STUDIA MATHEMATICA, T. XLIX. (1974)

An inequality for the Hardy-Littlewood maximal operator with respect to a product of differentiation bases

by

MIGUEL DE GUZMÁN (Madrid)

Summary. This paper considers the maximal operator with respect to intervals in $\boldsymbol{R}^2$ and shows how to obtain a weak type inequality for it by using the weak type inequality for the maximal operator with respect to intervals in $\boldsymbol{R}^1$. This is then applied to present an easy proof of the Jessen–Marcinkiewicz–Zygmund theorem on differentiation, and also of some other theorems of Zygmund.

1. INTRODUCTION

In $\boldsymbol{R}^n$ we shall call a *differentiation basis* $\mathscr{R}$ a family of open bounded sets such that for every $x \in \boldsymbol{R}^n$ there exists at least one sequence $\{R_k\} \subset \mathscr{R}$ such that R_k "contracts" to x (notation: $R_k \to x$) in the following sense: i) $x \in R_k$ for all k, and ii) given any neighborhood U of x, there is a subindex k_0 such that, if $k \geqslant k_0$, then $R_k \subset U$.

Given a differentiation basis $\mathscr{R}$ in $\boldsymbol{R}^n$ we define the (*Hardy–Littlewood*) *maximal operator* M with respect to $\mathscr{R}$ in the following way: for $f \in L_{\mathrm{loc}}(\boldsymbol{R}^n)$ (i.e. f real valued, measurable and for every compact set $K \subset \boldsymbol{R}^n$ $\int_K |f| < \infty$) we write

$$Mf(x) = \sup\left\{\big(1/m_n(R)\big)\int_R |f|:\ x \in R \in \mathscr{R}\right\}$$

where $m_n(R)$ means the n-dimensional Lebesgue measure of R. It is inmediate to show that Mf is a measurable function, since the set $\{x:\ Mf(x) > \lambda\}$ is open for every $\lambda > 0$. Hence the maximal operator M is a (sublinear) operator that maps $L_{\mathrm{loc}}(\boldsymbol{R}^n)$ into $M(\boldsymbol{R}^n)$ (real valued measurable functions on $\boldsymbol{R}^n$).

Let now $\mathscr{R}_1, \mathscr{R}_2$ be two given differentiation bases in $\boldsymbol{R}^{n_1} = X_1$, $\boldsymbol{R}^{n_2} = X_2$ and assume that the corresponding Hardy–Littlewood maximal operators M_1, M_2 satisfy the following weak type inequalities for every λ, $0 < \lambda < \infty$ (we write $m_1 = m_{n_1}$, $m_2 = m_{n_2}$):

$$m_i\big(\{x^i \in X_i:\ M_i f_i(x^i) > \lambda\}\big) \leqslant \int \varphi_i\left(\frac{|f_i(x^i)|}{\lambda}\right) dm_i(x^i), \qquad i = 1, 2$$

for each $f_i \in L_{\mathrm{loc}}(X^i)$, where φ_1 and φ_2 are strictly increasing continuous functions from $[0, \infty]$ to $[0, \infty]$ with $\varphi_1(0) = 0$, $\varphi_2(0) = 0$. We shall then say that M_1, M_2 *satisfy inequalities of type* φ_1, φ_2 respectively.

We can consider the space $\boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2}$ which we identify with $\boldsymbol{R}^{n_1+n_2} = \boldsymbol{R}^n = X$ and the differentiation basis R product of $\mathscr{R}_1$ and $\mathscr{R}_2$, that is

$$\mathscr{R} = \{R^1 \times R^2:\ R^1 \in \mathscr{R}_1,\ R^2 \in \mathscr{R}_2\}.$$

We also consider the corresponding maximal operator M.

In this paper we obtain a similar inequality for the operator M, more specifically, for $0 < \lambda < \infty$, $f \in L_{\mathrm{loc}}(\boldsymbol{R}^n)$ we have, setting $m_n = m$,

$$m\big(\{(x^1, x^2) \in X:\ Mf(x^1, x^2) > \lambda\}\big)$$
$$\leqslant \varphi_2(1) \int\limits_X \varphi_1\left(\frac{2|f|}{\lambda}\right) dm + \int\limits_X \left[\int\limits_1^{\frac{4|f|}{\lambda}} \varphi_1\left(\frac{4|f|}{\lambda\sigma}\right) d\varphi_2(\sigma)\right] dm.$$

Weak type inequalities of this kind are useful for several purposes. We shall apply them to the differentiation theory of integrals with respect to intervals in order to show how the theorem of Jessen–Marcinkiewicz–Zygmund [3] and also the more recent results of Zygmund [5] follow easily from this general theorem.

