PROBLEM: Hat die Fourier-Algebra der freien Gruppe mit zwei Erzeugenden approximierende Einheiten? Oder allgemeiner: Gibt es lokal kompakte Gruppen, deren Fourier-Algebra keine approximierenden Einheiten besitzt (auch keine unbeschränkten)?

Für einige Verbesserungen und Hinweise bin ich P. Eymard sehr zu Dank verpflichtet.

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Received August 10, 1973 (726)

STUDIA MATHEMATICA, T. LII. (1974)

Sums of independent Banach space valued random variables

by

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Abstract. Various generalizations of classical theorems about sums of independent real random variables to Banach space valued random variables are given: Let (X_n) be independent random variables with values in a Banach space, and let (S_n) be their partial sums. Then necessary and sufficient conditions for (S_n) to converge in L^p is given $(0 < p \leqslant \infty)$. For $p = \infty$ this gives a new characterization of those Banach spaces which do not contain c_0. Later we characterize the class of Banach spaces for which a.s. boundedness of (S_n) implies a.s. convergence of (S_n). Finally we prove that convergence in distribution of (S_n) in a weak topology of our Banach space implies a.s. norm-convergence.

1. Introduction. We shall in this paper study the properties of series of independent random variables with values in a Banach space, that is, series of the form

$$\sum_{n=1}^{\infty} X_n \tag{1.1}$$

where $X_1, X_2, \ldots$ are independent random variables with values in a Banach space E.

Such series have been studied by Nordlander [13], Kahane [9], and Ito and Nisio [8].

Section 2 contains the basic definitions and notation, and we list some known lemmas and theorems for reference in later sections.

In Section 3 we study boundedness and convergence of (1.1) in L^p $(0 \leqslant p \leqslant \infty)$. Theorem 3.1 is a generalization of Theorem 4, p. 17 in [9] and of Theorem 18.1.A, p. 254 in [12]. In [9] Kahane only considers the case $X_n = \varepsilon_n x_n$ where (ε_n) is a Bernoulli sequence and (x_n) is a non-random sequence of vectors. In [12] Loéve only considers the case where X_n is real and the X_n's are uniformly bounded. Theorem 3.1 gives new results even for real valued random variables. Theorem 3.5 and Corollary 3.7 give a new characterization of those Banach spaces that do not contain c_0.

In Section 4 we show that convergence or boundedness of (1.1) may imply convergence or boundedness of

$$\sum_{1}^{\infty} Y_n$$

where (Y_n) are independent Banach space valued random variables, such that Y_n is a random multiple of X_n, with the multipliers satisfying certain conditions. In Theorem 4.3 we use an idea of B. Maurey (oral communication) to strengthen our original version considerably.

It is well known that if X_n is symmetric and real for all $n \geqslant 1$, then boundedness of the partial sums of (1.1) implies convergence of (1.1). In Section 5 we show that the class of Banach spaces for which this holds is the class of Banach spaces for which c_0 is not contained in $L^p(I, \lambda, E)$ for some (or all) $1 \leqslant p < \infty$, where I is the unit interval and λ is the Lebesgue measure. I conjecture that if $c_0 \subseteq L^p(I, \lambda, E)$ for some $1 \leqslant p < \infty$ then E contains c_0. In Section 5 we show that relatively compactness of the partial sums of (1.1) in suitable topology τ on E implies convergence of (1.1).

In Section 6 we prove that weak convergence in a weak topology of the partial sums of (1.1) implies a.s. norm-convergence of (1.1), if X_n is symmetric. This result is closely related to that of Ito and Nisio [8].

2. Definitions and preliminary results. Let E denote a Banach space with norm $\|\cdot\|$ and $(\Omega, \mathscr{F}, P)$ a probability space in all of this paper.

An *E-valued random variable* $X\colon \Omega \frown E$ is a P-measurable function in the sense of Definition III. 2.10 in [4]. That is,

$$X^{-1}(A) = \{X \in A\} \in \mathscr{F} \quad \forall A \in \mathscr{B}(E), \tag{2.1}$$

$$\exists E_0 \text{ separable closed subspace of } E \text{ such that } P(X \in E_0) = 1 \tag{2.2}$$

where $\mathscr{B}(E)$ is the Borel σ-algebra on E.

