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Decompositions of operator-valued functions in Hilbert spaces

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Abstract. In the present paper we will prove some theorems concerning the canonical decompositions of operator-valued functions in Hilbert spaces. We consider positive definite, completely positive, completely contractive functions and representations of subalgebras of C^*-algebras. Moreover we give some corollaries about dilatable functions.

To begin with we introduce some notation and definitions. We denote by H the Hilbert space with the inner product $(,)$. $L(H)$ stands for the algebra of all linear, bounded operators in H. For $A \in L(H)$ we write $R(A) = \{Ax, x \in H\}$. I_H stands for the identity operator in H. If M is a closed subspace of H then $M^\perp$ denotes the orthogonal complement of M. An operator $P \in L(H)$ such that $P = P^2 = P^*$ is called a *projection*. If P is a projection onto the subspace M then P denotes the projection onto $M^\perp$. $A \in L(H)$ is a contraction (or contractive operator) if $\|A\| \leqslant 1$. Every involution preserving homomorphism of involutive Banach algebra B into $L(H)$ is called **-representation* of B. Every homomorphism of B into $L(H)$ is called a *representation* of B.

It is well known (see [11], ch. I.3.2) that for a contraction $T \in L(H)$ there are subspaces $H_0, H_1 = H_0^\perp$ reducing T such that the operator $T_0 = T|_{H_0}$ is the unitary operator in H_0 and $T_1 = T|_{H_1}$ is completely non-unitary. The decomposition $T = T_0 \oplus T_1$ is uniquely determined. It is called the *canonical decomposition of T*.

Every contraction $T \in L(H)$ induces a representation Π (by the J. von Neumann inequality) of the disc algebra $A(\Gamma)$ into $L(H)$ such that $\Pi(1) = I_H$, $\Pi(z) = T$ and $\|\Pi\| \leqslant 1$. $A(\Gamma)$ consists of all holomorphic functions in the open unit disc $\{|z| < 1\}$ continuous in its closure $\{|z| \leqslant 1\}$; $\Gamma = \{|z| = 1\}$. The representation Π has the following property: T is unitary if and only if there is a *-representation $\hat{\Pi}$: $C(\Gamma) \to L(H)$ which is an extension to $C(\Gamma)$ of the representation Π. If such $\hat{\Pi}$ exists then it is unique.

The reinterpretation of the canonical decomposition reads as follows: There exist subspaces H_0, $H_1 = H_0^\perp \subset H$ reducing $\Pi(u)$ for all $u \in A(\Gamma)$

such that the representation $\Pi_0\colon u\to\Pi(u)|_{H_0}$ has an extension to the *-representasion of $C(\Gamma)$ into $L(H)$ and no non-trivial part of $\Pi_1\colon u\to\Pi(u)|_{H_1}$ has an extension to $C(\Gamma)$ which is a *-representation. The decomposition $\Pi=\Pi_0\oplus\Pi_1$ is determined uniquely.

Let X be a compact Hausdorff space. A subalgebra $A\subset C(X)$ is called a *function algebra* (on X) if A is norm-closed, contains constants and separates the points of X. Let $T(.)\colon A\to L(H)$ be a representation of A. Following [5] we call the closed subspace $M\subset H$ *X-reducing* for $T(.)$ if M reduces all operators $T(u)$ for $u\in A$ and the representation $u\to T(u)|_M$ has an extension to a *-representation of the algebra $C(X)$ into $L(M)$. The representation $T(.)\colon A\to L(H)$ is *X-pure* if it has no nontrivial X-reducing subspace.

It has been proved by Foiaş and Suciu [5] that for every representation $T(.)$ of the function algebra A (on X) into $L(H)$ there is a largest subspace $M\subset H$ which X-reduces $T(.)$. The part of $T(.)$ in $M^\perp$ is X-pure. Notice that it is not necessary to assume that $T(.)$ is a bounded representation. An ingenious new proof of Foiaş–Suciu theorem has been given by Seever in [9]. The method of Seever as well the original proof given in [5] depend essentially on the commutativity of A.

The present paper deals with a general method of decompositions of arbitrary operator-valued functions. We use some elementary properties of von Neumann algebras, in particular the equivalence of projections. The crucial point of this method is contained in Lemma 2. In Part 2 we consider positive definite functions on semi-groups, groups and algebras, next—completely positive functions and some corollaries about dilations. In Part 3 we generalize the above Foiaş–Suciu result and give the canonical decomposition of completely contractive functions.

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1. Preliminaries. Let P,Q be two projections belonging to $L(H)$ onto the subspaces M,N of H respectively. Then $P\vee Q$, $P\wedge Q$ stand for the projections onto $M\vee N=\overline{\{x+y,\ x\in M,\ y\in N\}}$, $M\cap N$ respectively. It is well known that the following conditions hold true (see [6] for references):

(i) PQ is the projection (onto $M\cap N$) if and only if $PQ=QP$.

(ii) $P+Q$ is the projection (onto $M\oplus N=M\vee N$) if and only if $PQ=QP=0$.

(iii) $P-Q$ is the projection (onto $M\cap N^\perp$) if and only if $PQ=QP=Q$.

(iv) $P\wedge Q=\lim_{n\to\infty}(PQ)^n$ strongly.

(v) $(P^\perp\vee Q^\perp)^\perp=P\wedge Q$.

The operator $U\in L(H)$ is a partial isometry if U is an isometric operator on the orthogonal complement of its kernel. Equivalently: U is a partial isometry if and only if U^*U is the projection. If U is a partial isometry then the space $R(U^*U)$ is called the *initial* and $R(UU^*)$ – the *final space* of U. If U is a partial isometry, so does U^*.

Let $\mathfrak{A}$ be a strongly closed subalgebra of $L(H)$. $\mathfrak{A}$ is called a *von Neumann algebra* if it is symmetric (i.e. $A\in\mathfrak{A}\Rightarrow A^*\in\mathfrak{A}$) and contains the identity I_H of $L(H)$. We refer to [4], [12] for the theory of von Neumann algebras. Let $\mathscr{S}$ be an arbitrary subset of $L(H)$. The set $\mathscr{S}'=\{T\in L(H)\colon TS=ST$ for all $S\in\mathscr{S}\}$ is called the *commutant of* $\mathscr{S}$.

Bicommutant theorem. *For every von Neumann algebra $\mathfrak{A}\subset L(H)$*

$$\mathfrak{A}''=\mathfrak{A}.$$

Polar decomposition. *For every operator $T\in L(H)$ there are operators $|T|$, U such that $T=U|T|$, $|T|\geqslant 0$ and U is the partial isometry (we put $|T|=(T^*T)^{\frac12}$).*

We say that the projections P,Q which belong to the von Neumann algebra $\mathfrak{A}$ are *equivalent with respect to* $\mathfrak{A}$ if there is a partial isometry $U\in\mathfrak{A}$ such that $U^*U=P$ and $UU^*=Q$. We write then $P\sim Q$. We notice that $P\sim Q$ if and only if $Q\sim P$.