The idea of the proof is based on some elements of a paper of Burkill [1]. The results we present among the applications of Section 3 follow from a general theorem in the thesis of Rubio [4], which was obtained using the above-quoted theorems of Jessen–Marcinkiewicz–Zygmund and of Zygmund.

The present context could be easily generalized. Instead of Euclidean spaces a Lebesgue measure one could deal with more abstract measure spaces, one could consider the product of more spaces, etc. We confine ourselves in our presentation to a rather simple context where the features of the method can be made more intuitive.

2. THE INEQUALITY

In this section we prove the following theorem.

THEOREM. *Let* $\mathscr{R}_i$, $i = 1, 2$ *be two differentiation bases in* $\boldsymbol{R}^{n_i} = X_i$, *whose maximal operators are* M_i. *Assume* M_i *satisfies an inequality of type* φ_i. *Consider in* $\boldsymbol{R}^n = X$, $n = n_1+n_2$, *the product differentiation basis* $\mathscr{R}$ *of* $\mathscr{R}_1$ *and* $\mathscr{R}_2$. *Then the maximal operator* M *of* $\mathscr{R}$ *satisfies*

$$m(\{x \in X:\ Mf(x) > \lambda\})$$
$$\leqslant \varphi_2(1) \int\limits_X \varphi_1\left(\frac{2|f(x)|}{\lambda}\right) dm(x) + \int\limits_X \left[\int\limits_1^{4\frac{|f(x)|}{\lambda}} \varphi_1\left(\frac{4|f(x)|}{\lambda\sigma}\right) d\varphi_2(\sigma)\right] dm(x)$$

for each $f \in L_{\mathrm{loc}}(\boldsymbol{R}^n)$ *and each* λ, $0 < \lambda < \infty$.

Proof. In order to emphasize the main ideas of the proof we shall first disregard measurability problems. Let $f \geqslant 0$, $f \in L_{\mathrm{loc}}(\boldsymbol{R}^n)$. For $x^1 \in X_1$, $y \in X_2$ we define

$$T_1 f(x^1, y) = \sup\Big\{\big(1/m_1(J)\big) \int\limits_J f(z, y)\, dm_1(z):\ x^1 \in J \in \mathscr{R}_1\Big\}.$$

Consider, for $\lambda > 0$ the set

$$A = \left\{(x^1, y):\ T_1 f(x^1, y) > \frac{\lambda}{2}\right\}$$

and define again, for $x^1 \in X_1$, $x^2 \in X_2$,

$$T_2 f(x^1, x^2) = \sup\Big\{\big(1/m_2(H)\big) \int\limits_H \chi_A(x^1, z) T_1 f(x^1, z)\, dm_2(z):\ x^2 \in H \in \mathscr{R}_2\Big\}.$$

We first prove

$$B = \{(x^1, x^2):\ Mf(x^1, x^2) > \lambda\} \subset \left\{(x^1, x^2):\ T_2 f(x^1, x^2) > \frac{\lambda}{2}\right\} = C.$$

If (x^1, x^2) is such that $Mf(x^1, x^2) > \lambda$, there are J and H, $x^1 \in J \in \mathscr{R}_1$, $x^2 \in H \in \mathscr{R}_2$ such that, if $I = J \times H$,

$$\frac{1}{m(J \times H)} \int\limits_{J\times H} f(x^1, x^2)\, dm(x^1, x^2) > \lambda.$$

We partition $I = J \times H$ into two sets C_1, C_2 which are obtained by slicing $J \times H$ in the following way. Let $J \times \{\eta^2\}$ with $\eta^2 \in H$ be such a slice. Then $J \times \{\eta^2\} \subset C_1$ if for every $(y^1, \eta^2) \in J \times \{\eta^2\}$ we have $T_1 f(y^1, \eta^2) > \dfrac{\lambda}{2}$.