We shall for short denote $L^p(\Omega, \mathscr{F}, P, E)$ by $L^p(E)$ for $0 \leqslant p \leqslant \infty$. That is, if $0 < p < \infty$, then $L^p(E)$ is the set of E-valued random variables, X, with

$$\mathbf{E}(\|X\|^p) = \int_\Omega \|X(\omega)\|^p P(d\omega) < \infty$$

where $\mathbf{E}$ denotes the expectation ($=$ the integral with respect to P). $L^0(E)$ is the set of all E-valued random variables, and $L^\infty(E)$ is the set of E-valued random variables with $\operatorname{ess\,sup} \|X(\omega)\| < \infty$. We define the metric $\|\cdot\|_p$ (norm if $1 \leqslant p \leqslant \infty$) by

$$\begin{aligned}
\|X\|_0 &= \mathbf{E}\left\{\frac{\|X\|}{1+\|X\|}\right\}, && \forall X \in L^0(E),\\
\|X\|_p &= \mathbf{E}(\|X\|^p), && \forall X \in L^p(E),\ \forall\, 0 < p \leqslant 1,\\
\|X\|_p &= \{\mathbf{E}\|X\|^p\}^{1/p}, && \forall X \in L^p(E),\ \forall\, 1 \leqslant p < \infty,\\
\|X\|_\infty &= \operatorname{ess\,sup}\|X(\omega)\|, && \forall X \in L^\infty(E).
\end{aligned}$$

Then $L^p(E)$ is a Fréchet space for all $0 \leqslant p \leqslant \infty$, and if $1 \leqslant p \leqslant \infty$ then $L^p(E)$ is a Banach space.

It is well known that we have:

PROPOSITION 2.1. *If $K \subseteq L^p(E)$, $0 < p \leqslant \infty$, then K is a bounded subset of $L^p(E)$ if and only if*

$$\sup\{\|X\|_p\colon X \in K\} < \infty. \tag{2.3}$$

If $K \subseteq L^0(E)$, then the following 3 statements are equivalent:

$$K \text{ is a bounded subset of } L^0(E), \tag{2.4}$$

$$\forall \varepsilon > 0,\ \exists \delta > 0, \text{ such that } \|\delta X\|_0 \leqslant \varepsilon\ \forall X \in K, \tag{2.5}$$

$$\forall \varepsilon > 0,\ \exists K < \infty, \text{ such that } P(\|X\| \geqslant K) \leqslant \varepsilon\ \forall X \in K. \tag{2.6}$$

If (X_n) is a sequence in $L^0(E)$, then (X_n) converges in $L^0(E)$ to X if and only if

$$P(\|X_n - X\| \geqslant \varepsilon) \xrightarrow[n\to\infty]{} 0 \quad \forall \varepsilon > 0. \tag{2.7}$$

If $X_n \xrightarrow[n\to\infty]{} X$ in $L^0(E)$ we shall say that (X_n) *converges in probability* to X. If K is a bounded subset of $L^0(E)$ we shall say that K is *stochastically bounded.*

Let $(H, \mathscr{H})$ be a measurable space, that is, a set H and a σ-algebra $\mathscr{H}$. If $X\colon \Omega \frown H$ is a measurable map, then the image measure of P under X is denoted by P_X and called the *distribution law* of X. That is,

$$P_X(A) = P(X \in A) \quad \forall A \in \mathscr{H}.$$

An E-valued random variable X is called *symmetric* if $P_X = P_{-X}$, and a sequence (X_n) is called a *symmetric sequence* if $P_{\boldsymbol{X}} = P_{\boldsymbol{Y}}$ for all $\boldsymbol{Y} = (\pm X_n)$ with an arbitrary choice of $\pm$, where $\boldsymbol{X} = (X_n)$; (here $\boldsymbol{X}$ and $\boldsymbol{Y}$ are considered measurable maps from Ω into E^∞, with its product σ-algebra: $\mathscr{B}(E) \otimes \mathscr{B}(E) \otimes \ldots$).

It is clear that if $X_1, X_2, \ldots$ are symmetric independent E-valued random variables, then (X_n) is a symmetric sequence.

If X is an E-valued random variable, then X^* is called a *symmetrization* of X, if $X^* = X - X'$, where $P_X = P_{X'}$ and X and X' are independent. If (X_n) is a sequence of E-valued random variables then (X_n^*) is called a *symmetrization* of (X_n) if $X_n^* = X_n - X_n'$ where (X_n) and (X_n') are independent and equidistributed. A symmetrization of (X_n) will always exist at least on the product probability space $(\Omega \times \Omega, \mathscr{F} \otimes \mathscr{F}, P \times P)$. If (X_n) is a sequence of independent random variables, and (X_n^*) is a symmetrization of (X_n), then it is easily seen that (X_n^*) is a symmetric sequence.

