\}.$$

The last step of the construction is the addition of nondegenerate elements in the half-space $y<0$ in such a way that we have symmetry with respect to the z-axis. All the nondegenerate elements are indicated in Figure 11. They form two knit Cantor sets of nondegenerate elements. Let G_k denote this decomposition of E^3 and H_k denote its set of nondegenerate elements.

The space E^3/G_T, where G_T is the (2,1) toroidal decomposition, is shown by Bing in [4] to be homeomorphic to E^3. From this it can be

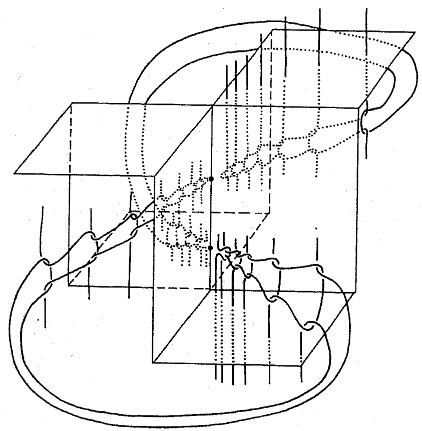

Fig. 11

shown that E^3/G_k is also homeomorphic to E^3. (In any element M_i of the defining sequence, it is possible to shrink the component containing g_0 to any prechosen small size. Then, since in each other component of M_i the decomposition is (2,1) toroidal, there is a shrinking homeomorphism in its interior.)

In E^3/G_k, the set $P(C)$ is a disk. It is the disk D_k that we claim is not ε-homeomorphic to any P-liftable disk for any positive distance ε less than a particular value ε_0. We choose this ε_0 by a method analogous to that used in the last section. For this we use the 3-manifold with boundary shown in Figures 12 and 13 as the first element of a defining sequence for H_k.

THEOREM 4.1. *For any $\varepsilon<\varepsilon_0$, there is no P-liftable disk D_ε that is ε-homeomorphic to D_k.*

Outline of a proof (Details are in [14]): Assume that in E^3/G_k there does exist a disk D_ε that is ε-homeomorphic to D_k and is the image of a disk $D'_\varepsilon\subset E^3$.

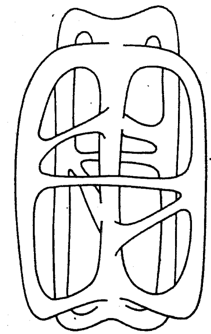

Fig. 12

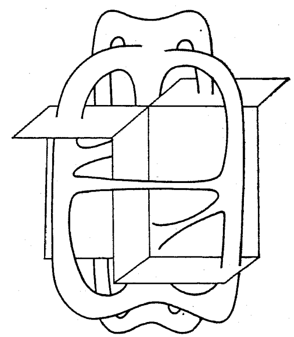

Fig. 13

The construction of G_k was based on a modified (2,1) toroidal decomposition, because that makes the following result provable: Each $g\in H_k$ intersects D'_ε. This proof is similar to the proof of Lemma 2.1. One assumes that there exists a $g_1\in H_k$ that does not intersect D'_ε. Then there is some component $R\supset g_1$ of an element of a defining sequence for H_k such that R does not intersect D'_ε. Analysis is broken into cases concerning whether R and the manifold at the previous stage containing R are tori or cubes-with-more-than-one-handle. It is shown that each case can be reduced to Lemma 2.1.

Recall that H_k consists of a countable collection of Cantor sets of arcs knit in a specific manner described above. Let $\mathcal{A}$ denote a set consisting of g_0 plus exactly one arc from each of these Cantor sets of arcs. Notice that $\mathcal{A}$ is the union of two countable sets, each of which is knit from $p \in g_0$ to $q \in g_0$. Associated with $\mathcal{A}$ there is a decomposition of E^3 having $\mathcal{A}$ as the set of nondegenerate elements. Denote this decomposition by $G_{\mathcal{A}}$ and let $P_{\mathcal{A}}\colon E^3 \to E^3/G_{\mathcal{A}}$. It can be shown that if the disk D'_ε exists, then $P_{\mathcal{A}}(D'_\varepsilon)$ is a disk.

By the definition of D'_ε, its boundary lies in a regular neighborhood, r_0, of $\mathrm{Bd}\,C$. Hence, $D'_\varepsilon \cup C \cup r_0$ separates E^3.

