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# Duality theory for the strict topology


by

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**Abstract.** Let $C_b(X)$ denote the bounded continuous functions on an arbitrary completely regular Hausdorff space $X$, and let $t_s$ denote the strict topology on $C_b(X)$. First we give a new proof of Conway's theorem that $t_s$ is a Mackey topology if $X$ is paracompact and locally compact. Secondly we describe the Mackey topology for $t_s$ for an entirely different class of $X$'s. Finally, we show that $C_b(X)$ with the $t_s$ topology is a dual if and only if $X$ is compact.


**1. Introduction.** The present paper grew out of our recent analysis of locally convex topologies related to the strict topology originally defined by Buck. In our previous paper we emphasized the interrelationships between those various topologies. In this paper we emphasize the strict topology, and study the Mackey topology for the strict topology, and we analyze when the space on which the strict topology is defined is a dual of some locally convex space.

To be more precise, let $C_b(X)$ be the space of all complex-valued bounded continuous functions on the arbitrary completely regular space $X$. In [1], Buck asked when the strict topology on $C_b(X)$ is a Mackey topology. After some years, Conway succeeded in giving a beautiful —but intricate— proof that whenever $X$ is locally compact and paracompact, the strict topology is a Mackey topology. In Section 2 we give an entirely new proof of this fact, which in outline at least appears more elementary than Conway's proof. We then go to an entirely different variety of $X$ and describe the Mackey topology for the strict topology whenever $X$ is a locally compact, non-compact space which has the property that all $\sigma$-compact subsets are relatively compact. This description can be used to generalize Conway's example showing that the strict topology is not always the Mackey topology.

Section 3 answers the following question: Under what conditions is $C_b(X)$ with the strict topology the dual of some locally convex space? Our answer is that it is a dual if and only if $X$ is compact. Finally we show that only under very special conditions a locally convex space $E$ with the weak topology is the dual of any locally convex space.

Before turning to the results of the paper, we concentrate on terminology and notations. In the first place, whenever $X$ appears it is completely regular and Hausdorff (though we may specify that $X$ have additional properties). If $A$ and $B$ are subsets of $X$, then $A \setminus B$ is the collection of elements in $A$ but not in $B$. A function $\varphi$ on $X$ vanishes at $\infty$ if for each $\varepsilon > 0$ there exists a compact subset $K_\varepsilon$ of $X$ such that $|\varphi(x)| < \varepsilon$ for each $x \in X \setminus K_\varepsilon$. The Stone–Čech compactification of $X$ is written $\beta X$, and it is the largest compactification of $X$. By $C_b(X)$ we mean the collection of complex-valued, bounded, continuous functions on $X$, as a vector space. On $C_b(X)$ there is a natural norm topology given by the supremum norm $\|\cdot\|$, where

$$\|f\| = \sup\{|f(x)|:\ x \in X\}.$$

The strict topology $t_s$ on $C_b(X)$ is described by saying that a net $(f_\lambda)_{\lambda \in \Lambda}$ in $C_b(X)$ converges in $t_s$ to $f \in C_b(X)$ if and only if for each function $\varphi$ on $X$ which is bounded and which vanishes at $\infty$, $f_\lambda \varphi \to f\varphi$ uniformly on $X$. We let the characteristic function of $A \subseteq X$ be written $\chi_A$.

If $E$ is any locally convex linear topological space, we let $E^*$ be the collection of continuous linear functionals on $E$, and call $E^*$ the *dual of $E$*. If we identify $f \in C_b(X)$ with the unique continuous extension $f'$ on $\beta X$, then the Riesz–Kakutani Theorem says in essence that $(C_b(X), \|\cdot\|)^* = M(\beta X)$, where $M(\beta X)$ consists of all countably additive regular bounded measures on $\beta X$. On the other hand, Theorem 2.6 of [3] tells us that $(C_b(X), t_s)^* = M(X)$, the collection of all countably additive regular bounded measures on $X$. If $\mu \in M(X)$, then $|\mu|$ denotes the total variation of $\mu$. If $A \subseteq X$, then $\|\mu\|_A$ and $\|\mu|_A\|$ are by definition $|\mu|(A)$, while by $\|\mu\|$ we mean $\|\mu\|_X$. If $B \subseteq M(X)$, then $\operatorname{supp} B$ means $\bigcup_{\mu \in B} \operatorname{supp} \mu$, where $\operatorname{supp} \mu$ denotes the support of $\mu$. If $E$ is any locally convex space and if $A \subseteq E$, then

$$A^{\flat} = \{x^* \in E^*:\ |x^*(x)| \leqslant 1, \text{ for all } x \in A\},$$

and $A^{\flat}$ is called the *polar* of $A$. The equicontinuous subsets of $E^*$ are by definition just those subsets contained in the polars of neighborhoods of 0 in $E$. On the other hand, if $B \subseteq E^*$, then we define $B^{\natural}$ by the equation

$$B^{\natural} = \{x \in E:\ |x^*(x)| \leqslant 1, \text{ for all } x^* \in B\},$$

and also call $B^{\natural}$ the *polar of $B$*. Furthermore, $A^{\flat\natural}$ means $(A^{\flat})^{\natural}$, etc.

**2. The Mackey topology for the strict topology.** The Mackey topology for a given locally convex topology on a given space is the strongest locally convex topology on the space which yields the same dual. The given topology is the *Mackey topology* if and only if every convex weak* compact subset of the dual is equicontinuous ([5], p. 206). It is this characterization of Mackey topologies that we use in proving that if $X$ is locally compact and paracompact, then $t_s$ is the Mackey topology for $(C_b(X), t_s)$. But first we need some information on weak* compact subsets, furnished by the following theorem.