If (ε_j) are independent random variables, so that $p(\varepsilon_j = 1) = p(\varepsilon_j = -1) = \frac{1}{2}$ for all $j \geqslant 1$, then (ε_j) is called a *Bernoulli sequence*, and we define

$$C(E) = \Big\{(x_j)\in E^\infty \,\Big|\, \sum_1^\infty \varepsilon_j x_j \text{ converges in } L^0(E)\Big\},$$

$$B(E) = \Big\{(x_j)\in E^\infty \,\Big|\, \Big(\sum_1^n \varepsilon_j x_j\Big)_{n=1}^\infty \text{ is bounded in } L^0(E)\Big\}.$$

It follows from Theorem 3.1 that we may substitute $L^0(E)$ with $L^p(E)$ for all $0\leqslant p<\infty$ in the definition of $C(E)$ and $B(E)$.

The following two propositions are well known (see, for example, Lemma 1, p. 12 in [9] and 17.1.A, p. 245 in [12]).

THEOREM 2.2. *If X^* is a symmetrization of X and $a\in E$, then we have*

$$P(\|X^*\|\geqslant t)\leqslant 2P(\|X-a\|\geqslant \tfrac{1}{2}t), \qquad \forall t\geqslant 0.$$

THEOREM 2.3. *Let (X_j) be a symmetric sequence and put $S_n = X_1+\ldots+X_n$. If M is an infinite subset of $\{1, 2, \ldots\}$, then*

(2.8) $$P(\max_{1\leqslant j\leqslant n}\|S_j\|\geqslant t)\leqslant 2P(\|S_n\|\geqslant t) \qquad \forall t\geqslant 0,$$

(2.9) $$P(\sup_j\|S_j\|\geqslant t)\leqslant 2\sup_{n\in M}P(\|S_n\|\geqslant t) \qquad \forall t\geqslant 0.$$

If S_n converges in probability to S then

(2.10) $$P(\sup_j\|S_j\|\geqslant t)\leqslant 2P(\|S\|\geqslant t) \qquad \forall t\geqslant 0.$$

A slight modification of the proofs of Theorem 1 and Theorem 2, p. 11 in [9] gives us:

THEOREM 2.4. *Let (X_n) be a symmteric sequence and $S_n = X_1+\ldots+X_n$, then the following 3 conditions are equivalent:*

(2.11) *(S_n) has a subsequence which is convergent in probability (stochastically bounded),*

(2.12) *(S_n) is convergent in probability (stochastically bounded),*

(2.13) *(S_n) is convergent a.s. (bounded a.s.).*

THEOREM 2.5. *Let (X_j) be independent E-valued random variables and put $S_n = X_1+\ldots+X_n$. Then (S_n) is convergent in probability (stochastically bounded) if and only if (S_n) is convergent a.s. (bounded a.s.).*

Let us conclude this section with the following simple but useful propositions:

THEOREM 2.6. *Let X and Y be two independent random variables both belonging to $L^p(E)$ for some $0\leqslant p\leqslant\infty$.*

If either X has mean 0 and $p\geqslant 1$, or X is symmetric, then we have

$$\tfrac{1}{2}\|2Y\|_p\leqslant\|X+Y\|_p.$$

In particular,

$$2^{p-1}\|Y\|_p\leqslant\|X+Y\|_p \quad \textit{for} \quad 0<p\leqslant 1,$$
$$\|Y\|_p\leqslant\|X+Y\|_p \quad \textit{for} \quad 1\leqslant p\leqslant\infty.$$

The proof is easy and we leave the verification to the reader.

PROPOSITION 2.7. *Let X be an E-valued random variable and φ an increasing continuous function on $[0,\infty)$. Let $R(t)$ be the tail probability function:*

$$R(t) = P(\|X\|\geqslant t), \qquad t\geqslant 0.$$

Then $\varphi(\|X\|)$ is integrable if and only if $R(t)$ is Lebesgue–Stieltjes integrable with respect to φ, and then we have

$$\mathbf{E}(\varphi(\|X\|)) = \varphi(0)+\int_0^\infty R(t)\,d\varphi(t).$$

This is simply integration by parts!