Let T be an operator belonging to $L(H)$. We write $\Pi[T]$ for the projection onto $\overline{R(T)}$. Using the bicommutant theorem it is not difficult to prove that for every operator $T\in L(H)$ with the polar decomposition $T=U|T|$, both U, $|T|$ belong to the von Neumann algebra $\mathfrak{A}(T)$ generated by T (i.e. the smallest von Neumann algebra containing T). The polar decomposition implies that $\Pi[T]\sim\Pi[T^*]=\Pi[|T|]$ with respect to $\mathfrak{A}(T)$. The last equality is true, because $R(T^*)=R(|T|)$.

Using this equivalence we prove first:

Lemma 1. *Let P,Q be two projections belonging to $L(H)$ onto $M,N\subset H$ respectively. Let $\mathfrak{A}$ be a von Neumann algebra generated by P,Q. Then $P\vee Q-Q\sim P-P\wedge Q$ with respect to $\mathfrak{A}$.*

Proof. The projections $P\vee Q-Q$, $P-P\wedge Q$ belong to $\mathfrak{A}$ by (iv), and (v). Moreover, $\ker(QP)=M^\perp\oplus(M\cap N^\perp)$ or, equivalently $\overline{R(PQ)}=M\cap(M\cap N^\perp)^\perp$. Hence $\Pi[PQ^\perp]=P-P\wedge Q$ and $\Pi[Q^\perp P]=Q^\perp-Q^\perp\wedge P^\perp=P\vee Q-Q$ (by (v)). The obvious equality: $(PQ^\perp)^*=(Q^\perp P)$ and the equivalence $\Pi[T^*]\sim\Pi[T]$ for all $T\in L(H)$ finish the proof. The proof of this lemma is a modification with suitable changes of the proof given in [12], page 19.

Let Ω be an arbitrary set and let $\varphi\colon\ \Omega\to L(H)$ be an operator function. Suppose we are given a property (W) concerning such φ's. We say that the closed subspace $M\subset H$ *(W)-reduces the function* φ, if M reduces every operator $\varphi(u)$ $(u\in\Omega)$ and the function $u\to\varphi(u)|_M$ has the property (W).

Let N be a closed subspace of H reducing all $\varphi(u)$ $(u \in \Omega)$. We say that the function $u \to \varphi(u)|_N$ is *completely non*-(W), if there is no non-zero subspace K of N which (W)-reduces this function. The closed subspace $M \subset H$ is called the *largest* (W)-reducing subspace for the function φ if M contains every closed subspace of H which (W)-reduces φ.

Assume now that the property (W) is *hereditary*, i.e. if the subspace M (W)-reduces the function φ and the subspace $N \subset M$ reduces φ then N (W)-reduces φ. The following simple property will be proved for the sake of completness.

(A) *Let M be the largest subspace of H (W)-reducing φ and let (W) be heredital. Then the function φ_1: $u \to \varphi(u)|_{M^\perp}$ is completely non-(W). If $\varphi_0(u) = \varphi(u)|_M$ then the decomposition $\varphi = \varphi_0 \oplus \varphi_1$ is determined uniquely.*

Proof. Let N be a subspace of H (W)-reducing φ and $N \subset M^\perp$. Then $N \subset M$ and $N = \{0\}$. Now we take the decomposition $H = M' \oplus M''$ where M', M'' reduce φ, M' (W)-reduces φ and the function $\varphi''(u) = \varphi(u)|_{M''}$ is completely non-(W). Then $M' \subset M$ and the space $M \ominus M'$ (W)-reduces φ by the heredity property of W. But $M \ominus M' \subset M''$ and consequently $M \ominus M' = \{0\}$ hence $M = M'$ and $M = M''$, q.e.d.

Let $\mathscr{P}$ be an arbitrary family of projections in $L(H)$. We assume that $\mathscr{P}$ is closed in the strong operator topology and the following condition holds true:

(1) *If $P, Q \in \mathscr{P}$ then $P \vee Q \in \mathscr{P}$.*

Then there is the least upper bound LUB $\mathscr{P} = \bigvee \mathscr{P}$ of the family $\mathscr{P}$ and $\bigvee \mathscr{P}$ belongs to $\mathscr{P}$ (see [6] for references). The following lemma is basic for the construction of decompositions of operator-valued functions:

LEMMA 2. *Let $\mathfrak{A}$ be a von Neumann algebra in $L(H)$ and $\mathscr{P} \subset \mathfrak{A}$ be a family of projections. We assume that $\mathscr{P}$ satisfies the following conditions:*

(I) *If $P, Q \in \mathscr{P}$ then $P \wedge Q \in \mathscr{P}$.*

(II) *If $P, Q \in \mathscr{P}$ and P, Q are pairwise orthogonal then $P + Q \in \mathscr{P}$.*

(III) *If $P, Q \in \mathscr{P}$ and $Q \leqslant P$ then $P - Q \in \mathscr{P}$.*

(IV) *If $P \in \mathscr{P}$, $Q \in \mathfrak{A}$, Q is a projection and $P \sim Q$ with respect to $\mathfrak{A}$ then $Q \in \mathscr{P}$.*

Under our assumptions $P \vee Q \in \mathscr{P}$ for $P, Q \in \mathscr{P}$. Hence LUB $\mathscr{P} = \bigvee \mathscr{P}$ exists, and consequently, if $\mathscr{P}$ is strongly closed, then $\bigvee \mathscr{P}$ belongs to $\mathscr{P}$.

Proof. We take two projections $P, Q \in \mathscr{P}$. By (I) $P \wedge Q \in \mathscr{P}$ and by (III) $P - P \wedge Q \in \mathscr{P} \subset \mathfrak{A}$. By Lemma 1 $P \vee Q - Q \sim P - P \wedge Q$ (with respect to the von Neumann algebra generated by P, Q and consequently, with respect to every von Neumann algebra containing P and Q). Hence, by (IV), $R = P \vee Q - Q$ is a projection belonging to $\mathscr{P}$. But $P \vee Q = R + Q$ belongs also to $\mathscr{P}$, by (II), because R and Q are two pairwise orthogonal projections from $\mathscr{P}$. This completes the proof.

We come back to the function φ on a set Ω into $L(H)$ and to the property (W) of this function. We assume that (W) is hereditary. Using Lemma 2 we will give the method of decomposition of this function.

PROPOSITION 1. *Let φ be an arbitrary function on the set Ω into $L(H)$. Let the considered property (W) be hereditary. We define the set*

$$\mathscr{P} = \{P\colon P \text{ is a projection, } P \in \mathfrak{A} \text{ and the function } u \to \varphi(u)|_{R(P)} \text{ has the property (W)}\},$$

where $\mathfrak{A} = (\varphi(\Omega) \cup \varphi(\Omega)^*)'$.