THEOREM 2.1. *If $X$ is paracompact and locally compact, and if $B$ is a uniformly bounded subset of $M(X)$ but $\operatorname{supp} B$ is not relatively $\sigma$-compact, then $B$ cannot be relatively weak* compact in $(C_b(X), t_s)^*$.*

Proof. Since $X$ is paracompact and locally compact, we may assume that $X = \bigcup_{\lambda \in \Lambda} X_\lambda$, where each $X_\lambda$ is $\sigma$-compact, closed and open in $X$, for the appropriate index set $\Lambda$. Since $B$ is assumed to be uniformly bounded, we may as well assume that if $\mu \in B$, then $\|\mu\| \leqslant 1$. Since bounded regular countably additive measures on $X$ must have $\sigma$-compact supports, if $\operatorname{supp} B$ is not relatively $\sigma$-compact, then there exists an uncountable subset $\Lambda^\circ$ in $\Lambda$ such that if $\lambda \in \Lambda^\circ$, then there is a $\mu_\lambda \in B$ such that $|\mu_\lambda|(X_\lambda) > 0$. Next, well-order $\Lambda^\circ$. Since countable unions of countable sets are countable, we can furthermore say that if $\operatorname{supp} B$ is not relatively $\sigma$-compact, then there exists a $c \in (0, 1)$ and an uncountable subset $\Lambda_0 \subseteq \Lambda$ such that for each $\lambda \in \Lambda_0$ there is a $\mu_\lambda \in B$ for which not only $|\mu_\lambda|(X_\lambda) \geqslant c$, but also $|\mu_\lambda|(X_{\lambda'}) = 0$ for all $\lambda' > \lambda$ with $\lambda' \in \Lambda_0$. Henceforth we will without loss of generality assume that $\Lambda_0$ is identified with the ordinals less than the first uncountable, since $\Lambda_0$ is at any rate uncountable and a suitable subset of $\Lambda_0$ can be so identified.

Let $\lambda_0$ be the first element of $\Lambda_0$ for which there exists an associated cofinal net $N_{\lambda_0} \subseteq \Lambda_0$ such that $\lambda \in N_{\lambda_0}$ and $\lambda > \lambda_0$ together imply that $|\mu_\lambda|(X_{\lambda_0}) < c/2$. Such a $\lambda_0$ exists. (In fact, if $n$ is an integer $> 2/c$ and if $\lambda_1, \ldots, \lambda_n \in \Lambda_0$ and if each $\lambda_i$ lacks the just-named property, then for any $\lambda \in \Lambda_0$ such that $\lambda > \lambda_i$, $i = 1, 2, \ldots, n$, we have $|\mu_\lambda|(X_{\lambda_i}) \geqslant c/2$, for $i = 1, 2, \ldots, n$. Consequently,

$$|\mu_\lambda|\Big(\bigcup_{i=1}^{n} X_{\lambda_i}\Big) > 1,$$

contradicting the assumption that the elements of $B$ have norm at most 1.) Define $\lambda_0$ to be in $\Lambda_1$, and call $N_{\lambda_0}$ an *associated net corresponding to* $\lambda_0$.

Now inductively assume that $\lambda' \in \Lambda_1$, and let $N_{\lambda'}$ be an associated cofinal net for $\lambda'$. We will find the next element of $\Lambda_1$ and an associated cofinal net. Let $\lambda''$ be the smallest element of $N_{\lambda'}$ such that $\lambda'' > \lambda'$. We define $\lambda''$ to be in $\Lambda_1$ iff there exists a cofinal $N_{\lambda''} \subseteq N_{\lambda'}$ such that

$$\text{(a)} \qquad |\mu_\nu|\Big(\bigcup_{\substack{\lambda_0 \leqslant \lambda < \lambda'' \\ \lambda \in \Lambda_0}} X_\lambda\Big) \geqslant c/2, \quad \text{all } \nu \in N_{\lambda_i'} \text{ with } \nu \geqslant \lambda''.$$

We call $N_{\lambda''}$ an *associated net to* $\lambda''$. If there is no such cofinal net, then begin all over again in the construction of $\Lambda_1$, with $\Lambda_0$ replaced by $\{\lambda \in N_{\lambda'}: \lambda > \lambda''\}$ and with $\lambda_0$ replaced by the first element of this latter set. Such a new beginning to the process of creating $\Lambda_1$ can occur no more than $2/c$ times at most. (Otherwise, let $n > 2/c$. Then there exist $\lambda_1'', \ldots, \lambda_n''$ in $\Lambda_0$ with $\lambda_1'' < \ldots < \lambda_n''$, and corresponding nets $N_{\lambda_1''} \supseteq \ldots \supseteq \ldots \supseteq N_{\lambda_n''}$, such that for each $i$, $i = 1, 2, \ldots, n$, there is a $\nu_i \in N_{\lambda_i''}$ for which

$$|\mu_\nu|\Big(\bigcup_{\substack{\lambda_{i-1}'' \leqslant \lambda < \lambda_i'' \\ \lambda\in\Lambda_0}} X_\lambda\Big) \geqslant c/2, \quad \text{for all } \nu \in N_{\lambda_i''} \text{ with } \nu \geqslant \nu_i,$$

where we denote $\lambda_0$ by $\lambda_0''$. But then for any $\nu \in N_{\lambda_n''}$ such that $\nu \geqslant \sup\{\nu_i: i = 1, 2, \ldots, n\}$ we have

$$|\mu_\nu|\Big(\bigcup_{\substack{\lambda<\lambda_n'' \\ \lambda\in\Lambda_0}} X_\lambda\Big) \geqslant \sum_{i=1}^{n} |\mu_\nu|\Big(\bigcup_{\substack{\lambda_{i-1} \leqslant \lambda < \lambda_i'' \\ \lambda\in\Lambda_0}} X_\lambda\Big) > (c/2)(2/c) = 1.$$

But this contradicts the unit boundedness of all elements of $B$.) Moreover, because the process can start at most a finite number of times, the method yields the inductive step, and we obtain $\Lambda_1$. If we proceed transfinitely and use the total order of $\Lambda_0$, then the largest possible $\Lambda_1$ is cofinal in $\Lambda_0$. From now on we denote the first element of $\Lambda_1$ by $\lambda_0$, and note that inequality (a) says in particular that for any $\lambda'' \in \Lambda_1$,

$$\text{(b)} \qquad |\mu_{\lambda''}|\Big(\bigcup_{\substack{\lambda_0 \leqslant \lambda < \lambda'' \\ \lambda\in\Lambda_1}} X_\lambda\Big) < c/2.$$