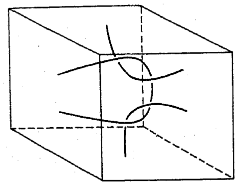

Fig. 14

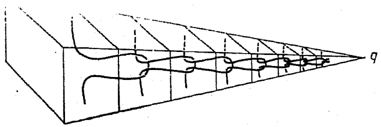

Fig. 15

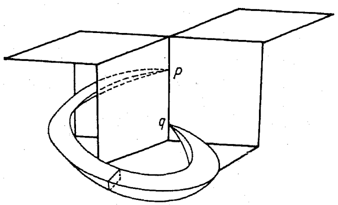

Fig. 16

It is possible to define the knit portions of $\mathcal{A}^*$ in a manner analogous to that used by Fox and Artin for wild arcs [8]. An infinite number of copies of a cube (Fig. 14) containing arcs are mapped into frusta of a pyramid (Fig. 15). Figure 16 indicates the manner in which such pyramids can "contain" the knitting of the set $\mathcal{A}$. Let π denote the pair of pyramids shown in Figure 16. We will use frusta faces that contain maps

of the left and right faces of the cube in Figure 14. One of these frusta faces lies in the unbounded component of $E^3-(D'_\varepsilon \cup C \cup r_0)$. Let f_i denote this face and let f_j denote any other frustum face of π. By analyzing possible intersections of D'_ε with π between f_i and f_j, it can be shown that f_j also lies in the unbounded component of $E^3-(D'_\varepsilon \cup C \cup r_0)$. This result depends on the fact that $P_{\mathcal{A}}(D'_\varepsilon)$ is a disk.

It can now be shown that there must be four disjoint arcs in D'_ε, each from a point in $\mathrm{Bd}\,D'_\varepsilon$ to p or $q \in g_0$. These arcs can be ordered by their endpoints in $\mathrm{Bd}\,D'_\varepsilon$. In this ordering, they alternate with respect to having p and q as their second endpoint. This set of four arcs unioned with g_0 separate D'_ε into four subdisks. A segment of g_0 is a common set in the four boundaries of these disks. This contradicts the assumption that D'_ε is a disk and proves the theorem. ∎

5. Concerning a question asked by Armentrout. In [1] Armentrout asks (Question 8): Suppose G is a pointlike decomposition of E^3. If S is a 2-sphere in E^3/G, does there exist a 2-sphere S' in E^3 such that $P[S']$ is a 2-sphere homeomorphically close to S? We now show that the answer is negative.

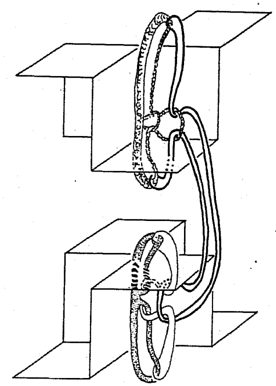

Fig. 17

In the manner that is obvious in Figure 3, a disk in E^3-M_1 can be added to the punctured disk K in such a way that the resulting set becomes a punctured 2-sphere. This immediately gives a counterexample to the question.

It is conjectured that a counterexample can be based on the example in section 4. Because of the orientation of the knitting, it is not possible to add a disk with the same boundary as the set C. Two copies of C can be used and then the nondegenerate elements and the knitting so constructed that the boundaries of the two copies are homotopic in the degenerate points of E^3. A portion of such an arrangement is shown in Figure 17. Similar knitting is repeated in the other four quadrants. It seems very likely that a proof similar to the one outlined in section 4 could be given for this 2-sphere counterexample.

6. Examples in equivalent decompositions. The following definition is made by Armentrout, Lininger, and Meyer [2]:

Definition. If G is any monotone decomposition of E^3, let H_G denote the union of the nondegenerate elements of G, and let P_G denote the projection map from E^3 onto the decomposition space E^3/G associated with G. Suppose that F and G are monotone decompositions of E^3 such that each of $\mathrm{Cl}(P_F[H_F])$ and $\mathrm{Cl}(P_G[H_G])$ is compact and 0-dimensional. Then F and G are *equivalent* decompositions of E^3 if and only if there is a homeomorphism h from E^3/F onto E^3/G such that $h[\mathrm{Cl}(P_F[H_F])] = \mathrm{Cl}(P_G[H_G])$.

For each of the decompositions G_T and G_k above there is an equivalent decomposition having only a countable number of nondegenerate elements. These can be defined using a technique due to Bing [6]. We briefly describe it for G_T. For the (2,1) toroidal decomposition recall that each component of each element of the defining sequence is denoted by T with a subscript consisting of the digits one and two. We can required that the diameter of any torus be less than one-half raised to the power equal to the number of twos in the subscript notation. This method results in only a countable number of nondegenerate elements.