If $\mathscr{P}$ is closed in the strong operator topology and (I)–(IV) are true for $\mathscr{P}$, then there is a largest subspace $H_0 \subset H$ which (W)-reduces φ. The function $\varphi_1(u) = \varphi(u)|_{H_1 = H_0^\perp}$ is completely non-(W) and the decomposition $\varphi = \varphi_0 \oplus \varphi_1$ (where $\varphi_0(u) = \varphi(u)|_{H_0}$) is uniquely determined.

Proof. By the basic lemma 2 there is a projection LUB $\mathscr{P} = \bigvee \mathscr{P} \in \mathscr{P}$ Every space $R(P)$ $(P \in \mathscr{P})$ (W)-reduces the function φ. We put $H_0 = R(\bigvee \mathscr{P})$ If N is a closed subspace of H (W)-reducing φ, then the projection Q onto N belongs to $\mathscr{P}$ and $Q \leqslant \bigvee \mathscr{P}$, which implies that $N \subset H_0$. We use now (A) and complete the proof.

2. Positive definite and completely positive functions. We consider an arbitrary set Ω and a complex Hilbert space H. $F(\Omega, H)$ stands for the linear space (with pointwise addition and multiplication by scalars) of all functions $f\colon \Omega \to H$ having finite supports. The support of $f = \{u \in \Omega\colon f(u) \neq 0\}$.

The function $K\colon \Omega \times \Omega \to L(H)$ is called *positive definite* if for all $f(\cdot) \in F(\Omega, H)$ the following inequality holds true:

$$\sum_{u,v \in \Omega} (K(u,v)f(v), f(u)) \geqslant 0.$$

The function $K\colon \Omega \times \Omega \to L(H)$ is completely non-positive definite if there is no non-zero subspace of H which reduces K to the positive definite function.

THEOREM 1. *For every function $K\colon \Omega \times \Omega \to L(H)$ there is a largest subspace $H_0 \subset H$ reducing K to the positive definite function. The function $K_1(u,v) = K(u,v)|_{H_1 = H_0^\perp}$ is completely non-positive definite. The decomposition $K = K_0 \oplus K_1$ (where $K_0(u,v) = K(u,v)|_{H_0}$) is uniquely determined.*

We will give two proofs of this theorem (Proof I and Proof II). In the first proof we use Lemma 2 and Proposition 1. The second proof is direct but longer than the first one and we do not use in it the von Neumann algebras. Notice that the method of the second proof is very similar in its first steps to the proof of bicommutant theorem.

Proof I of Theorem 1. We define the von Neumann algebra $\mathfrak{A} = \big(K(\Omega\times\Omega)\cup K(\Omega\times\Omega)^*\big)'$. Let P be a projection belonging to $\mathfrak{A}$. We introduce the following condition:

$$\sum_{u,v\in\Omega}\big(K(u,v)Pf(v),Pf(u)\big)\geqslant 0 \quad \text{for } f(.)\in F(\Omega,H) \tag{2}$$

and we define the set

$\mathscr{P} = \{P\colon P$ is a projection belonging to $\mathfrak{A}$ such that for every $f(.)\in F(\Omega,H)$, (2) holds true$\}$.

It is plain that $\mathscr{P}$ is strongly closed. We will prove that $\mathscr{P}$ satisfies the conditions (I)–(IV) of Lemma 2.

Let $P\in\mathscr{P}$ and $T\in L(H)$. For $f(.)\in F(\Omega,H)$ the function $g(u) = Tf(u)$ belongs to $F(\Omega,H)$.

Hence

$$\sum\big(K(u,v)PTf(v),PTf(u)\big) = \sum\big(K(u,v)Pg(v),Pg(u)\big)\geqslant 0.$$

Hence if $P,Q\in\mathscr{P}$, then $PQ\in\mathfrak{A}$ and PQ satisfies (2). It follows that $(PQ)^n\in\mathfrak{A}$ and $(PQ)^n$ satisfy (2) for all integers $n\geqslant 0$. Hence, by (iv) $P\wedge Q =$ strong-$\lim(PQ)^n$ belongs to $\mathscr{P}$. We just proved that our $\mathscr{P}$ satisfies (I) of Lemma 2. We next prove (II). If $P,Q\in\mathscr{P}$ and $PQ = QP = 0$ then $P+Q$ is the projection from $\mathfrak{A}$ and for $f(.)\in F(\Omega,H)$ we have

$$\sum\big(K(u,v)(P+Q)f(v),(P+Q)f(u)\big) = \sum\big(K(u,v)(P+Q)f(v),f(u)\big)$$
$$= \sum\big(K(u,v)Pf(v),Pf(u)\big)+\sum\big(K(u,v)Qf(v),Qf(u)\big)\geqslant 0.$$

Hence $P+Q\in\mathscr{P}$. To prove (III) we proceed as follows: if $P,Q\in\mathscr{P}$, $Q\leqslant P$, then $PQ = QP = Q$ and $P-Q = P-PQ = P(I-Q)\in\mathscr{P}$.

The crucial step is the proof of (IV). We take $P\in\mathscr{P}$, a projection $Q\in\mathfrak{A}$ and assume that $P\sim Q$. Then there is a partial isometry $U\in\mathfrak{A}$ such that $U^*U = P$, $UU^* = Q$. In other words, U is the isometric operator from $R(P)$ onto $R(Q)$. We take $f(.)\in F(\Omega,H)$ and $u\in\Omega$. Then there is $y_u = Py_u\in R(P)$ such that $Qf(u) = UPy_u = Uy_u$. The equality $\|Qf(u)\| = \|UPy_u\| = \|y_u\|$ implies that the function $g(u) = y_u$ is well-defined. Moreover, if $f(u) = 0$ then $Qf(u) = 0$ and $\|y_u\| = \|g(u)\| = 0$. Hence $g(u) = 0$ which shows that $g(.)$ belongs to $F(\Omega,H)$. We have finally:

$$\sum\big(K(u,v)Qf(v),Qf(u)\big) = \sum\big(K(u,v)\,UPg(v),\ UPg(u)\big)$$
$$= \sum\big(UPK(u,v)Pg(v),UPg(u)\big) = \sum\big(K(u,v)Pg(v),Pg(u)\big)\geqslant 0$$

which implies $Q\in\mathscr{P}$. Using Proposition 1 we finish the proof.

A simple illustration of Theorem 1 is at hand.

The operator T is called *completely non-positive* if there is no a non-zero subspace of H reducing T to the positive operator.