PROPOSITION 2.8. *Let (X_j) be a symmetric sequence; then we have*

$$P\Big(\sum_1^\infty X_j \textit{ converges}\Big) = P\big((X_j)\in C(E)\big),$$

$$P\Big(\Big(\sum_1^n X_j\Big)_{n=1}^\infty \textit{ is bounded}\Big) = P\big((X_j)\in B(E)\big).$$

This follows easily from Fubini's theorem and the fact that (X_j) and $(\varepsilon_j X_j)$ are equidistributed if (ε_j) is a Bernoulli sequence, which is independent of (X_j).

3. Convergence and boundedness of (S_n) in $L^p(E)$. Let (X_n) be a sequence of independent E-valued random variables, and put

$$S_n = \sum_{j=1}^n X_j; \qquad N = \sup_n\|X_n\|; \qquad M = \sup_n\|S_n\|.$$

We shall in this section study convergence and boundedness of (S_n) in $L^p(E)$ $(0\leqslant p\leqslant\infty)$. We know already from Theorem 2.5 that convergence or boundedness of (S_n) in $L^p(E)$ implies a.s. convergence respectively a.s. boundedness of (S_n). Our first theorem tells us when the converse holds.

THEOREM 3.1. *Let (X_j) be independent E-valued random variables such that*

(3.1) $$(S_n) \textit{ is stochastically bounded},$$

(3.2) $$N\in L^p(\mathbf{R}),$$

for some $0<p<\infty$. Then $M\in L^p(\mathbf{R})$.

Proof. Notice first that $M < \infty$ a.s. by Theorem 2.5, and $N \leqslant 2M < \infty$. Let us first assume that X_j is symmetric for $j \geqslant 1$.

Let $R_k(t) = P(\|S_k\| \geqslant t)$, $R(t) = P(M \geqslant t)$ and $Q(t) = P(N \geqslant t)$. We shall then prove that

(3.3) $$R_k(2t+s) \leqslant Q(s) + 4R_k(t)^2 \quad \forall\, t, s \geqslant 0 \ \forall\, k \geqslant 1.$$

Let T be the stopping time defined by

$$T = \inf\{n \geqslant 1 \mid \|S_n\| \geqslant t\}$$

where $\inf(\emptyset) = \infty$. Then $\|S_k\| \geqslant 2t+s$ implies that $T \leqslant k$, and so we have

$$R_k(2t+s) = \sum_{j=1}^{k} P(\|S_k\| \geqslant 2t+s,\ T = j).$$

If $T = j$ and $\|S_k\| \geqslant 2t+s$ then $\|S_{j-1}\| < t$, and so

$$\|S_k - S_j\| \geqslant \|S_k\| - \|S_{j-1}\| - \|X_j\| \geqslant t+s-N.$$

Hence we have

$$\begin{aligned} P(T = j,\ \|S_k\| \geqslant 2t+s) &\leqslant P(T = j,\ \|S_k - S_j\| \geqslant t+s-N) \\ &\leqslant P(T = j,\ N \geqslant s) + P(T = j,\ \|S_k - S_j\| \geqslant t) \\ &= P(T = j,\ N \geqslant s) + P(T = j)P(\|S_k - S_j\| \geqslant t) \end{aligned}$$

since $\{T = j\}$ and $\{\|S_k - S_j\| \geqslant t\}$ are independent events. Hence we have that

$$R_k(2t+s) \leqslant Q(s) + \sum_{j=1}^{k} P(T = j)P(\|S_k - S_j\| \geqslant t).$$

Now let $Y_1 = S_k - S_j$ and $Y_2 = S_j$; then Y_1 and Y_2 are symmetric independent random variables and $Y_1 + Y_2 = S_k$, so by Theorem 2.3 we have that

$$P(\|S_k - S_j\| \geqslant t) \leqslant 2P(\|S_k\| \geqslant t),$$

and since $\{T \leqslant k\} = \{\max_{1 \leqslant j \leqslant k} \|S_j\| \geqslant t\}$, we have that

$$\begin{aligned} R_k(2t+s) &\leqslant Q(s) + 2R_k(t) \sum_{j=1}^{k} P(T = j) \\ &= Q(s) + 2R_k(t) P(\max_{1 \leqslant j \leqslant k} \|S_j\| \geqslant t) \\ &\leqslant Q(s) + 4R_k(t)^2, \end{aligned}$$

where we have used Theorem 2.3. Hence (3.3) is proved. From (3.1) and Theorem 2.3 we find that

$$R(2t+s) \leqslant 2Q(s) + 8R(t)^2 \quad \forall\, t, s \geqslant 0$$

since $R_k(t) \leqslant R(t) \leqslant 2 \sup_k R_k(t)$.