As stated previously, E^3/G_T and E^3/G_k are each homeomorphic to E^3.

A theorem proved in [16] states that if a decomposition G has a countable number of nondegenerate elements and E^3/G is homeomorphic to E^3, then for each disk $D \subset E^3/G$ and each $\varepsilon > 0$ there exists a P-liftable disk D_ε that is ε-homeomorphic to D. Hence, the new equivalent decompositions differ from G_T and G_k in the property concerning the existence of the P-liftable approximating disk D_ε.

7. Concluding remarks. The Kline sphere characterization of the 2-sphere states: A nondegenerate, locally connected, compact continuum which is separated by each of its simple closed curves but by no pair of its points is homeomorphic with the 2-sphere [3]. Consider the two examples (one conjectured) in section 5 of 2-spheres that are not arbitrarily close to P-liftable 2-spheres. The one based on the example in section 3 uses a set in E^3 that is not locally connected; the one based on section 4 uses a set in E^3 that is not separated by each of its simple closed curves. If one does not require that the decomposition be pointlike, then one can easily find other examples of spheres in E^3/G that are not arbitrarily close to P-liftable spheres. Try one circle as a nondegenerate element, or use disconnected nondegenerate elements.

The author constructed the example in section 4 when she was attempting to prove a theorem concerning the existence of a P-liftable disk near a given disk. For such a theorem, it seemed useful to hypothesize that the decomposition satisfies a new property. For this, we make the following definitions.

Definition (Michael [11]). A collection G of closed points sets filling a metric space is said to be *equi*-LC^m provided that it is true that, if y is a point of an element g_0 of G and ε is a positive number, there is a positive number δ such that if g is an element of G, then any mapping of a k-sphere ($k \leqslant m$) onto a subset of $g \cap S(y, \delta)$ is homotopic to a constant on a subset of $g \cap S(y, \varepsilon)$. (The notation $S(y, \delta)$ denotes a δ-neighborhood of y.)

Observe that a knit Cantor set is equi-LC^0. We now make new definitions, which are not satisfied by a knit Cantor set at a point p from which the set is knit.

Definition. A collection G of closed point sets in E^3 is said to be *equi-locally connected* provided that, if y is a point of an element g_0 of G and ε is a positive number, there is in the ε-neighborhood of y a topological 3-cell B containing y in its interior and such that if g is an element of G, then $g \cap \mathrm{Int}B$ is connected.

Definition. A collection G of closed point sets in E^3 is said to be *strongly equi-locally connected* provided that, if y is a point of an element g_0 of G and ε is a positive number, there is in the ε-neighborhood of y a topological 3-cell B containing y in its interior and such that if g is an element of G, then $g \cap \mathrm{Int}B$ and $g \cap B$ are connected.

Definition. A collection G of closed point sets in E^3 is said to be *equi-locally connected* and *equi-semi-connected* provided that, if y is a point of an element g_0 of G and ε is a positive number, there is in the ε-neighborhood of y a topological 3-cell B containing y in its interior and such that if g is an element of G, then $g \cap \mathrm{Int}B$ is connected and $g \cap \mathrm{Ext}B$ has no more than two components.

Question. Is the following true?

Let G be an upper semicontinuous decomposition of E^3 with nondegenerate elements H. Suppose that H is a continuous, strongly equi-locally connected collection and that $P(H)$ is 0-dimensional in E^3/G. Furthermore, suppose that a disk $D \subset E^3/G$ is the image of a locally connected set in E^3. Then for any positive number ε there is a disk D_ε that is ε-homeomorphic to D and is the image of a disk under the projection mapping P.

Questions. Is the above true if we substitute one of the other three definitions of equi-local connectedness? Can the condition that $P(H)$ be 0-dimensional be dropped from the hypotheses?

Notice that the (2,1) toroidal decomposition space and Bing's dogbone space [5] each satisfy the hypotheses in the above question.

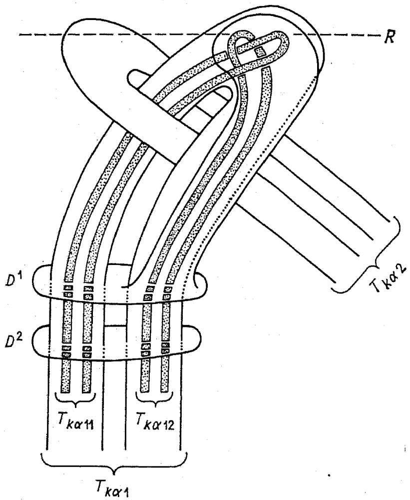

Fig. 18

Because the dogbone space is not homeomorphic to E^3, the method used in section 3 can not be used with this space. Hence, we ask the following question.