Corollary 1. *For every operator $K\in L(H)$ there is the largest subspace $H_0\subset H$ reducing K to the positive operator $K_0 = K|_{H_0}$. The operator $K_1 = K|_{H_1=H_0^\perp}$ is completely non-positive and the decomposition $K = K_0\oplus K_1$ is uniquely determined.*

Proof. Let $K\in L(H)$. We take an arbitrary set $\Omega = \{u_0\}$ and we define the function $K(u_0,u_0) = K$, which is positive definite if and only if K is the positive operator. Applying Theorem 1 we complete the proof.

But we can prove direct this corollary as follows: It is wellknown (see [2], I) that for every operator $T\in L(H)$ there is the largest subspace of H, which reduces T to the self-adjoint operator. Let M be a such subspace for the operator K and let K' be the part of K in M. K' has a suitable spectral measure $E(.)$ in M defined on the Borel subsets of the real line. We define the subspace $H_0 = R\big(E(\sigma)\big)$ of M (where $\sigma = [0,\infty)$), which is the largest subspace of H reducing K to the positive operator. Using (A) we complete the proof.

Proof II of Theorem 1. We write $H_n = \underbrace{H\oplus\ldots\oplus H}_{n\text{-times}}$ and for $A\in L(H)$

$$\tilde{A}_n = \begin{pmatrix} A & 0 & \ldots & 0\\ 0 & A & 0 \ldots & 0\\ \cdot & \cdot & \cdot\ \cdot & \cdot\\ 0 & \ldots & 0 & A\end{pmatrix}\in L(H_n)$$

It is easy to verify that if P and Q are two projections belonging to $L(H)$, then $\tilde{P}_n$, $\tilde{Q}_n$ and $(\widetilde{P\vee Q})_n$ are projections belonging to $L(H_n)$, and $\tilde{P}_n\vee\tilde{Q}_n = (\widetilde{P\vee Q})_n$. We observe now that a function $K\colon \Omega\times\Omega\to L(H)$ is positive definite if and only if for all integers $n\geqslant 0$ and $u_1,\ldots,u_n\in\Omega$ the operator

$$\tilde{K} = \tilde{K}(u_1,\ldots,u_n) = \begin{pmatrix} K(u_1,u_1),\ldots,K(u_1,u_n)\\ K(u_2,u_1),\ldots,K(u_2,u_n)\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ K(u_n,u_1),\ldots,K(u_n,u_n)\end{pmatrix}$$

is positive in H_n.

We fix the integer $n\geqslant 0$ and the finite subset $\omega = \{u_1,\ldots,u_n\}$ of Ω. We take projections $P,Q\in L(H)$. If P,Q commute with all $K(u_i,u_j)$ $(i,j = 1,\ldots,n)$ then the projections $\tilde{P}_n$, $\tilde{Q}_n$ and $\tilde{P}_n\vee\tilde{Q}_n$ commute with $\tilde{K}$. Moreover, if $R(P)$ and $R(Q)$ reduce $K|_{\omega\times\omega}$ to the positive definite function on $\omega\times\omega$ then $\tilde{K}\tilde{P}_n\geqslant 0$ and $\tilde{K}\tilde{Q}_n\geqslant 0$. Corollary 1 yields that $\tilde{K}(\tilde{P}_n\vee\tilde{Q}_n)\geqslant 0$ and hence $\tilde{K}(\widetilde{P\vee Q})_n\geqslant 0$. It follows that $R(P\vee Q)$ reduces $K|_{\omega\times\omega}$ to the positive definite function on $\omega\times\omega$. Hence we get that for every finite subset $\omega\subset\Omega$ there is a largest subspace $M_\omega\subset H$ reducing the function $K|_{\omega\times\omega}$ to the positive definite function. Next we observe that if ω_1 and ω_2 are two finite subsets of Ω and $\omega_1\subset\omega_2$ then $M_{\omega_1}\supset M_{\omega_2}$. Indeed, M_{ω_2}

reduces all operators $K(u,v)$ $(u,v\in\omega_1)$ because it reduces all operators $K(u,v)(u,v\in\omega_2)$. Taking zeros for suitable vectors in the definition of the positive definiteness, namely for vectors, corresponding to these elements of ω_2 which are not in ω_1, we get that the part of $K|_{\omega_1\times\omega_1}$ in M_{ω_2} is the positive definite function. Hence $M_{\omega_2}\subset M_{\omega_1}$, because M_{ω_1} is the largest subspace reducing $K|_{\omega_1\times\omega_1}$ to the positive definite function. We take finally $H_0=\bigcap M_\omega$, where $\omega\subset\Omega$ is runing over all finite subsets of Ω. We must show that H_0 reduces every operator $K(u,v)$, $u,v\in\Omega$. To prove this we take $u,v\in\Omega$ and the finite subset $\gamma\subset\Omega$. It is plain that if $u,v\in\gamma$ then M_γ reduces $K(u,v)$. Assume now that $(u,v)\notin\gamma\times\gamma$. Then we consider the finite subset $\gamma_1=\gamma\cup\{u,v\}$ of Ω and observe that for $x\in H_0\subset M_{\gamma_1}$ we have $K(u,v)x\in M_{\gamma_1}\subset M_\gamma$ by the above remarks. Hence $K(u,v)$ leaves H_0 invariant. Using similar arguments we prove that H_0 is invariant for $K(u,v)^*$. Thus H_0 reduces $K(u,v)$ $(u,v\in\Omega)$. The definition of H_0 implies that H_0 is the largest subspace of H reducing K to the positive definite function and the proof is finished.

Now we will consider completely positive functions. Let B, B' be C^*-algebras and n—an arbitrary positive integer number. Suppose that $S\subset B$ is a symmetric subspace of B. B_n denotes the C^*-algebra of $n\times n$ matrices $(u_{ij})_{i,j=1,\dots,n}$, where $u_{ij}\in B$ with the natural, hermitian involution $(u_{ij})^*=(u_{ji}^*)$. S_n stands for the subspace of B_n of $n\times n$ matrices (u_{ij}) over S. The linear function $\varphi\colon B\to B'$ (resp. $\varphi\colon S\to B'$) is called positive if for every positive element $u\in B$ (resp. $u\in S$) $\varphi(u)$ is positive. The linear function $\varphi\colon B\to B'$ (resp. $\varphi\colon S\to B'$) is completely positive if for every positive integer n the function $\varphi_n\colon B_n\to B_n'$ (resp. $\varphi_n\colon S_n\to B_n'$) defined as follows:

$$\varphi_n\big((u_{ij})\big)=\big(\varphi(u_{ij})\big)$$

is positive. This definition was introduced by Stinespring (for C^*-algebras) and generalized by Arveson (for symmetric subspaces of C^*-algebras) — see [1]. If we assume that B is an involutive Banach algebra and B' is a C^*-algebra then the definition of completely positivity may be performed following Paschke [8] as follows: A linear function $\varphi\colon B\to B'$ is called positive if for all $u\in B\colon\varphi(u^*u)$ is positive in B'. The function $\varphi\colon B\to B'$ is completely positive if for every integer $n\geqslant 0$ and $(u_{ij})\in B_n$: $\varphi_n\big((u_{ij})^*(u_{ij})\big)\geqslant 0$. If B is a C^*-algebra then an element $u\in B$ is positive if and only if it has the form v^*v for some $v\in B$. The last definition of completely positivity implies immediately that:

(B) *If B is a C^*-algebra then the linear function $\varphi\colon B\to L(H)$ is completely positive if and only if φ is positive definite.*

Let B be a C^*-algebra and $S\subset B$ its symmetric subspace. Let $\varphi\colon S\to L(H)$ be a linear function. We define the sets

$\mathscr{P}_1=\{P$: *P is a projection in $\big(\varphi(S)\cup\varphi(S)^*\big)'$ such that for every $u\in B$, $u\geqslant 0\colon\varphi(u)P\geqslant 0$*$\}$.

$\mathscr{P}_2=\{P$: *P is a projection in $\big(\varphi(S)\cup\varphi(S)^*\big)'$ such that for every integer $n\geqslant 0$ and for every positive element $(u_{ij})\in S_n$, $\varphi_n\big((u_{ij})\big)\tilde P_n$ is the positive operator*$\}$.

Using the same arguments as in the first proof of Theorem 1, we verify easily that the properties (I)–(IV) of Lemma 2 hold true for $\mathscr{P}_1$ and $\mathscr{P}_2$. We get therefore the following:

THEOREM 2. *If S is a symmetric subspace of C^*-algebra B and $\varphi\colon S\to L(H)$ is a linear function then there is the largest subspace $H_0\subset H$ reducing the function φ to the completely positive (resp. positive) function $\varphi_0(a)=\varphi(a)|_{H_0}$ $(a\in S)$. The function $\varphi_1(a)=\varphi(a)|_{H_1=H_0^\perp}$ is completely non-completely positive (resp. positive) and the decomposition $\varphi=\varphi_0\oplus\varphi_1$ is uniquely determined.*

Now we will give some corollaries about dilatable functions. We consider an involutive semi-group G (i.e. $(uv)^*=v^*u^*$ for all $u,v\in G$) and the function $T(.)$ on G into $L(H)$. Let K be a Hilbert space, $R\colon H\to K$ be a linear, bounded operator and $T[.]\colon G\to L(K)$ be a*-representation (i.e. involution preserving homomorphism) of G into $L(K)$. We will say that $(K,R,T[.])$ is an *R-dilation* of the function $T(.)$ if for all $x\in H$, $u\in G$: $T(u)x=R^*T[u]Rx$. $(K,T[.])$ is called a *dilation* of $T(.)$ if $K\supset H$ and $T(u)x=PT[u]x$ for all $u\in G$, $x\in H$, where P stands for the projection from K onto H. Following Sz.-Nagy we call the function $T(.)\colon G\to L(H)$ positive definite if the function $K(u,v)=T(u^*v)$ is positive definite on $G\times G$. It is easy to observe that if G has the unit then every positive definite function on G preserves involution.

The following theorem has been proved by Sz.-Nagy [10].

DILATION THEOREM 1. *Let G be a multiplicative, involutive semi-group with the unit e and let $T(.)\colon G\to L(H)$ be an arbitrary function.*

a) *If $T(.)$ has an R-dilation $(K,R,T[.])$ then the function $T(.)$ is positive definite.*

b) *If the function $T(.)$ is positive definite on G and if there is a function $c(.)\geqslant 0$ on G such that for all $f(.)\in F(G,H)$ and $w\in G$:*

$$\sum_{u,v\in G}\big(T(u^*w^*wv)f(v),f(u)\big)\leqslant c(w)\sum_{u,v\in G}\big(T(u^*v)f(v),f(u)\big) \tag{3}$$

then there is an R-dilation $(K,R,T[.])$ of the function $T(.)$.

Moreover, if for all $u,v\in G$: $T(uzv)=T(uz_1v)+T(uz_2v)$ for some z,z_1,z_2 then $T[z]=T[z_1]+T[z_2]$.

Notice that if $T(e)=I_H$ in the above theorem then we can identify H with the subspace of K and R^* is interpreted as the projection from

K onto H. Consequently, in this case $(K, T[.])$ is simply the dilation of $T(.)$ and $T[e] = I_K$.

It is well known that for an operator $A \in L(H)$ there is the largest subspace M which reduces A to the zero operator (this is also a trivial consequence of Lemma 2). Let M be a closed subspace of H and let P, Q be two projections such that $AP = PA$, $AQ = QA$ and $R(P) \subset M$, $R(Q) \subset M$. Then $A(P \vee Q) = (P \vee Q)A$ and $R(P \vee Q) \subset M$. Hence there is a largest subspace N of H reducing the operator A and $N \subset M$. Suppose that the function $T(.)\colon G \to L(H)$ has an R-dilation $(K, R, T[.])$ and consider the operator $T(e) - I_H$. It follows that there is the largest subspace M_0 of H which reduces $T(e) - I_H$ to the zero operator in M_0. Hence $T(e)|_{M_0} = I_{M_0}$. Moreover, there is the largest subspace $M \subset M_0$ which reduces all operators $T(u)$ $(u \in G)$. It is clear that $T(e)|_M = I_M$. This implies that M is the largest subspace reducing $T(.)$ to the function which has a dilation. Summing up and applying Theorem 1 and the Dilation Theorem 1, we get

Proposition 2. *Let G be a multiplicative, involutive semigroup with the unit. Let $T(.)\colon G \to L(H)$ be an involution preserving function. Assume that there is a function $c(.) \geqslant 0$ on G, satisfying the condition* (3) *for all $f(.) \in F(G, H)$ and $w \in G$. Then*

a) *there is a largest subspace $H_0 \subset H$ which reduces the function $T(.)$ to the function $T_0(.) = T(.)|_{H_0}$ such that $T_0(.)$ has an R-dilation;*

b) *there is a largest subspace $M \subset H_0$, which reduces the function $T_0(.)$ to the function, having a dilation.*

Some simple applications of Theorems 1 and 2 are now in order. Take a group G and a function $T(.)\colon G \to L(H)$. The mapping $u \to u^{-1}$ is the involution in G. The function $T(.)$ is positive definite if the function $K(u, v) = T(u^{-1}v)$ is positive definite. The theorem below which is due to Naimark (see [11]) may be derived from Dilation Theorem 1.