Next we show that 0 is not a weak* cluster point of $(\mu_\lambda)_{\lambda\in\Lambda_1}$. Define $f \in C_b(X)$ by making

$$\|f\|_\infty \leqslant 1/c \text{ and } |\mu_\lambda(\chi_{X_\lambda} f)| \geqslant 3/4, \quad \text{for all } \lambda \in \Lambda_1,$$
$$f(x) = 0, \quad \text{for all } x \in X_\lambda, \text{ for all } \lambda \in \Lambda \setminus \Lambda_1.$$

Since $\lambda \in \Lambda_1$ implies that $|\mu_\lambda|(X_\lambda) \geqslant c$, and since the $X_\lambda$'s are open and closed in $X$, the Riesz–Kakutani Theorem assures us of the existence of a function $f$ so defined. Then the definition of $\mu_{\lambda''}$ and of $f$, along with inequality (b), imply that for any $\lambda'' \in \Lambda_1$, we have

$$|\mu_{\lambda''}(f)| \geqslant |\mu_{\lambda''}(\chi_{X_{\lambda''}} f)| - \sum_{\substack{\lambda'\in\Lambda_1 \\ \lambda'<\lambda''}} |\mu_{\lambda''}(\chi_{X_{\lambda'}} f)| - \sum_{\substack{\lambda'\in\Lambda_1 \\ \lambda'>\lambda''}} |\mu_{\lambda''}(\chi_{X_{\lambda'}} f)|$$
$$\geqslant 3/4 - (c/2)(1/c) = 1/4.$$

Thus $(\mu_\lambda)_{\lambda\in\Lambda_1}$ does not have 0 as a weak* cluster point. To complete the proof we note that if there existed a $\mu$ in $M(X)$ which was a weak* cluster point of the net $(\mu_\lambda)_{\lambda\in\Lambda_1}$, then

$$\operatorname{supp}\mu \subseteq \bigcup_{\lambda<\lambda'} X_\lambda,$$

where $\lambda'$ corresponds to some countable ordinal in $\Lambda_0$. Now if

$$X' = \bigcup_{\substack{\lambda\in\Lambda_1 \\ \lambda>\lambda'}} X_\lambda,$$

then 0 would be a weak* cluster point of $(\mu_\lambda|_{X'})_{\substack{\lambda\in\Lambda_1 \\ \lambda>\lambda'}}$ for the space $(C_b(X'), t_s)^*$. But that cannot be, by the preceding argument applied to $X'$. Consequently not every net in $B$ has a weak* cluster point in $M(X)$, which means precisely that $B$ is not relatively weak* compact. ∎

Now we are ready for the promised theorem, due originally to Conway.

THEOREM 2.2. [2]. *Let $X$ be locally compact and paracompact. Then $t_s$ is a Mackey topology for $C_b(X)$.* (1)

Proof. Let $B$ be a weak* compact subset of $(C_b(X), t_s)^*$. We wish to show that $B$ is equicontinuous. In the first place, since $B$ is weak* compact, $B$ must be weak* bounded. Next, the local compactness of $X$ implies that $(C_b(X), t_s)$ is complete ([1], Theorem 1), so that the bounded sets and the weak* bounded sets of $(C_b(X), t_s)^* = M(X)$ coincide ([5], p. 210). Thus $B$ is bounded in $(C_b(X), t_s)^*$, which means that it is bounded in norm, by Theorem 1 of [1]. We may as well assume that $\|\mu\| \leqslant 1$, for all $\mu \in B$. As a result, the previous theorem says that $\operatorname{supp} B$ must be relatively $\sigma$-compact, and thus the proof reduces to the case in which $X$ is $\sigma$-compact and locally compact, which means that $X$ is indeed hemicompact. In other words, there is a sequence $(K_n)_{n=1}^\infty$ of compact subsets of $X$ such that $K_n \subseteq K_{n+1}$, for all $n$, such that $X = \bigcup_{n=1}^\infty K_n$, and such that if $K$ is compact in $X$, then there is an $n$ such that $K \subseteq K_n$. To complete the proof we will assume that $B$ is not equicontinuous in order to derive a contradiction.

The non-equicontinuity of $B$ means that there exists an $\varepsilon > 0$ with $\varepsilon < 1$, such that for any compact $K \subseteq X$, there exists a $\mu \in B$ such that $|\mu|(X \setminus K) > \varepsilon$ ([2], Theorem 2.2). At this time we define a sequence $(\mu_i, D_i, K_{n_i}, U_i)_{i=1}^\infty$, inductively as follows. Let $K_{n_1} = D_1 = K_1$ and let $\mu_1 \in B$ be arbitrary. Since $X$ is locally compact, there exists an open and relatively compact $U_1 \subseteq X$ such that $U_1 \supseteq D_1 = K_{n_1}$. For the induction we assume that we already have selected $(\mu_i, D_i, K_{n_i}, U_i)_{i=1}^j$ with $D_i$

(1) Added in proof: R. F. Wheeler has recently generalized this result.

compact, $U_i$ open and relatively compact, and such that

$$D_i \subseteq X \setminus \overline{U}_{i-1},$$

$$U_i \subseteq \overline{U}_{i-1} \cup D_i \cup K_{n_i}, \quad (\text{so } D_i \subseteq U_i \setminus \overline{U}_{i-1})$$

and also

(a) $\qquad |\mu_i|(D_i) > \varepsilon \quad$ and $\quad |\mu_i|(X \setminus K_{n_i}) < \varepsilon/8, \quad$ for $i \geqslant 2$.

By assumption on $B$, there exists a $\mu_{j+1} \in B$ such that $|\mu_{j+1}|(X \setminus \overline{U}_j) > \varepsilon$ and hence there exists a compact $D_{j+1} \subset X \setminus \overline{U}_j$ such that $|\mu_{j+1}|(D_{j+1}) > \varepsilon$. By the hemicompactness of $X$ and the boundedness of $\mu_{j+1}$, there exists an $n_{j+1}$ with $n_{j+1} > n_j$, such that $|\mu_{j+1}|(X \setminus K_{n_{j+1}}) < \varepsilon/8$, so we now have $(\mu_{j+1}, D_{j+1}, K_{n_{j+1}}, U_{j+1})$, confirming the existence of the sequence. Now let us select open sets $V_i$ and $W_i$ in $X$ such that

(b) $$D_i \subseteq V_i \subseteq \overline{V}_i \subseteq W_i \subseteq U_i \setminus \overline{U}_{i-1},$$

where we take $U_0$ to be empty by definition. Note for future use that the $W_i$'s are pairwise disjoint.