Question. Is every disk D in Bing's dogbone space ε-homeomorphic to a P-liftable disk for any $\varepsilon > 0$?

The following leads me to think that the answer to this question may be affirmative. In the (2,1) toroidal decomposition it is not true that every disk can be approximated by a P-liftable disk. This fact is related to the existence of a non-locally connected set that projects onto a disk. One can use the construction indicated in Figure 18 to get such a non-locally connected set. Shown are subdisks D^1 and D^2 that each contain two subdisks intersecting a toroidal component $T_{k\alpha 1}$. For each of the two subdisks of D^1 there must be a torus $T_{k\alpha 1i}$ with $i = 1$ or 2 that D^1 intersects in two meridional subdisks. As shown, it is possible for $T_{k\alpha 11}$ to intersect both subdisks of D^1 and $T_{k\alpha 12}$ to miss both. Similarly, D^2 can

intersect only $T_{k\alpha 12}$. The desired set is obtained by iteration of this construction. In the dogbone space there does not seem to be a similar construction of a non-locally connected set that projects onto a disk. Figure 19 indicates pushing subdisks of D^1, D^2, D^3, and D^4 in an attempt

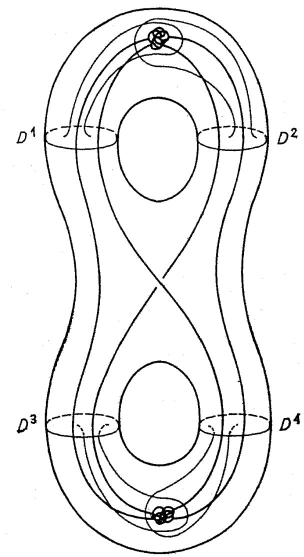

Fig. 19

to have each dogbone at the next stage intersect only one of these subdisks. Obviously, all the pushes shown can not be performed. Of course, this only shows that one can not visualize a counterexample to the question by this method. It would be quite exciting to find that the dogbone space shares with the trivial decomposition a property that the (2,1) toroidal decomposition lacks.

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A characterization of locally connectedness by means of the set function T

by

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Abstract. In this paper the connective properties of the set function T are investigated. In particular, the images of closed sets under T are shown to contain closed connected subsets which are also in the image of T. These results are used to give a characterization of locally connectedness in unicoherent continua. This characterization generalizes a result of Kuratowski which concerned continua contractible with respect to S^1.

A continuum is a compact connected topological space. Throughout this paper X will denote a continuum. If $A \subset X$, then the interior of A in X will be denoted by $\mathrm{int}_X A$ and 2^X will denote the collection of all non-empty closed subsets of X. If $A \in 2^X$ and $p \in X-A$, then X is said to be *aposyndetic at p with respect to A* provided there is a subcontinuum M of X such that $p \in \mathrm{int}_X M \subset M \subset X-A$ [3]. The set function T is a mapping from 2^X into 2^X such that for each $A \in 2^X$, $T(A) = A \cup \{x \in X \mid X$ is not aposyndetic at x with respect to $A\}$.

For terms used but not defined herein, the reader is referred to [4] and [6].

It is easily seen that for each $A \in 2^X$, $T(A)$ is closed in X. In [1] it is shown that if A is connected, then $T(A)$ is also connected. In [5] Vought proved that if X is n-aposyndetic and A is a set consisting of $n+1$ points then $T(A)$ is connected. We shall extend these results concerning the connective properties of T.

The proof of the following lemma parallels that of Lemma 3.1 of [5].

LEMMA 1. *Suppose $S \in 2^X$, S is totally disconnected, $p \in T(S)-S$, and for each closed proper subset S' of S, $p \notin T(S')$. Then $T(S)$ is connected.*

Proof. Let S_0 be a non-empty subset of S which is both open and closed in S. Since $p \notin T(S-S_0)$, there is a subcontinuum H such that $p \in \mathrm{int}_X H \subset H \subset X-(S-S_0)$. Let $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ be decreasing sequences of open sets such that for each positive integer n, $S-S_0 \subset U_n$, $S_0 \subset V_n$, $U_1 \cap \overline{V}_1 = U_1 \cap H = \overline{V}_1 \cap \{p\} = \emptyset$, and $S-S_0 = \bigcap_{n=1}^{\infty} U_n$ while $S_0 = \bigcap_{n=1}^{\infty} V_n$.