If there is at least one $(y^1, \eta^2) \in J \times \{\eta^2\}$ such that $T_1 f(y^1, \eta^2) \leqslant \dfrac{\lambda}{2}$ then $J \times \{\eta^2\} \subset C_2$. We get, from the definition of C_2,

$$\int\limits_{C_2} f \leqslant \frac{\lambda}{2} m(C_2) \leqslant \frac{\lambda}{2} m(I)$$

and since $\int\limits_{C_1} f + \int\limits_{C_2} f = \int\limits_I f > \lambda m(I)$, we obtain $\int\limits_{C_1} f > \dfrac{\lambda}{2} m(I)$. Now we have:

$$T_2 f(x^1, x^2) \geqslant \big(1/m_2(H)\big) \int\limits_H \chi_A(x^1, z) T_1 f(x^1, z)\, dm_2(z)$$
$$\geqslant \big(1/m_2(H)\big) \int\limits_H \chi_A(x^1, z) \Big[\big(1/m_1(J)\big) \int\limits_J f(y, z)\, dm_1(y)\Big] dm_2(z)$$
$$\geqslant \big(1/m(I)\big) \int\limits_{C_1} f(y, z)\, dm(y, z) > \frac{\lambda}{2}.$$

Hence $B \subset C$ as we wished to prove. We now establish the desired inequality for the measure of the set C. We can assume the second member finite, since otherwise there is nothing to prove. For almost every fixed x^1 we have

$$m_2\big(\{(x^1, x^2):\ T_2f(x^1, x^2) > \lambda/2\}\big) \leqslant \int_{X_2} \varphi_2\left(\frac{\chi_A(x^1, z)\,T_1f(x^1, z)}{\lambda/2}\right) dm_2(z).$$

So we have, integrating with respect to x^1 and interchanging the order of integration,

$$\begin{aligned} m\big(\{(x^1, x^2):\ T_2f(x^1, x^2) > \lambda/2\}\big) &\leqslant \int_{X_2}\int_{X_1} \varphi_2\left(\frac{\chi_A(x^1, z)\,T_1f(x^1, z)}{\lambda/2}\right) dm_1(x^1)\,dm_2(z) \\ &= \int_{X_2}\left[\int_0^\infty m_1\left(\left\{x^1:\ \varphi_2\left(\frac{\chi_A(x^1, z)\,T_1f(x^1, z)}{\lambda/2}\right) > \sigma\right\}\right) d\sigma\right] dm_2(z) \\ &= \int_{X_2}\Big[\int_0^{\varphi_2(1)} + \int_{\varphi_2(1)}^\infty\Big]. \end{aligned}$$

Now we have:

$$\begin{aligned} \int_{X_2}\int_0^{\varphi_2(1)} m_1\left(\left\{x^1:\ \varphi_2\left(\frac{\chi_A(x^1, z)\,T_1f(x^1, z)}{\lambda/2}\right) > \sigma\right\}\right) d\sigma\, dm_2(z) &\leqslant \int_{X_2} \varphi_2(1)\, m_1\left(\left\{x^1:\ T_1f(x^1, z) > \frac{\lambda}{2}\right\}\right) dm_2(z) \\ &\leqslant \int_{X_2} \varphi_2(1) \int_{X_1} \varphi_1\left(\frac{f(x^1, x^2)}{\lambda/2}\right) dm_1(x^1)\,dm_2(x^2). \end{aligned}$$

Also, with the change $\sigma = \varphi_2(\tau)$, we have:

$$\begin{aligned} \int_{X_2}\int_{\varphi_2(1)}^\infty m_1\left(\left\{x^1:\ \varphi_2\left(\frac{\chi_A(x^1, z)\,T_1f(x^1, z)}{\lambda/2}\right) > \sigma\right\}\right) d\sigma\, dm_2(z) \\ = \int_{X_2}\int_1^\infty m_1\left(\left\{x^1:\ \frac{\chi_A(x^1, z)\,T_1f(x^1, z)}{\lambda/2} > \sigma\right\}\right) d\varphi_2(\tau)\,dm_2(z). \end{aligned}$$