Dilation theorem 2. *Let G be a group and $T(.)\colon G \to L(H)$ be a positive definite function such that $T(e) = I_H$. Then there exists a dilation $(K, T[.])$ of the function $T(.)$.*

Notice that $T[.]\colon G \to L(K)$ is the unitary representation of G into $L(K)$. Applying Theorem 1 we get

Proposition 3. *For every group G and every function $T(.)\colon G \to L(H)$ which preserves involution (i.e. $T(u^{-1}) = T(u)^*$ for $u \in G$) there is the largest subspace $H_0 \subset H$ reducing the function $T(.)$ to the function having a dilation.*

Now we consider an arbitrary operator $T \in L(H)$ and we define the function $T(.)$ on the additive group $\boldsymbol{Z}$ of integer numbers as follows:

$$T(n) = T^n \text{ for } n \geqslant 0, \; T(n) = T^{*|n|}, \text{ for } n < 0.$$

This is the celebrated theorem of Sz.-Nagy [11] that the operator T is contractive if and only if $T(n)$ is the positive definite function. Applying Theorem 1 with $\Omega = \boldsymbol{Z}$ and $K(n, m) = T(n-m)$ $(n, m \in \boldsymbol{Z})$ we get

Corollary 2. *For every operator $T \in L(H)$ there exists the largest subspace $H_0 \subset H$ reducing T to the contractive operator $T_0 = T|_{H_0}$. The operator $T_1 = T|_{H_0^\perp}$ is completely non-contractive and the decomposition $T = T_0 \oplus T_1$ is uniquely determined.*

Consider now an involutive Banach algebra B and a linear, positive definite function $T(\cdot)$ on B into $L(H)$. $T(\cdot)$ is positive definite as the function on the involutive, multiplicative semigroup B.

Dilation theorem 3 (see [7]). *If the involutive Banach algebra B has an approximative unit and the linear, positive definite function $T(.)$ is bounded then $T(.)$ has an R-dilation $(K, R, T[.])$ where, as usually, $T[\cdot]\colon B \to L(K)$ is a *-representation of B.*

If an involutive Banach algebra has unit, then the boundedness of every positive functional on B implies that every positive definite function on B is bounded. Moreover, if B is a C^*-algebra then it has an approximative unit and every positive functional on B is bounded. Hence every linear, positive definite function on a C^*-algebra is bounded. Dilation Theorem 3 implies immediately that:

(C) *If B is an involutive Banach algebra with unit or B is a C^*-algebra and $T(.)$ is a linear, positive definite function on B, then $T(.)$ has an R-dilation $(K, R, T[.])$.*

Combining the Dilation Theorem 3 and Theorem 1 (taking $\Omega = B$ and $K(a, b) = T(a^*b)$ for $a, b \in B$) we get

Proposition 4. *Let B be an involutive Banach algebra with an approximative unit and $T(.)$ be a linear, bounded function on B into $L(H)$. Then there is a largest subspace H_0 of H which reduces $T(.)$ to the function having a R-dilation $(K, R, T[.])$.*

Moreover, (C) implies the following corollary:

Corollary 3. *If B is an involutive Banach algebra with unit or B is a C^*-algebra and if $T(.)\colon B \to L(H)$ is a linear function then there exists a largest subspace H_0 of H which reduces $T(.)$ to the function having an R-dilation $(K, R, T[.])$.*

It has been proved by Arveson ([1], th. 1.2.3.) that if B is a C^*-algebra with the unit e and S is its norm closed, symmetric subspace containing e then every completely positive function $\varphi\colon S \to L(H)$ has a completely positive extention $\varphi_1\colon B \to L(H)$ such that $\varphi = \varphi_1|_S$. The following corollary is a generalization of the Corollary 3:

Corollary 4. *Let B be a C^*-algebra with the unit e and let S be its* *norm closed, symmetric subspace containing e. Then for every linear function $T(.)$: $S\to L(H)$ there is a largest subspace $H_0 \subset H$ reducing $T(.)$ to the function $T_0(.) = T(.)|_{H_0}$, which has a completely positive extention $T_1(.)$: $B\to L(H_0)$.*

Proof. By Theorem 2 there is the largest subspace H_0 of H, which reduces the function $T(.)$ to the completely positive function $T_0(.) = T(.)|_{H_0}$ on S. The Arveson result above implies that $T_0(.)$ has a completely positive extention $T_1(.)$: $B\to L(H_0)$, and the proof is finished.

Notice that the extention $T_1(.)$ in the last Corollary is positive definite, by the remark (B), and, by Corollary 3, it has an R-dilation $(K, R, T[.])$.

3. Decomposition of representations of subalgebras of C^*-algebras. We take a C^*-algebra A with the unit e and its norm closed subalgebra B containing e. Let $P(A)$ be the set of all pure states on A and $\overline{P(A)}$—its closure in the weak topology of the dual A^*. We say that B separates the points of $\overline{P(A)}$ if for all $\varphi_1, \varphi_2 \in \overline{P(A)}$ $\varphi_1|_B = \varphi_2|_B$ implies $\varphi_1 = \varphi_2$. First we prove afore-going generalization of the Foiaş–Suciu theorem in the context of subalgebras of C^*-algebras in the noncommutative case. We need the general Stone–Weierstrass theorem for C^*-algebras—see [3] for references.

THEOREM (STONE–WEIERSTRASS). *Let A, B be as above and suppose that B is the symmetric subalgebra separating the points of $\overline{P(A)}$. Then $B = A$.*

Now we consider a function $T(.)$ on a closed subalgebra B of the C^*-algebra A with unit $e\in B$. The closed subspace $M \subset H$ is called *A-reducing for $T(.)$* if M reduces every operator $T(b)$ $(b\in B)$ and there is a $*$-representation $T[.]$: $A\to L(M)$ such that $T(b)|_M = T[b]$ for $b\in B$. The function $T(.)$ is *A-pure* if there is no non-zero A-reducing subspace for $T(.)$.

Now we will prove the following lemma:

LEMMA 3. *Let $T\in L(H)$ be an arbitrary operator and c_1, c_2 two real positive numbers. Then there is a largest subspace M of H which reduces T and such that for $x\in M$: $\|Tx\| \leqslant c_1\|x\|$, $\|T^*x\| \leqslant c_2\|x\|$.*

Proof. We write $\mathcal{T} = \{T, T^*\}$ and

$$\mathcal{P} = \{P\colon P\in\mathcal{T}',\ P \text{ is a projection and } \|TP\| \leqslant c_1,\ \|T^*P\| \leqslant c_2\}.$$

$\mathcal{P}$ is strongly closed in $L(H)$. We will prove that $\mathcal{P}$ satisfies the conditions (I)–(IV) of Lemma 2.