Now let $t_0 > 0$ be chosen so that $R(t_0) < \dfrac{1}{16 \cdot 3^p}$. If $A > 3t_0$ we then have

$$\begin{aligned} \int_0^A px^{p-1}R(x)\,dx &= 3^p \cdot p \int_0^{A/3} x^{p-1}R(3x)\,dx \\ &\leqslant 3^p \cdot p \cdot 2 \int_0^{A/3} x^{p-1}\, Q(x)\,dx + 8p3^p \int_0^{A/3} x^{p-1}R(x)^2\,dx \\ &\leqslant 2 \cdot 3^p \mathbf{E}(N^p) + 8 \cdot 3^p \cdot t_0^p \cdot p + 8p3^p \int_0^{A/3} x^{p-1}R(t_0)R(x)\,dx \\ &\leqslant C + \tfrac{1}{2} \int_0^A px^{p-1}R(x)\,dx \end{aligned}$$

where $C = 2 \cdot 3^p \mathbf{E}(N^p) + 8 \cdot 3^p \cdot t_0^p$, where we have used Lemma 2.7. Hence

$$\int_0^A px^{p-1}R(x)\,dx \leqslant 2C \qquad \forall\, A > 3t_0$$

and so $M \in L^p(E)$ by Lemma 2.7.

If (X_j) are not symmetric, then we consider a symmetrization $X_j^* = (X_j - X_j')$ of (X_j). Let us define

$$\begin{aligned} S_n' &= \sum_{j=1}^{n} X_j'; & S_n^* &= \sum_{j=1}^{n} X_j^* = S_n - S_n'; \\ M' &= \sup_n \|S_n'\|; & M^* &= \sup_n \|S_n^*\| \leqslant M + M'; \\ N' &= \sup_n \|X_n'\|; & N^* &= \sup_n \|X_n^*\| \leqslant N + N'. \end{aligned}$$

Since M and M' are equidistributed and N and N' are equidistributed, we find that $M^* < \infty$ a.s., and $N^* \in L^p(E)$. Thus from the argument above we have that $M^* \in L^p(E)$, and so

$$\infty > \mathbf{E}\{(M^*)^p\} = \int_{E^\infty} \mathbf{E}\Big\{\sup_n \Big\|S_n - \sum_{j=1}^{n} x_j\Big\|^p\Big\}\mu(dx),$$

where μ is the distribution law of (X_n'). Now since (S_n') is bounded a.s., we can find $(x_j) \in E^\infty$ so that

$$c = \sup_n \Big\|\sum_{j=1}^{n} x_j\Big\| < \infty,$$

$$\mathbf{E}\{(\sup_n \|S_n - s_n\|)^p\} < \infty,$$

where $s_n = x_1 + \ldots + x_n$. Now

$$M \leqslant c + \sup_n \|S_n - s_n\|,$$

and so $M \in L^p(E)$, which proves the theorem.

Corollary 3.2. *Let (X_j) be independent E-valued random variables, so that (S_n) is stochastically bounded, and let $0 < p < \infty$. Then the following 3 statements are equivalent:*

(3.4) (S_n) *is bounded in* $L^p(E)$,

(3.5) $M = \sup_n \|S_n\| \in L^p(\mathbf{R})$,

(3.6) $N = \sup_n \|X_n\| \in L^p(\mathbf{R})$.

Furthermore (S_n) is bounded in $L^\infty(E)$ if and only if $M \in L^\infty(\mathbf{R})$.

Proof. The last statement is obvious.

We know from Theorem 3.1 that (3.6) implies (3.5), and (3.5) obviously implies (3.4) and (3.6). Hence it suffices to prove that (3.4) implies (3.5). To do this we shall first assume that X_j is symmetric for all $j \geqslant 1$.