We can set

$$f^*(x^1, x^2) = \begin{cases} f(x^1, x^2) & \text{if } f(x^1, x^2) > \dfrac{\lambda\tau}{4}, \\ 0 & \text{if } f(x^1, x^2) \leqslant \dfrac{\lambda\tau}{4}, \end{cases}$$

and $f = f^* + f_*$. It is then clear that $T_1f \leqslant T_1f^* + T_1f_*$ and $T_1f_* \leqslant \dfrac{\lambda\tau}{4}$. So we obtain

$$\begin{aligned} \int_{X_2}\int_1^\infty m_1\left(\left\{x^1:\ \frac{T_2f(x^1, z)}{\lambda/2} > \tau\right\}\right) d\varphi_2(\tau)\,dm_2(z) &\leqslant \int_{X_2}\int_1^\infty m_1\left(\left\{x^1:\ T_1f^*(x^1, z) > \frac{\lambda\tau}{4}\right\}\right) d\varphi_2(\tau)\,dm_2(z) \\ &\leqslant \int_{X_2}\int_1^\infty\int_{X_1} \varphi_1\left(\frac{f^*(x^1, z)}{\lambda\tau/4}\right) dm_1(x^1)\,d\varphi_2(\tau)\,dm_2(z) \\ &= \int_{X_1\times X_2}\left[\int_1^{\frac{4f}{\lambda}} \varphi_1\left(\frac{4f}{\lambda\tau}\right) d\varphi_2(\tau)\right] dm_1\,dm_2. \end{aligned}$$

Adding up we get the statement of theorem.

In order to treat with more ease the measurability problems that arise in the foregoing argument we first try to show that everything we have done works out well for a function of the type $f(x^1, x^2) = \sum_{k=1}^{N_0} a_k \chi_{A_k}(x^1, x^2)$ with $a_k > 0$, A_k being the product of two open bounded intervals A_k^1, A_k^2 of X_1, X_2 respectively. We can disregard the closed set N of m-measure zero of all points on the affine subspaces bordering the sets A_k. It is then easy to see that for every $a > 0$ the set

$$\{(x^1, y) \in X - N:\ T_1f(x^1, y) > a\}$$

is open in X and so $T_1f\colon X \to \boldsymbol{R}^+$ is measurable. For convenience we define $A = \left\{(x^1, y) \in X - N:\ T_1f(x^1, y) > \dfrac{\lambda}{2}\right\}$ and remark that A is open in X. Moreover, if we call π_1, π_2 the projections of X onto X_1, X_2 we have that for every $x^1 \in X_1 - \pi_1(N)$ the function $T_1f(x^1, \cdot)\colon X_2 \to \boldsymbol{R}^+$ is also measurable since the set

$$\{x^2 \in X_2:\ T_1f(x^1, x^2) > a\}$$

is open in X_2 for every $a > 0$. Therefore, for such an x^1 and for every x^2 we can define $T_2f(x^1, x^2)$ as we have done. We now prove that $T_2f\colon X \to \boldsymbol{R}^+$ is measurable by showing that also the set

$$\{(x^1, x^2) \in X - N:\ T_2f(x^1, x^2) > a\}$$

is open for every $a > 0$. In fact, if $T_2f(x^1, x^2) > a$, then there is $H \in R_2$

such that $x^2 \in H$ and

$$\frac{1}{m_2(H)} \int_H \chi_A(x^1, z) T_1 f(x^1, z)\, dm_2(z) > a + \eta$$

for some $\eta > 0$. Take any compact subset K of $H - \pi_2(N)$ containing a neighborhood of x^2 in X_2 and such that

$$\frac{1}{m_2(H)} \int_{H-K} \chi_A(x^1, z) T_1 f(x^1, z)\, dm_2(z) \leqslant \frac{\eta}{2}.$$

For every (x^1, z) with $z \in K$ there is in X an n-dimensional open interval $U(x^1, z)$ centered at (x^1, z) so that for every $(y^1, y^2) \in U(x^1, z)$ we have

$$\chi_A(x^1, z) T_1 f(x^1, z) - \frac{\eta}{2m_2(H)} \leqslant \chi_A(y^1, y^2) T_1 f(y^1, y^2).$$