Next we create a suitable subsequence of $(\mu_i)_{i=1}^{\infty}$, proceeding along the same lines as in the proof of Theorem 2.1. Let $X_i = U_i \setminus \overline{U}_{i-1}$, for each $i$, so the $X_i$'s are pairwise disjoint, and assume without loss of generality that each $X_i$ is non-empty. Let $N$ constitute the integers $\geqslant 2$. Let $m_1$ denote the smallest element of $N$ such that there exists a subsequence $N_{m_1}$ of $N$ of integers larger than $m_1$, such that

$$|\mu_n|(X_{m_1}) < \varepsilon/16, \qquad n \in N_{m_1}.$$

Such an $m_1$ exists. (Otherwise, for each $i$, there is an $n_i$ such that $n \geqslant n_i$ implies that $|\mu_n|(X_i) \geqslant \varepsilon/16$. Now if $k > 16/\varepsilon$ and $n > \sup(n_1, \ldots, n_k)$, then

$$|\mu_n|\Big(\bigcup_{i=1}^{k} X_i\Big) = \sum_{i=1}^{k} |\mu_n|(X_i) > (\varepsilon/16)(16/\varepsilon) = 1,$$

so $\|\mu_n\| > 1$, a contradiction.) We say $N_{m_1}$ is associated with $\mu_{m_1}$. Assume that $(\mu_{m_i})_{i=1}^{k}$ has been defined, and let $N_{m_k}$ be a subsequence of $N_{m_{k-1}}$ associated with $\mu_{m_k}$. Let $p$ be the smallest number in $N_{m_k}$ with $p > m_k$. We let $p = m_{k+1}$ provided that there exists a subsequence $N_{m_{k+1}}$ of $N_{m_k}$ of numbers larger than $p$, such that

$$|\mu_j|\Big(\bigcup_{\substack{m_1 \leqslant i < p \\ i \in N}} X_i\Big) < \varepsilon/16, \qquad \text{all } j \in N_{m_{k+1}}.$$

If there is no such subsequence $N_{m_{k+1}}$, then begin all over again in search for $(\mu_{m_i})_{i=1}^{\infty}$, with $N$ replaced by $N_{m_k}$. Such a new beginning cannot occur more than $16/\varepsilon$ times. (Otherwise, let $k > 16/\varepsilon$. Then there exist $n_1, \ldots, n_k$ with $n_1 < \ldots < n_k$, and there exists also an $r_k$, such that if $r \geqslant r_k$, then

$$|\mu_r|\Big(\bigcup_{n_{j-1} \leqslant i < n_j} X_i\Big) \geqslant \varepsilon/16, \qquad j = 1, \ldots, k.$$

Evidently $\|\mu_r\| > 1$ for all such $r$, since the $X_i$'s are pairwise disjoint. This is a contradiction.) Thus after at most a finite number of false starts, we can apply the induction hypothesis henceforth, and in this way we obtain distinguished subsequences $N_0 = (m_j)_{j=1}^{\infty} \subseteq N$ and $(\mu_{m_j})_{j=1}^{\infty}$ of $(\mu_i)_{i=1}^{\infty}$ for which

$$|\mu_j|\Big(\bigcup_{\substack{m_1 \leqslant i < p \\ i \in N_0}} X_i\Big) < \varepsilon/16, \qquad \text{all } j \in N_0 \text{ with } j \geqslant p.$$

In particular,

(c) $$|\mu_{m_j}|\Big(\bigcup_{\substack{m_1 \leqslant i < m_j \\ i \in N_0}} X_i\Big) < \varepsilon/16, \qquad \text{all } j \in N_0.$$

Recall that the inequalities of (a) hold for all $i \geqslant 2$ and hence for all elements of $N_0$. In what follows the notation will be markedly simpler if we perform all operations with respect to $N$, instead of *all* with respect to $N_0$. So assume from now on that $N_0 = N$. Then the sequence $(\mu_{m_j})_{j=1}^{\infty}$ becomes $(\mu_i)_{i=1}^{\infty}$ and satisfies the following sets of relations:

(d) $\quad |\mu_i|\big(\bigcup_{j<i} X_j\big) < \varepsilon/16$, all $i \in N$ (from (c)), so $|\mu_i|\big(\bigcup_{j<i} V_j\big) < \varepsilon/16$, $i \in N$,

(e) $$|\mu_i|(D_i) > \varepsilon, \ i \in N \quad (\text{from (a)}),$$

(f) $\quad |\mu_i|(X \setminus K_{n_i}) < \varepsilon/8$ (from (a)), so $|\mu_i|\big(\bigcup_{j>i} V_j\big) < \varepsilon/8$, $i \in N$.

Now we must manufacture an appropriate function $f$. Since the $W_i$'s are pairwise disjoint, and by relation (e), the Riesz–Kakutani Theorem ensures the existence of a function $f$ continuous on $\bigcup_{i=2}^{\infty} D_i$ such that

(g) $$|\mu_i(\chi_{D_i} f)| > 1 - \varepsilon/8, \qquad \text{with } \|f\| < 1 - \varepsilon.$$

Utilizing relation (b) we can extend $f$ to a continuous function on $\bigcup_{i=2}^{\infty} W_i$, with the stipulations that $\|f\| < 1/\varepsilon$, $f(x) = 0$ for all $x \in \big(\bigcup_{i=2}^{\infty} W_i\big) \setminus \big(\bigcup_{i=2}^{\infty} V_i\big)$, and also

(h) $$|\mu_i(\chi_{V_i \setminus D_i} f)| < \varepsilon/8.$$

If we extend $f$ to the remainder of $X$ by letting $f(x) = 0$, then $f$ is continuous. The only question is when $x \in X \setminus \bigcup_{i=2}^{\infty} W_i$, since the $W_i$'s are all open. But note that in this case $x \in U_i$ for some minimal $i$, and if