Let $R(t) = P(M \geqslant t)$, $R_k(t) = P(\|S_k\| \geqslant t)$; then by Theorem 2.3 we have

$$R(t) \leqslant 2 \liminf_{k\to\infty} R_k(t) \qquad \forall\, t \geqslant 0.$$

Now let $A > 0$; then by Fatou's lemma we have

$$\int_0^A px^{p-1} R(x)\,dx \leqslant 2 \liminf_{k\to\infty} \int_0^A px^{p-1} R_k(x)\,dx \leqslant 2 \liminf_{k\to\infty} \mathbf{E}\|S_k\|^p < \infty.$$

Thus, by Lemma 2.7, we have that $\mathbf{E}(M^p) < \infty$. The general case follows by a symmetrization procedure similar to the one in the last part of Theorem 3.1.

Corollary 3.3. *Let (X_j) be independent E-valued random variables, so that (S_n) converges a.s. to S, and let $0 < p < \infty$. Then the following 5 statements are equivalent:*

(3.7) $S_n \to S$ *in* $L^p(E)$,

(3.8) $S \in L^p(E)$,

(3.9) $M \in L^p(\mathbf{R})$,

(3.10) $N \in L^p(\mathbf{R})$,

(3.11) (S_n) *is bounded in* $L^p(E)$.

Furthermore $S \in L^\infty(E)$ if and only if $M \in L^\infty(E)$.

Remark. It is easy to construct an example where S belongs to $L^\infty(c_0)$, but $S_n \nrightarrow S$ in $L^\infty(c_0)$. However, in Theorem 3.5 we shall prove that if $S \in L^\infty(E)$ and E does not contain c_0, then $S_n \to S$ in $L^\infty(E)$.

Proof. (3.9), (3.10) and (3.11) are equivalent by Corollary 3.2, (3.7) obviously implies (3.8), and (3.9) implies (3.7) (use Lebesgue's theorem on dominated convergence). Hence it suffices to prove that (3.8) implies (3.9). To do this we shall assume that X_j is symmetric for all $j \geqslant 1$. In this case we have by Theorem 2.3 that

$$P(M \geqslant t) \leqslant 2P(\|S\| \geqslant t),$$

and so if $S \in L^p(E)$ then by Lemma 2.7 we have that $M \in L^p(\mathbf{R})$ (note that this result also holds for $p = \infty$).

Hence (3.8) implies (3.9), and we have also proved the last part of Corollary 3.3 under the assumption of symmetry of X_j. The general case is proved by a standard symmetrization procedure.

Corollary 3.4. *Let (X_j) be independent E-valued random variables and (a_j) a decreasing sequence of non-negative real numbers. Let*

$$U_n = a_n \sum_{j=1}^n X_j; \qquad V = \sup_n \|U_n\|; \qquad W = \sup_n \|a_n X_n\|.$$

Suppose that $V < \infty$ a.s. Then $W < \infty$ a.s., and if $W \in L^p(\mathbf{R})$ for some $0 < p < \infty$, then $V \in L^p(\mathbf{R})$.

Proof. Let us define

$$\tilde{X}_j = (0, \ldots, 0, a_j X_j, a_{j+1} X_j, \ldots, a_n X_j, \ldots).$$

Then $\tilde{X}_j$ is a $l^\infty(E)$-valued random variable, such that $\tilde{X}_1, \tilde{X}_2, \ldots$ are independent, and

$$\|\tilde{X}_j(\omega)\|_\infty = a_j \|X_j(\omega)\| \qquad \forall\, j \geqslant 1,\ \forall\, \omega \in \Omega.$$

If $\tilde{S}_j = \tilde{X}_1 + \ldots + \tilde{X}_j$ and S_{jk} is the kth coordinate of $\tilde{S}_j$, then

$$S_{jk} = \begin{cases} a_k \sum_{v=1}^{j} X_v & \text{if} \quad j < k, \\ U_k & \text{if} \quad j \geqslant k. \end{cases}$$

Now since $a_k \leqslant a_j$ for $j < k$, we find that

$$\|\tilde{S}_j(\omega)\|_\infty = \sup_k \|S_{jk}(\omega)\| = \max_{1\leqslant k\leqslant j} \|U_k\| \leqslant V(\omega).$$

Hence $(\tilde{S}_j(\omega))$ is a.s. bounded and Corollary 3.4 is an immediate consequence of Theorem 3.1.