In fact, if $(x^1, z) \in A$, then the inequality is trivial. Assume $(x^1, z) \in A$. Observe first that, since A is open, we have a neighborhood $V(x^1, z)$ of (x^1, z) in X contained in A. Observe further that if we form the number $\chi_A(x^1, z) T_1 f(x^1, z) - \frac{\eta}{2m_2(H)}$ we then have $J \in R_1$, $x^1 \in J$ such that

$$\frac{1}{m_1(J)} \int_J f(u, z)\, dm_1(u) > \chi_A(x^1, z) T_1 f(x^1, z) - \frac{\eta}{2m_2(H)}.$$

Notice also that, in virtue of the simple form of f we can translate a little the set $J \times \{z\}$ in X so that we obtain $J \times \{v\}$ with $|v - z| < \varepsilon$, $\varepsilon > 0$, in such a form that we still have

$$\frac{1}{m_1(J)} \int_J f(u, v)\, dm_1(u) > \chi_A(x^1, z) T_1 f(x^1, z) - \frac{\eta}{2m_2(H)}.$$

The intersection of the two neighborhoods of (x^1, z) just considered contains an interval $U(x^1, z)$ as indicated.

Hence, by the Heine–Borel theorem, we have a neighborhood of $\{x^1\} \times K$ in X such that for all its points of the form with y^1 fixed we have in particular

$$\chi_A(x^1, z) T_1 f(x^1, z) - \frac{\eta}{2m_2(H)} \leqslant \chi_A(y^1, z) T_1 f(y^1, z).$$

Hence

$$\frac{1}{m_2(H)} \int_H \chi_A(y^1, z) T_1 f(y^1, z)\, dm_2(z) \geqslant \frac{1}{m_2(H)} \int_K \chi_A(y^1, z) T_1 f(y^1, z)\, dm_2(z)$$

$$\geqslant \frac{1}{m_2(H)} \int_K \chi_A(y^1, z) T_1 f(y^1, z)\, dm_2(z) - \frac{\eta m_2(K)}{2m_2(H)}$$

$$= \frac{1}{m_2(H)} \int_H \chi_A(x^1, z) T_1 f(x^1, z)\, dm_2(z) -$$

$$- \frac{1}{m_2(H)} \int_{H-K} \chi_A(x^1, z) T_1 f(x^1, z)\, dm_2(z) - \frac{\eta m_2(K)}{2m_2(H)}$$

$$> a + \eta - \frac{\eta}{2} - \frac{\eta}{2} = a.$$

So we have that $T_2 f\colon X \to \boldsymbol{R}^+$ is measurable.

The measurability of the set C_2 is proved in the same way. If $(y^1, \eta^2) \in J \times \{\eta^2\} - N$ is such that $T_1 f(y^1, \eta^2) \leqslant \frac{\lambda}{2}$ then there is an $\varepsilon > 0$ such that if (y^1, η^*) is such that η^* is in the X_2-ball of center η^2 and radius ε we have

$$T_1 f(y^1, \eta^2) = T_1 f(y^1, \eta^*) \leqslant \frac{\lambda}{2}$$

and so the projection $\pi_2(C_2)$ is open and C_2 is measurable.

We shall now indicate how the restriction imposed on f can be removed.

(a) Assume first that h is a nonnegative linear combination of characteristic functions of disjoint open bounded sets. We can take a sequence of functions $\{f_k\}$ of the type of the one which has already been treated such that $f_k \nearrow h$ a.e. For f_k the inequality is valid and we have

$$\{x\colon Mh(x) > \lambda\} \subset \bigcup_k \{x\colon Mf_k(x) > \lambda\}$$

since, if $Mh(x) > \lambda$, we have a k such that $Mf_k > \lambda$. The sequence of sets $\{x\colon Mf_k(x) > \lambda\}$ is increasing and so we have

$$m(\{x\colon Mh(x) > \lambda\}) \leqslant \lim_{k\to\infty} m(\{x\colon Mf_k(x) > \lambda\})$$

$$\leqslant \lim_{k\to\infty} \Phi\left(\frac{f_k}{\lambda}\right) \leqslant \Phi\left(\frac{h}{\lambda}\right),$$

where $\Phi\left(\frac{f}{\lambda}\right)$ denotes the second member of the inequality in the statement of the theorem.