$U = U_i \setminus \bigcup_{j=2}^{i} \overline{V}_j$, then $U$ is a neighborhood of $x$, and $f = 0$ on $U$. By computation we obtain, for $i \geqslant 2$,

$$\begin{aligned} |\mu_i(f)| &= \Big|\sum_{j=2}^{\infty} \mu_i(\chi_{V_j} f)\Big| \\ &\geqslant |\mu_i(\chi_{V_i} f)| - \Big|\sum_{j<i} \mu_i(\chi_{V_j} f)\Big| - \Big|\sum_{j>i} \mu_i(\chi_{V_j} f)\Big| \\ &\geqslant |\mu_i(\chi_{V_i} f)| - [|\mu_i|(\bigcup_{j<i} V_j)]\,\|f\| - [|\mu_i|(\bigcup_{j>i} V_j)]\,\|f\| \\ &> [(1-\varepsilon/8) - \varepsilon/8] - (\varepsilon/8)(1/\varepsilon) - (\varepsilon/8)(1/\varepsilon) \\ &> \tfrac{1}{2}, \end{aligned}$$

by virtue of inequalities (d)–(h). Since $B$ was assumed to be weak* compact, the sequence $(\mu_i)_{i=1}^{\infty}$ has weak* cluster points in $M(X)$. Take such a $\mu$. Then there must exist a positive integer $p$ such that $|\mu|(X \setminus V_p) < \varepsilon/16$. On the one hand, $(\mu_i|_{X\setminus V_p})_{i=1}^{\infty}$ has the weak* cluster point $\mu|_{X\setminus V_p}$, and $\|\mu|_{X\setminus V_p}\| < \varepsilon/16$. On the other hand, if $i > p$, then

$$\big|\mu_i|_{V_p}(\chi_{V_p} f)\big| \leqslant \|\mu_i|_{V_p}\|\,\|f\| < (\varepsilon/8)(1/\varepsilon) = 1/8,$$

since $|\mu_i|(\bigcup_{j<i} V_j) < \varepsilon/8$, by relation (d), where we note that $\bigcup_{j<i} X_j \supseteq \bigcup_{j<i} V_j$. Therefore

$$\tfrac{1}{2} < |\mu_i(f)| \leqslant \big|\mu_i|_{X\setminus V_p}(\chi_{X\setminus V_p} f)\big| + \big|\mu_i|_{V_p}(\chi_{V_p} f)\big| \leqslant \big|\mu_i|_{X\setminus V_p}(\chi_{X\setminus V_p} f)\big| + 1/8.$$

Hence

$$\big|\mu|_{X\setminus V_p}(\chi_{X\setminus V_p} f)\big| \geqslant \liminf_i \big|\mu_i|_{X\setminus V_p}(\chi_{X\setminus V_p} f)\big| \geqslant 1/8,$$

which means that $\|\mu|_{X\setminus U_p}\| \geqslant \varepsilon/8$, a contradiction. Thus $(\mu_i)_{i=1}^{\infty}$ has no weak* cluster point in $M(X)$, so evidently $B$ cannot be weak* compact in $(C_b(X), t_s)^* = M(X)$. Thus the proof is complete. ■

What we have actually shown in the proof of Theorem 2.2 is that if $X$ is locally compact and paracompact, then $(C_b(X), t_s)$ is a strong Mackey space, which by Conway's definition means that any weak* compact subset of the dual—and not just the convex ones—is equicontinuous. It is this stronger statement which Conway proved in [2].

If we let $t_\sigma$ denote the topology on $C_b(X)$ of uniform convergence on $\sigma$-compact subsets of $X$, then we conjecture that one can adapt the proof of Theorem 2.1 to prove that if $X$ is locally compact and paracompact, then $t_\sigma$ is a Mackey topology for $C_b(X)$. In order to use the criterion for Mackey topologies mentioned at the outset of this section, we need first to know what the equicontinuous subsets of the dual of $(C_b(X), t_\sigma)$ look like. We show presently that they have a simple form.



THEOREM 2.3. *Any equicontinuous set in $(C_b(X), t_\sigma)^*$ is contained in a set of the form*

$$N_{A,n} = \{\mu \in M(\beta X)\colon\ \mu \text{ concentrated on } \overline{A}^{\beta X},\ \|\mu\| \leqslant n\},$$

*where $A$ is a $\sigma$-compact subset of $X$ and $n$ is some positive integer independent of $\mu$. Moreover, all such $N_{A,n}$ are equicontinuous.*

Proof. We will first show that $N_{A,n}$ is equicontinuous. Let

$$U_{A,n} = \{f \in C_b(X)\colon\ \|f\|_A \leqslant 1/n\},$$

a veritable neighborhood of 0 for the topology $t_\sigma$ in $C_b(X)$. If $f'$ is the continuous extension of $f$ to $\beta X$, then $\mu \in N_{A,n}$ and $f \in U_{A,n}$ together imply that

$$\Big|\int_{\overline{A}^{\beta X}} f'\,d\mu\Big| \leqslant \|\mu\|/n \leqslant 1,$$

which means that $N_{A,n}$ is equicontinuous. For the other half of the proof, we let $U_{A,n}$ be given. We will first prove that if $\mu$ is in the polar of $U_{A,n}$ then $\mu$ is concentrated on $\overline{A}^{\beta X}$. Assuming the contrary, we take such a $\mu$ in the polar of $U_{A,n}$ and find a compact $D \subseteq \beta X \setminus \overline{A}^{\beta X}$ such that $|\mu| D > 0$. Without loss of generality we may assume that $\mu(D) = \alpha > 0$. By an application of the Riesz–Kakutani Theorem we can obtain an $f \in C_b(X)$ and an open set $U$ with $D \subseteq U \subseteq \beta X \setminus \overline{A}^{\beta X}$ such that the continuous extension $f'$ of $f$ on $\beta X$ is 0 on $\beta X \setminus U$ and $\|f'\|_{\beta X} = f'(x) = 2/\alpha$, for all $x \in D$, and such that $|\mu|(U \setminus D) < \alpha/2$. Then $f \in U_{A,n}$ and

$$|\mu(f)| \geqslant \Big|\int_D f'\,d\mu\Big| - \int_{U\setminus D} |f'|\,d|\mu| > (2/\alpha)\,\alpha - \frac{2}{\alpha}\,\frac{\alpha}{2} = 1,$$

so that $\mu$ is not in the polar of $U_{A,n}$. In other words, we have just proved that the polar of $U_{A,n}$ is a subset of $M(\overline{A}^{\beta X})$. Using the Riesz–Kakutani Theorem once again, we can easily prove that the polar of $U_{A,n}$ is precisely $N_{A,n}$. Then since the collection of all possible $U_{A,n}$ forms a basis for the neighborhood system of 0 in $t_\sigma$, the proof is complete. ■

Because of Theorem 2.3, it seems that in order to prove that $t_\sigma$ is a Mackey topology, one would only need to emulate the proof of Theorem 2.1, and require that $X$ be locally compact and paracompact but *not* $\sigma$-compact. However, this is a tall order, for it appears that in order to obtain the $A_1$ defined within the proof of Theorem 2.1, it is necessary to have a very sensitive analysis of the relationships between the closures in $\beta X$ of the various $\sigma$-compact subsets $\bigcup_{i=1}^{\infty} X_{\lambda_i}$.