THEOREM 3.5. *Let (X_j) be independent E-valued random variables such that the series*

$$S = \sum_{j=1}^{\infty} X_j \tag{3.12}$$

converges a.s. and $S \in L^\infty(E)$. If E does not contain a subspace isomorphic to c_0, then the series (3.12) *converges in $L^\infty(E)$.*

Proof. Suppose that $S \in L^\infty(E)$. From Corollary 3.3 it follows that

$$s_n = \sum_{j=1}^{n} \mathbf{E} X_j$$

converges to $\mathbf{E}S$. Let $X_j' = X_j - \mathbf{E}X_j$ and put $S_n' = X_1' + \ldots + X_n'$. Then $S_n' \xrightarrow[n\to\infty]{} S' = S - \mathbf{E}S$ a.s., $S' \in L^\infty(E)$, and $S_n' \to S'$ in $L^\infty(E)$ if and only if $S_n \to S$ in $L^\infty(E)$.

This shows that it is no loss of generality assuming that $\mathbf{E}X_j = 0$ for all $j \geq 1$, which we shall assume for the rest of the proof.

Now suppose that $S_n \not\to S$ in $L^\infty(E)$; we shall then show that E contains a closed subspace isomorphic to c_0. Since $S_n \to S$ a.s., we see that (S_n) is not a Cauchy sequence in $L^\infty(E)$. Hence we can find $a > 0$ and a sequence of integers: $0 = n_0 < n_1 < \ldots$ such that (put $S_0 = 0$)

$$\|S_{n_{j+1}} - S_{n_j}\|_\infty > a \quad \forall\, j = 0, 1, \ldots$$

Let $Y_j = S_{n_{j+1}} - S_{n_j}$ for $j = 0, 1, \ldots$ Then

$$\sum_{j=0}^{k} Y_j = S_{n_k}$$

and so $\sum_0^\infty Y_j = S$ a.s. Now Y_j takes values in a bounded separable subset of E with probability 1. Hence we can find random variables Z_j such that $\mathbf{E}Z_j = \mathbf{E}Y_j = 0$ and

$$Z_j = \sum_{v=1}^{\infty} x_{jv} 1_{A_{jv}} \quad \forall\, j \geq 0,$$

$$\|Z_j - Y_j\|_\infty \leq a 2^{-j-1} \quad \forall\, j \geq 0,$$

where $\{A_{jv} \mid v = 1, 2, \ldots\}$ are disjoint sets in the σ-algebra $Y_j^{-1}(\mathscr{B}(E))$ and $x_{jv} \in E$. Since

$$\sum_{j=1}^{\infty} \|Z_j - Y_j\|_\infty \leq a < \infty,$$

we find that $\sum_0^\infty Z_j = T$ exist a.s. and $T \in L^\infty(E)$. Also we have that

$$\|Z_j\|_\infty \geq \|Y_j\|_\infty - a 2^{-j-1} > \tfrac{1}{2} a \quad \forall\, j \geq 0.$$

Hence we can find $v(j) \geq 1$ such that $\|x_{jv(j)}\| \geq \frac{1}{2} a$ and $P(A_{jv(j)}) > 0$. Let us put $y_j = x_{jv(j)}$ and $B_j = A_{jv(j)}$ for $j \geq 0$.

Since $Y_0, Y_1, \ldots$ are independent random variables, we have that $Z_0, Z_1, \ldots$ are independent and so, by Theorem 3.5, we can find a constant $K > 0$ such that

$$\Big\| \sum_{j=1}^{n} Z_j \Big\|_\infty \leq K \quad \forall\, n \geq 1.$$

Since $\mathbf{E}Z_j = 0$, it follows from Theorem 2.6 that whenever $\sigma \subseteq \{0, \ldots, n\}$, then we have

$$\Big\| \sum_{j\in\sigma} Z_j \Big\|_\infty \leq \Big\| \sum_{j=1}^{n} Z_j \Big\|_\infty \leq K.$$

That is, there exists an event $\Omega_0 \subseteq \Omega$ with $P(\Omega_0) = 1$ and

$$\Big\| \sum_{j\in\sigma} Z_j(\omega) \Big\| \leq K \quad \forall\, \omega \in \Omega_0$$

and for all finite subsets σ of $\{0, 1, \ldots\}$ (there is at most countably many finite subsets of $\{0, 1, \ldots\}$).