(b) If g is a nonnegative linear combination of characteristic functions of disjoint compact sets we can take a sequence $\{h_k\}$ of functions as in (a) such that $h_k \searrow g$ a.e. For these functions we have

$$m(\{x\colon Mh_k(x) > \lambda\}) \leqslant \Phi\left(\frac{h_k}{\lambda}\right) < \infty.$$

We also have

$$m(\{x\colon\ Mg(x) > \lambda\}) \leqslant m(\{x\colon\ Mh_k(x) > \lambda\})$$

for each k and making $k \to \infty$ we obtain

$$m(\{x\colon\ Mg(x) > \lambda\}) \leqslant \Phi\left(\frac{g}{\lambda}\right).$$

(c) If l is an L_0^∞ function (i.e. essentially bounded and with compact support) we proceed as in (a) taking now functions $\{g_k\}$ as in (b) such that $g_k \nearrow l$ a.e.

(d) If $j \in L_{\text{loc}}$, $j \geqslant 0$, we consider for $k = 1, 2, \ldots$

$$j_k(x) = \begin{cases} j(x) & \text{if } |x| \leqslant k,\ j(x) \leqslant k, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $j_k \nearrow j$ a.e. and $j_k \in L_0^\infty$. Proceeding as in (a) we obtain the inequality for j.

3. APPLICATION TO DIFFERENTIATION THEORY

3.1. Some particular cases of the preceding inequality. Consider in the preceding theorem $n_1 = n_2 = 1$, $\mathscr{R}_1 = \mathscr{R}_2$ being the basis of open bounded intervals. Then $\mathscr{R}$ is the basis of two-dimensional open bounded intervals. We know, according to the classical inequality for the one-dimensional Hardy–Littlewood maximal operator that we can take $\varphi_1(u) = \varphi_2(u) = 4u$ and so we easily get for $f \in L_{\text{loc}}(\boldsymbol{R}^2)$, $\lambda > 0$

$$m(\{x\colon\ Mf(x) > \lambda\}) \leqslant c \int\limits_{\boldsymbol{R}^2} \frac{|f|}{\lambda}\left(1 + \lg^+ \frac{|f|}{\lambda}\right) dm$$

where c does not depend on f, λ.

An easy computation of the same type shows that, if $\varphi_1(u) = c_1 u(1+\lg^+ u)^s$ and $\varphi_2(u) = c_2 u$, then we get from the general theorem

$$m(\{x\colon\ Mf(x) > \lambda\}) \leqslant \int c \frac{|f|}{\lambda}\left(1+\lg^+ \frac{|f|}{\lambda}\right)^{s+1} dm.$$

Hence for the basis of open bounded intervals in $\boldsymbol{R}^n$ we get

$$m(\{x\colon\ Mf(x) > \lambda\}) \leqslant \int c \frac{|f|}{\lambda}\left(1+\lg^+ \frac{|f|}{\lambda}\right)^{n-1} dm.$$

If we consider in the general theorem in $\boldsymbol{R}^{n_1}$ the basis of all open cubic intervals and in $\boldsymbol{R}^{n_2}$ the basis of all open bounded intervals, then we know by an easy covering property of the cubic intervals that $\varphi_1(u) = c_1 u$ and by the preceding reasoning $\varphi_2(u) = c_2 u(1+\lg^+ u)^{n_2-1}$. Hence $\varphi(u) = cu(1+\lg^+ u)^{n_2}$ and so for the basis $\mathscr{R}$ in $\boldsymbol{R}^n$, $n = n_1 + n_2$, of intervals whose n_1 "first edges" (ordering the edges corresponding to one vertex according to the coordinate axes) are of the same length and the others n_2 are of arbitrary length we get

$$m(\{x\colon\ Mf(x) > \lambda\}) \leqslant \int c \frac{|f|}{\lambda}\left(1+\lg^+ \frac{|f|}{\lambda}\right)^{n_2}.$$

It is immediate to see that if we consider in $\boldsymbol{R}^n$ the basis of intervals such that n_1 arbitrary edges are of the same length we obtain the same inequality with another different constant c.

3.2. Differentiation. Given a basis $\mathscr{R}$ in $\boldsymbol{R}^n$ and a function $f \in L_{\text{loc}}(\boldsymbol{R}^n)$ we can define the *upper derivative of* $\int f$ *with respect to* $\mathscr{R}$ *at* x

$$\overline{D}\left(\int f, x\right) = \sup \limsup_{k\to\infty} \frac{1}{m(R_k)} \int\limits_{R_k} f$$

where the sup is taken over all sequences $\{R_k\} \subset \mathscr{R}$ with $R_k \to x$. In the same way we define the lower derivative

$$\underline{D}\left(\int f, x\right) = \inf \liminf_{k\to\infty} \frac{1}{m(R_k)} \int\limits_{R_k} f.$$

When the upper and lower derivative coincide at x we call this number the *derivative of* $\int f$ *with respect to* $\mathscr{R}$ *at* x.