On the other side of the ledger, it is known [2] that if $X$ is the space of ordinals less than the first uncountable $\Omega_0$, then $t_s \doteq t_\sigma$ is not a Mackey topology for $C_b(X)$. What we describe next is a Mackey topology for $t_s$ on $C_b(X)$ when $X$ is a rather special type of space (having certain qualities of the ordinals less than $\Omega_0$). Such $X$ are very non-paracompact. We wish to thank Professor Robert F. Wheeler for pointing out an error in an earlier version of the next theorem.

To set the stage we let, as usual, $f'$ denote the continuous extension to $\beta X$ of $f \in C_b(X)$. We then consider sets $F \subseteq C_b(X)$ which have the property that for each compact $K \subseteq X$, there exists an $F_K \subseteq F$ such that $|f(x)| \leqslant 1$ for all $x \in K$ and $f \in F_K$, and such that if $\mu \in M(\beta X \setminus X)$ and if there exists an $n$ such that $|\mu(\chi_{\beta X \setminus X} f')| \leqslant n$ for all $f \in F_K$, then $\mu = 0$. The set of all such $F \subseteq C_b(X)$ we denote by $\mathfrak{F}$. If $B$ is the norm-bounded unit ball of $C_b(X)$ under the supremum norm, and if $U_F$ denotes the convex balanced hull of $B \cup F$, then let $t_F$ denote the topology generated by $t_s$ and $U_F$, and let $t$ denote the topology generated by $t_s$ and $\{U_F : F \in \mathfrak{F}\}$. Moreover, let $t_m$ be the Mackey topology for $t_s$ on $C_b(X)$. Finally, for any compact $K \subseteq X$, let

$$V_K = \{f \in C_b(X) : |f(x)| \leqslant 1, \text{ for all } x \in K\}.$$

Then $V_K$ is a generic neighborhood for the compact-open topology.

Now we can state and prove the proposed theorem.

THEOREM 2.4. *Let $X$ be locally compact and non-compact, and assume that each $\sigma$-compact subset of $X$ is relatively compact. Then $t$ is the Mackey topology $t_m$ for $(C_b(X), t_s)$.*

Proof. Let $F \in \mathfrak{F}$. We will first show that $(C_b(X), t_F)^* \subseteq M(X)$. Since $t_F$ is weaker than the norm topology, surely $(C_b(X), t_F)^* \subseteq M(\beta X)$, so that we can take an arbitrary element of $(C_b(X), t_F)^*$ to be $\mu \in M(\beta X)$. In addition, the local compactness of $X$ allows us to assume that $\mu = \mu_1 + \mu_2$, where $\mu_1$ is concentrated on $X$ and $\mu_2$ is concentrated on $\beta X \setminus X$. Since $\mu \in (C_b(X), t_F)^*$, and since by Proposition 2.2 of [3], $t_s$ is the compact-open topology, there is a neighborhood $V_{K_0}$ of 0 in the compact-open topology, and there is an $M < \infty$, such that $|\mu(f)| < M$ for all $f \in V_{K_0} \cap U_F$. Now if $\mu_1$ is concentrated on $K_1$, then by hypothesis on $X$ we may assume that $K_1$ is compact, since it has to be relatively $\sigma$-compact. Next we let $K = K_0 \cup K_1$, so $K$ is evidently a compact subset of $X$. Since $V_K \subseteq V_{K_0}$, this means that $|\mu(f)| \leqslant M$, for all $f \in V_K \cap U_F$. However, the Riesz–Kakutani Theorem tells us that $\sup |\mu_1(f)| \leqslant \|\mu_1\| < \infty$, for all $f \in V_K \cap U_F$, and by hypotheses on $F$ and $F_K$, we know that

$$\sup_{f \in F} |\mu_2(f)| \geqslant \sup_{f \in F_K} |\mu_2(f)| = \infty$$

unless $\mu_2 = 0$. Thus $\mu_2$ must be 0, so that $\mu = \mu_1$, which proves that $(C_b(X), t_F)^* \subseteq M(X)$. Using this result, and the fact that the Mackey topology (like any topology) is closed under finite intersections, we infer that $t \subseteq t_m$, where $t_m$ is the Mackey topology for $t_s$. In order to gain the reverse inclusion $t_m \subseteq t$, we let $U \in t_m$, and assume without loss of generality that $U$ is convex and balanced and contains the norm unit ball $B$ of $C_b(X)$. Since $(C_b(X), t_m)^* = (C_b(X), t_s)^* = M(X)$, we find that if $U \in t_m$, then $|\mu(U)|$ is unbounded for all non-zero $\mu \in M(\beta X \setminus X)$. Next, for any compact $K \subseteq X$, we observe that $V_K \cap U \in t_m$, so that $|\mu(V_K \cap U)|$ is unbounded for all non-zero $\mu \in M(\beta X \setminus X)$, by the statement just preceding. For each compact $K \subseteq X$, let $F_K = V_K \cap U$, and let $F = \bigcup_{K \text{ compact}} F_K$. Then by its very definition, $F \in \mathfrak{F}$. In addition, $U_F \subseteq U$, so that $U \in t$. Consequently $t_m \subseteq t$, which completes the proof that $t = t_m$. ∎



For a certain class of $X$'s, Theorem 2.4 characterizes the Mackey topology of $(C_b(X), t_s)$ intrinsically, without utilizing the dual or the weak* topology on the dual. Of course, it does not tell us when $t_s$ is or is not a Mackey topology. Nevertheless, with Theorem 2.4, and without much trouble, we can show that if $X$ is the ordinals $\Omega$ less than the first uncountable $\Omega_0$, then $(C_b(X), t_s)$ does not have the Mackey topology. To show it we will presently create a $U_F \in t_m$ such that $U_F \notin t_s$.