Let $P\in\mathcal{P}$, $S\in L(H)$, $\|S\|\leqslant 1$. Then $\|TPS\| \leqslant \|TP\|\ \|S\| \leqslant c_1$ and $\|T^*PS\| \leqslant c_2$. It follows that if $P, Q\in\mathcal{P}$ then $PQ\in\mathcal{P}$ and consequently for all integers $n \geqslant 0$ $(PQ)^n\in\mathcal{P}$. Hence $P\wedge Q\in\mathcal{P}$, which proves (I). To prove (II) we take projections $P, Q\in\mathcal{P}$ such that $PQ = QP = 0$. Then for $x\in H$ we have:

$$\|T(P+Q)x\|^2 = \|TPx\|^2 + \|TQx\|^2 \leqslant \|TP\|^2\,\|Px\|^2 + \|TQ\|^2\,\|Qx\|^2$$
$$\leqslant c_1^2\|(P+Q)x\|^2 \leqslant c_1^2\|x\|^2 \qquad \text{q.e.d.}$$

By similar token $\|T^*(P+Q)x\| \leqslant c_2\|x\|$ if $P, Q\in\mathcal{P}$, $x\in H$. Now we prove (III). If $P, Q\in\mathcal{P}$, $Q\leqslant P$ then $PQ = QP = Q$ and $P-Q = P-PQ = P(I-Q)\in\mathcal{P}$.

We take $P\in\mathcal{P}$, $Q\in\mathcal{T}'$, $Q\sim P$ with respect to $\mathcal{T}'$. Let U be the partial isometry belonging to $\mathcal{T}'$ such that $U^*U = P$, $UU^* = Q$. For $x\in H$ we can find $y\in R(P)$ such that $UPy = Uy = Qx$. Hence $\|y\| = \|Uy\| = \|x\|$. Finally we get: $\|TQx\| = \|TUPy\| = \|UTPy\| \leqslant \|TPy\| \leqslant c_1\|x\|$ and $\|T^*Q\| \leqslant c_2$ which shows that $Q\in\mathcal{P}$. Applying Proposition 1 we finish the proof.

Notice that if we put $c_1 = c_2 = 1$ then we get immediately from Lemma 3 Corollary 2.

The following theorem generalizes the Foiaş–Suciu result of [5]

THEOREM 3. *Let A be C^*-algebra with unit e, B—its closed subalgebra containing e and separating the points of $\overline{P(A)}$. Let $T(.)$: $B\to L(H)$ be an arbitrary function. Then there is the largest A-reducing for $T(\cdot)$, subspace $H_0 \subset H$. The function $T_1(u) = T(u)|_{H_0^\perp}$ $(u\in B)$ is A-pure and the decomposition $T(.) = T_0(.)\oplus T_1(.)$ where $T_0(u) = T(u)|_{H_0}$ $(u\in B)$ is uniquely determined.*

Proof. By Stone–Weierstrass theorem, the algebra A is generated by $B\cup B^*$. We write $\hat{T}(b) = T(b)$ and $\hat{T}(b^*) = T(b)^*$ if $b\in B$ and $\mathcal{T} = T(B)\cup T(B)^*$. Let $b_{i_1}, \ldots, b_{i_n}$ be elements of $B\cup B^*$ and let P be a projection belonging to $\mathcal{T}'$. We consider the inequality

$$\text{(4)} \qquad \Big\|\sum_{i_1,\ldots,i_n} \hat{T}(b_{i_1})\ldots\hat{T}(b_{i_n})P\Big\| \leqslant \Big\|\sum_{i_1,\ldots,i_n} b_{i_1}\ldots b_{i_n}\Big\|$$

and define the set $\mathcal{P}$ of all projections $P\in\mathcal{T}'$ for which the condition (4) holds true for every choice of $b_{i_j}\in B\cup B^*$, $n = 1, 2, \ldots$ $\mathcal{P}$ is closed in the strong operator topology. We take $P, Q\in\mathcal{P}$ and for $b_{i_j}\in B\cup B^*$ the operator $Z = \sum\hat{T}(b_{i_1})\ldots\hat{T}(b_{i_n})$. If $c_1 = \|\sum b_{i_1}\ldots b_{i_n}\|$, $c_2 = \|\sum b_{i_n}^*\ldots b_{i_1}^*\|$ then $\|Z(P\vee Q)\| \leqslant c_1$ and $\|Z^*(P\vee Q)\| \leqslant c_2$ by Lemma 3. Hence, LUB $\mathcal{P} = \bigvee\mathcal{P}$ belongs to $\mathcal{P}$. We fix now the projection $P\in\mathcal{P}$ and take the space $N = R(P)$, which reduces the function $T(.)$. Since P satisfies the condition (4), we may extend the function $T_N(b) = T(b)|_N$ to the well-defined $*$-representation of A into $L(N)$. This extention is given by the following formula: $T_N(\sum b_{i_1}\ldots b_{i_n})x = \sum\hat{T}(b_i)\ldots\hat{T}(b_i)x$ for $b_{i_1}, \ldots, b_{i_n}\in B\cup B^*$ and $x\in N$ and we denote it also by T_N. We must verify that this definition does not depend on the form of the element $\sum b_{i_1}\dots b_{i_n}$. More precisely, we must show that if $\sum b_{i_1}\dots b_{i_n} = \sum c_{j_1}\dots c_{j_m}$ for $b_{i_1},\dots,b_{i_n}$, $c_{j_1},\dots,c_{j_m}\in B\cup B^*$ then $\sum\hat{T}(b_{i_1})\dots\hat{T}(b_{i_n})x = \sum\hat{T}(c_{j_1})\dots\hat{T}(c_{j_m})x$ for $x\in N$. Using the condition (4) we get:

$$\Big\|\sum\hat{T}(b_{i_1})\dots\hat{T}(b_{i_n})x-\sum\hat{T}(c_{j_1})\dots\hat{T}(c_{j_m})x\Big\| \leqslant \Big\|\sum b_{i_1}\dots b_{i_n}-\sum c_{j_1}\dots c_{j_m}\Big\|\,\|x\| = 0.$$

This proves that T_N is the well-defined *-representation of A into $L(N)$ and consequently N is the A-reducing subspace for $T(.)$. We put $H_0 = R(\vee\mathscr{P})$. The preceding consideration implies that H_0 A-reduces $T(.)$ because $\vee\mathscr{P}\in\mathscr{P}$. Now we will prove that H_0 is the largest such subspace of H. Let $L\subset H$ be A-reducing subspace for $T(\cdot)$. It is well known that every *-homomorphism from an involutive Banach algebra to a C^*-algebra is contractive [3]. Hence, if P_L stands for the projection onto L then $P_L\in\mathscr{P}$, which proves that H_0 is the largest A-reducing subspace for $T(.)$. We apply (A) and complete the proof.

Now we will give a decomposition theorem for completely contractive functions. The definition of complete contractivity was introduced by Arveson [1]. Let B be a C^*-algebra with the unit e and S—its symmetric subspace containing e. The linear function φ from S to the Banach algebra B' is called *contractive* if $\|\varphi(u)\|\leqslant 1$ for all $u\in S$. The linear function $\varphi\colon S\to B'$ is completely contractive if for all $n=1,2,\dots$ the function $\varphi_n\colon S_n\to B'_n$ is contractive.