3.3. The theorem of Jessen–Marcinkiewicz–Zygmund and the theorem of Zygmund. The inequalities we have proved lead to an immediate proof of the following theorem.

THEOREM (Zygmund, 1967. For $n_1 = 1$, $n_2 = n-1$, Jessen–Marcinkiewicz–Zygmund, 1935). *Consider in* $\boldsymbol{R}^n$, $n = n_1 + n_2$, *the basis* $\mathscr{R}$ *of open bounded intervals such that, fixing one vertex and considering the edges corresponding to that vertex,* n_1 *of such edges are of the same length and the others are of arbitrary lengths. Let* f *be a function in* $L(\lg^+ L)^{n_2}(\boldsymbol{R}^n)$, *i.e. such that* $\int |f|(\lg^+|f|)^{n_2} < \infty$. *Then the derivative of* $\int f$ *with respect to* $\mathscr{R}$ *is* f *almost everywhere.*

Proof. Without any loss of generality we can reduce us to prove the theorem for $f \geqslant 0$ and $f \in L(1+\lg^+ L)^{n_2}$. The basis of intervals (no matter how many edges are of the same length) is such that if $g \in L^\infty$ then the derivative of $\int g$ with respect to R is g almost everywhere. (This fact can be derived from a general criterion of Busemann and Feller [2] and from the inequalities we have obtained.) We can take a sequence of simple functions $\{g_k\}$ such that $g_k \nearrow f$ at every point. For g_k we have the differentiability property of the theorem. Let $h_k = f - g_k$. We can write for every $\alpha > 0$

$$G = \left\{x\colon \left|\overline{D}\left(\int f, x\right) - f(x)\right| > \alpha\right\} = \left\{x\colon \left|\overline{D}\left(\int h_k, x\right) - h_k(x)\right| > \alpha\right\}$$

$$\subset \left\{x\colon \left|\overline{D}\left(\int h_k, x\right)\right| > \frac{\alpha}{2}\right\} \cup \left\{x\colon |h_k(x)| > \frac{\alpha}{2}\right\} = A_k \cup B_k,$$

For A_k we have $A_k \subset \left\{x\colon Mh_k(x) > \frac{\alpha}{2}\right\}$ and so $m(A_k) \leqslant \Phi_{n_2}\left(\frac{2h_k}{\alpha}\right)$ where $\Phi_{n_2}\left(\frac{f}{\lambda}\right)$ denotes the second member of the last inequality in 3.1. For B_k we have $m(B_k) \leqslant \frac{1}{c}\Phi_{n_2}\left(\frac{2h_k}{\alpha}\right)$. Hence the exterior measure of G must be zero since $m(A_k) + m(B_k) \to 0$ as $k \to \infty$. This proves $\overline{D}\left(\int f, x\right) = f(x)$ a.e. In the same way $\underline{D}\left(\int f, x\right) = f(\alpha)$ a.e. and so the theorem is proved.

Acknowledgement. I wish to thank I. Peral for some very helpful remarks on a previous version of this paper.

References

[1] J. C. Burkill, *On the differentiability of multiple integrals*, J. London Math. Soc. 26 (1951), pp. 244–249.

[2] H. Busemann und W. Feller, *Zur Differentiation der Lebesgueschen Integrale*, Fund. Math. 22 (1934), pp. 226–256.

[3] B. Jessen, J. Marcinkiewicz and A. Zygmund, *Note on the differentiability of multiple integrals*, Fund. Math. 25 (1935), pp. 217–234.

[4] B. Rubio, *Propiedades de derivación y el operador maximal de Hardy–Littlewood* (Doctoral thesis, Universidad de Madrid, 1971).

[5] A. Zygmund, *A note on the differentiability of multiple integrals*, Colloq. Math. 16 (1967), pp. 199–204.

UNIVERSIDAD DE MADRID

Received January 8, 1973 (631)