First of all, let $L$ denote the collection of limit ordinals in $X$, and let

$$g_{x+2n} = 2n\chi_{[x+2n, \Omega_0)}, \quad \text{for all } x \in L \text{ and } n = 1, 2, 3, \ldots$$

If $L_0 = \{x + 2n : x \in L, n = 1, 2, 3, \ldots\}$, then we let $F = \{g_y : y \in L_0\}$. After a moment's reflection, it is obvious that $F \in \mathfrak{F}$, so that $U_F \in t_m$. To show that $U_F \notin t_s$, we note that

$$g_y(z) = g_y(z+1), \quad \text{for all } y, z \in L_0.$$

Consequently for each $f \in U_F$,

$$|f(z) - f(z+1)| \leqslant 2, \quad \text{for all } z \in L_0.$$

You cannot say this of *any* neighborhood of 0 in $t_s$. Therefore $t_s \neq t_m$.

The preceding argument works if $X$ is any locally compact, non-compact space for which the $\sigma$-compact subsets are relatively compact, provided that some element of $\beta X \setminus X$ has a totally ordered neighborhood basis. In such a case, once again $t_s \neq t_m$.

Regarding $t_m$, we have the following curious theorem, which shows that if $X = \Omega$, then convergence in $t_m$ is not given by the order structure of $X$.

THEOREM 2.5. *Let $X$ be the ordinals less than the first uncountable. If $(f_\lambda)_{\lambda \in X} \subseteq (C_b(X), t_m)$ and $f_\lambda \to 0$ in $t_m$, then $f_\lambda \to 0$ in the uniform norm.*

Proof. If $(f_\lambda)_{\lambda \in X}$ does not converge to 0 uniformly, we may as well assume that $\|f_\lambda\| \geqslant 1$, for all $\lambda \in X$. Since $f_\lambda \to 0$ in $t_m$, of course $f_\lambda \to 0$ in the compact-open topology, so that for each $x \in X$, there is a $\lambda_x$ such that whenever $\lambda \geqslant \lambda_x$, we have $|f_\lambda(y)| < 1/2$, for all $y \in X$ with $y \leqslant x$. But each $f_\lambda$ has the property that $\|f_\lambda\| \geqslant 1$, so that for each $x \in X$, there is a $y_x > x$ such that $|f_{\lambda_x}(y_x)| > 1/2$, and we may assume that if $x < z$, then $y_x < y_z$. Let

$$d_{\lambda_x} = 2 \sup_{y \geqslant x} |f_{\lambda_x}(y) - f_{\lambda_x}(y+1)|.$$

By the preceding assertion, $d_{\lambda_x} > 0$, for all $x \in X$. Thus there exists an $\varepsilon > 0$ such that $d_{\lambda_x} \geqslant \varepsilon$ for a cofinal subnet $(f_{\lambda_\nu})_{\nu \in \Lambda_1}$. Let $y_{\lambda_\nu}$ be such that $|f_{\lambda_\nu}(y_{\lambda_\nu}) - f_{\lambda_\nu}(y_{\lambda_\nu}+1)| \geqslant d_{\lambda_\nu}$, and without loss of generality (or by taking a suitable cofinal subnet of $(f_{\lambda_\nu})_{\nu \in \Lambda_1}$), assume that $y_{\lambda_\nu}+1 < y_{\lambda_{\nu'}}$, if $\nu < \nu'$ in $\Lambda_1$. We let $F = \{f \in C(X): |f(y_{\lambda_\nu}) - f(y_{\lambda_\nu}+1)| \leqslant 2$, for all $\nu \in \Lambda_1\}$. Then $F \in \mathfrak{F}$, and $U_F$, which is the convex balanced hull of $(B \cup F)$, has the property that if $g \in (\varepsilon/4) U_F$, then $|g(y_{\lambda_\nu}) - g(y_{\lambda_\nu}+1)| < \varepsilon/2 < d_{\lambda_\nu}$, for all $\nu \in \Lambda_1$. Thus

$$(f_{\lambda_\nu})_{\nu \in \Lambda_1} \cap (\varepsilon/4) U_F = \emptyset,$$

contradicting the hypothesis that $f_\lambda \to 0$ in $t_m$. ■

Of course this does not mean that convergence in $t_m$ is uniform, which it is not since the dual of $C_b(X)$ with the uniform norm is $M(\beta X)$, while the dual of $C_b(X)$ with $t_m$ is merely $M(X)$, and if $X = \Omega$, then $X \neq \beta X$.

**3. A locally convex space as a dual.** By saying that a given locally convex space $(E, t)$ is a dual we mean that there exists a locally convex space $(E_0, t_0)$ and an identification of $E$ with the vector space of all continuous linear functionals on $E_0$, and furthermore, the topology $t$ on $E$ is precisely the dual topology of uniform convergence on the collection of all bounded subsets of $E_0$.

If $E$ is given, with dual $E^*$, we designate by $t_w$ the weak topology on $E$, by $t_{w^*}$ the weak* topology on $E^*$, and by $t_b$ the dual topology on $E^*$.

LEMMA 3.1. *Let $E$ be an arbitrary barreled space under the topology $t$. Then $(E^*, t_{w^*})^* = (E, t)$, so that $(E, t)$ is a dual.*

Proof. By Grothendieck's Theorem ([5], p. 250), the set $(E^*, t_{w^*})^*$ can be identified as a point-set with $E$. By an equivalent to the notion of barreledness, the equicontinuous subsets of $E$ are precisely the $t_{w^*}$-bounded subsets of $E^*$, which just means that the original topology $t$ on $E$ is the dual topology for $(E^*, t_{w^*})^*$. ■

We observe that every Banach space, and even some incomplete normed spaces, are barreled. As a consequence, every Banach space is a dual (though not necessarily the dual of some Banach space). We will return to this topic later, after Theorem 3.3.