Let S, B be as above and let $\varphi\colon S\to L(H)$ be a linear function. We define the sets: $\mathscr{T}=\varphi(S)\cup\varphi(S)^*$ and

$$\mathscr{P}_1=\{P\colon P\in\mathscr{T}', P \text{ is a projection such that for } u\in S, \|\varphi(u)P\|\leqslant 1\},$$

$$\mathscr{P}_2=\{P\colon P\in\mathscr{T}', P \text{ is a projection such that for every } n=1,2,\dots \text{ and } (u_{ij})\in S_n \text{ we have } \|\varphi_n((u_{ij}))\tilde{P}_n\|\leqslant 1\}.$$

Using the same arguments as in the proof of the Lemma 3 we get:

THEOREM 4. *Let B be a C^*-algebra with the unit e and S—its symmetric subspace containing e. Let $\varphi\colon S\to L(H)$ be a linear function. Then there is the largest subspace $H_0\subset H$ reducing φ to the completely contractive (or contractive) function.*

The following result has been proved by Arveson [1].

If S, B are as above and $\varphi\colon S\to L(H)$ is a completely contractive function satisfying $\varphi(e)=I_H$ then there is a completely positive extention $\varphi_1\colon B\to L(H)$ of $\varphi|_S$ such that $\varphi_1|_S=\varphi$. Using this result we will prove now:

COROLLARY 5. *Let B be a C^*-algebra with the unit e and S its symmetric subspace containing e. Let $\varphi\colon S\to L(H)$ be a linear function such that $\varphi(e)=I_H$. Then there is the largest subspace H_0 of H reducing φ to the function $\varphi_0(a)=\varphi(a)|_{H_0}$ for $a\in S$ having a completely positive extention $\varphi_1\colon B\to L(H_0)$. Moreover, the function φ_1 has an R-dilation.*

Proof. By Theorem 4 there is the largest subspace H_0 of H, reducing the function φ to the function $\varphi_0(a)=\varphi(a)|_{H_0}$, which is completely contractive. The above Arveson result implies that there is an extention $\varphi_1\colon B\to L(H_0)$, which is the completely positive function. By (B), this extention is positive definite on B. (C) finishes the proof.

4. Localization theorems. Let us point out that Seever in [9] besides proving the canonical decomposition proved that the projection on its parts belong to the von Neumann algebra $\mathfrak{A}$ generated by the range of the representation in question. Now a little trivial heuresis is in order. Since this concerns a commutative case, Seever's result means that the projections on the parts of canonical decomposition belongs to $\mathfrak{A}'$ and consequently they belong to the center $\mathfrak{A}\cap\mathfrak{A}'$. Now a natural question appears: whether an analogous property holds true for decompositions discussed in the present paper. To be more precise we define von Neumann algebras $\mathfrak{A}_K$, $\mathfrak{A}_B$ where $K\colon \Omega\times\Omega\to L(H)$ and $T(.)\colon B\to L(H)$ are the functions appearing in Theorem 1 and Theorem 3 respectively, as follows:

$$\mathfrak{A}_K=\big(K(\Omega\times\Omega)\cup K(\Omega\times\Omega)^*\big)' \quad\text{and}\quad \mathfrak{A}_B=\big(T(B)\cup T(B)^*\big)'.$$

Let $\mathfrak{A}$ be $\mathfrak{A}_K$ or $\mathfrak{A}_B$. Now the question is: whether the largest projections $P_0=\vee P$ involved in proofs of Theorem 1 and Theorem 3 belong to the center $\mathfrak{A}\cap\mathfrak{A}'$ of the algebra $\mathfrak{A}$. The basic Theorem 5 below gives the positive answer to this question. We start with the following lemma:

LEMMA 4. *Let $\mathfrak{A}$ be a von Neumann algebra in $L(H)$ and let $\mathscr{P}\subset\mathfrak{A}$ be a family of projections. We assume that $\mathscr{P}$ satisfies the condition (IV) of Lemma 2, that is: if $P\in\mathscr{P}$ and Q is a projection belonging to $\mathfrak{A}$, equivalent to P with respect to $\mathfrak{A}$, then $Q\in\mathscr{P}$. If $P\in\mathscr{P}$ and X is an unitary operator from $\mathfrak{A}$ then the projection $Q=\Pi[XP]$ is equivalent to the projection P and $Q\in\mathscr{P}$.*

Proof. We take $P\in\mathscr{P}$ and an unitary operator $X\in\mathfrak{A}$. Since X^* is also the unitary operator, then X^* maps H onto H. Hence $R(XP) = R(XPX^*)$ and $Q=\Pi[XP]=\Pi[XPX^*]$. But the operator XPX^* is self-adjoint and $(XPX^*)^2 = XPX^*XPX^* = XPX^*$ which proves that XPX^* is a projection and hence $Q=XPX^*\in\mathfrak{A}$. Moreover, the operator $U=XP$ is the partial isometry belonging to $\mathfrak{A}$ and $U^*U=P$, $UU^*=Q$. Hence $P\sim Q$ with respect to $\mathfrak{A}$ and $Q\in\mathscr{P}$, which finishes the proof.

THEOREM 5. *Let $\mathfrak{A}$ be a von Neumann algebra in $L(H)$ and let $\mathscr{P}\subset\mathfrak{A}$ be a family of projections, satisfying the condition (IV) of Lemma 2. Then if there is the LUB $\mathscr{P}=P_0$ of the family $\mathscr{P}$ and $P_0\in\mathscr{P}$ then P_0 belongs to $\mathfrak{A}'$. Consequently, P_0 belongs to the center $\mathfrak{A}\cap\mathfrak{A}'$ of the algebra $\mathfrak{A}$.*

Proof. It is well known that the von Neumann algebra is the linear span of its unitary elements. We will prove that every unitary operator X belonging to $\mathfrak{A}$ commutes with P_0. Let X be an unitary operator from $\mathfrak{A}$. Then, by Lemma 4, the projection $\Pi[XP_0]$ belongs to $\mathscr{P}$. Hence $\Pi[XP_0] \leqslant P_0$ and $R(XP_0) \subset R(P_0)$, which proves that $R(P_0)$ is the invariant subspace for X. Using the same arguments for the unitary operator X^* we derive finally that $R(P_0)$ reduces X. But hence $XP_0 = P_0X$ which proves that $P_0 \in \mathfrak{A}'$, q.e.d.

All sets of projections defined in Theorems 1–4 satisfy the assumption of Theorem 5. Thus, the projections onto the parts of canonical decompositions described by these theorems belong to the centers of the suitable von Neumann algebras.

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