LEMMA 3.2. *If $(E, t)$ is locally convex and is a dual, and if $(E^*, t_b)$ is normed, then $(E, t)$ is normed as well.*

Proof. Assume that $(E_0, t_0)^* = (E, t)$, and let $\pi$ denote the canonical injection of $E_0$ into $E^*$. If $B$ is bounded in $(E_0, t_0)$, then $\pi(B)$ is bounded in $(E^*, t_b)$ since $\pi(B) \subseteq B^{\circ\circ}$. On the other hand, if $\pi(B)$ is bounded in $(E^*, t_b)$, then since $t_b$ restricted to $\pi(E_0)$ is finer than $t_0$, we find that $B$ is bounded in $t_0$. Consequently the $t_b$-bounded sets in $\pi(E_0)$ correspond to the $t_0$-bounded sets in $(E_0, t_0)$. But $(E^*, t_b)$ is by hypothesis normed. Thus if $S$ is the unit ball in $(E^*, t_b)$, then $S \cap \pi(E_0)$ corresponds to a generic bounded set in $(E_0, t_0)$, which means that $(E, t)$ has a generic neighborhood which generates the neighborhood system of 0, so $t$ is normed. ■

THEOREM 3.3. *$(C_b(X), t_s)$ is a dual if and only if $X$ is compact.*

Proof. If $X$ is compact, then $t_s$ equals the supremum norm, so that $(C_b(X), t_s)$ is obviously complete, and thus is a Banach space. Therefore Lemma 3.1 applies to show that $(C_b(X), t_s)$ is a dual. On the other hand, irrespective of the compactness of $X$, the bounded sets of $(C_b(X), t_s)$ are precisely the uniformly bounded sets ([3], p. 5), so that the dual $(C_b(X), t_s)^*$ is always normed, regardless of the $X$. Thus Lemma 3.2 tells us that $(C_b(X), t_s)$ can only be a dual when it is normed, and we know that this can only happen when $X$ is compact, by Theorem 2.8 of [3]. ■

Dr. John V. Ryff asked under what conditions $(C_b(X), t_\sigma)$ would be a dual. Noting that the $t_\sigma$-bounded sets in $C_b(X)$ are precisely the uniformly bounded sets ([3], p. 170), and noting also that $t_\sigma$ is normed if and only if it is the supremum norm topology ([3], Proposition 4.6), we see immediately from the proof of Theorem 3.3 above that $(C_b(X), t_\sigma)$ is a dual if and only if $t_\sigma$ is normed (which by Proposition 4.2 of [3] means that there exists a $\sigma$-compact subset $A$ in $X$ whose closure is $X$).

Now let us focus on some biproducts of Lemmas 3.1 and 3.2. First of all, each Banach space is a dual, by Lemma 3.1. Is each Banach space automatically a bidual (dual of a dual)? In general, the answer is no. For it is easy to deduce from Lemmas 3.1 and 3.2 together that a Banach space is a bidual only if it is a dual of some normed—and hence some Banach—space. Thus $l_1$ is a bidual, while $c_0$ is not a bidual, though it is a dual. Characterizing those locally convex spaces which are biduals would be very interesting. Even finding in terms solely of $X$ those $(C_b(X), t_s)$ which are biduals is surely no mean task.

Since not every Banach space is a bidual, and since $(E^*, t_{w^*})$ has a bona fide weak topology, we infer from Lemma 3.1 that there exist locally convex spaces with weak topologies which are not duals. In fact, we can say something far stronger: if $F$ is any locally convex space, and if $(F, t_w)$ is a dual, then the strongest locally convex topology on $F^*$ with the same bounded sets as has $t_{w^*}$ is itself the strongest existent locally convex topology on $F^*$. This means that if $(F, t)$ is an infinite dimensional Banach space, then $(F, t_w)$ cannot be a dual. On the other hand, in [4] we exhibit non-trivial examples of spaces of continuous functions which, when endowed with the weak topology of simple convergence, are duals.

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# A multiplier counter-example for mixed-norm spaces

by

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**Abstract.** If $G = \Gamma H$ is a semi-direct product of an amenable locally compact group $H$ with an arbitrary locally compact group $\Gamma$, and if $T$ is an operator on $L^p(G) = L^p(\Gamma; L^p(H))$ (left-invariant Haar measure) of norm $\|T\|_{p,p}$ which commutes with all right translations, then Herz and Rivière have proved the theorem that for $q$ between 2 and $p$, $T$ is bounded on $L^p(\Gamma; L^q(H))$ with norm $\|T\|_{p,q} \leqslant \|T\|_{p,p}$. In this note, we show by example that in the simple case $G = \boldsymbol{R}^2 = \boldsymbol{R}^1 \times \boldsymbol{R}^1$ this theorem fails if $q$ is not between 2 and $p$. One consequence is that certain spaces of multipliers do not interpolate by the method of Riesz.

**1. Introduction.** Suppose that the function $m(\xi)$ is known to be a multiplier of Fourier transforms of $L^p(\boldsymbol{R}^1)$ functions; that is, the transformation $T_m$ defined by

$$(T_m f)\hat{\ }(\xi) = m(\xi)\hat{f}(\xi)$$

is a continuous mapping of $L^p(\boldsymbol{R}^1)$ into itself. We shall be concerned here with the extension of $T_m$ to the spaces $L^p(L^q)$ consisting of those measurable functions $f(x, y)$ defined on $\boldsymbol{R}^2$ for which the norm

$$\|f\|_{p,q} = \Big[\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}|f(x, y)|^q dy\Big)^{\frac{p}{q}} dx\Big]^{\frac{1}{p}}$$

is finite. The extension of $T_m$ from an operator on $L^p$ to the operator $\tilde{T}_m$ on $L^p(L^q)$ is that it should operate on the first variable:

$$(\tilde{T}_m f)\hat{\ }(\xi, \eta) = m(\xi)\hat{f}(\xi, \eta);$$

or, if $T_m$ is given by convolution with the function $M(x)$:

$$(T_m f)(x) = \int_{-\infty}^{\infty} M(x-t)f(t)\,dt,$$

then the extension $\tilde{T}_m$ is given by

$$(\tilde{T}_m f)(x, y) = \int_{-\infty}^{\infty} M(x-t)f(t, y)\,dt.$$

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