We know that $P(B_j) > 0$ and that $B_0, \ldots, B_n, \ldots$ are independent. Hence we have that $P\big(\bigcap_0^n B_j\big) > 0$ and so we can find an $\omega_n \in \Omega_0 \cap B_0 \cap \ldots \cap B_n$. Since $Z_j(\omega_n) = y_j$ for $0 \leq j \leq n$, we find that

$$\Big\| \sum_{j\in\sigma} y_j \Big\| \leq K, \quad \sigma \subseteq \{1, \ldots, n\} \quad \forall\, n \geq 1.$$

But this implies that

$$\sum_{j=1}^{\infty} |\langle x', y_j \rangle| \leq 2K \|x'\| \quad \forall\, x' \in E',$$

and since $\|y_j\| \geq \frac{1}{2} a$, it follows from Theorem 5 in [1] that E contains a subspace isomorphic to c_0.

THEOREM 3.6. *Let (X_j) be a sequence of independent integrable E-valued random variables, with $\mathbf{E}X_j = 0$ for all $j \geq 1$. Then the following 4 conditions are equivalent:*

$$\Big(\sum_{1}^{n} X_j\Big) \textit{ is bounded in } L^\infty(E), \tag{3.13}$$

(3.14) $\exists K > 0$, $\exists \Omega_0 \in \mathscr{F}$, *such that* $P(\Omega_0) = 1$ *and*

$$\sum_{j=1}^{\infty} |\langle x', X_j(\omega) \rangle| \leq K \|x'\| \quad \forall\, x' \in E', \forall\, \omega \in \Omega_0,$$

(3.15) $\forall\, x' \in E', \exists\, K(x') > 0$, *such that* $P\Big(\omega \Big| \sum_{j=1}^{\infty} |\langle x', X_j(\omega)\rangle| \leqslant K(x')\Big) = 1,$

(3.16) $\forall x' \in E',\ \exists\, K(x') > 0,$ *such that* $P\Big(\omega \Big| \Big|\sum_{j=1}^{n} \langle x', X_j(\omega)\rangle\Big| \leqslant K(x')\Big)$

$= 1 \quad \forall\, n \geqslant 1.$

If integrability of X_j *and* $\mathbf{E}X_j = 0$ *is not assumed then* (3.14) *and* (3.15) *are equivalent,* (3.15) *implies* (3.16), *and* (3.16) *is equivalent to* (3.13).

Proof. Suppose that (3.13) holds and $\mathbf{E}X_j = 0$. Let K be a finite constant such that

$$\Big\|\sum_{j=1}^{n} X_j\Big\|_\infty \leqslant K.$$

Then, by Theorem 2.6, we have that

$$\Big\|\sum_{j\in\sigma} X_j\Big\|_\infty \leqslant K$$

for every finite subset σ of $\{1, 2, \ldots\}$. Since there is only countably many finite subsets of $\{1, 2, \ldots\}$, we can find $\Omega_0 \in \mathscr{F}$ with $P(\Omega_0) = 1$ and

$$\Big\|\sum_{j\in\sigma} X_j(\omega)\Big\| \leqslant K \qquad \forall\, \omega \in \Omega_0.$$

Hence we have

$$\sum_{j=1}^{\infty} |\langle x', X_j(\omega)\rangle| \leqslant 2K\|x'\| \qquad \forall\, \omega \in \Omega_0,\ \forall\, x' \in E'.$$

Now let us drop the assumptions on integrability of X_j and $\mathbf{E}X_j = 0$ for the rest of this proof.

It is evident that (3.14) implies (3.15), (3.15) implies (3.16), and (3.13) implies (3.16).

Suppose that (3.16) holds. Since X_j is essentially separably valued for all $j \geqslant 1$, it is no loss of generality to assume that E is separable. Let U' be the unit ball in E' equipped with the w^*-topology. Then U' is metrizable and compact. Let T' be a countable w^*-dense subset of U'.

We know by assumption that $\langle x', S_n\rangle$ belongs to $L^\infty(\mathbf{R})$ for all $x' \in E'$, hence we may define the linear map

$$A_n x' = \langle x', S_n\rangle = \sum_{j=1}^{n} \langle x', X_j\rangle$$

from E' into $L^\infty(\mathbf{R})$. If $x'_k \to x'$ and $A_n x'_k \to f$ in $L^\infty(\mathbf{R})$, then

$$\langle x'_k, S_n(\omega)\rangle \xrightarrow[k\to\infty]{} \langle x', S_n(\omega)\rangle \qquad \forall\, \omega \in \Omega$$

and so $f = A_n x'$. Hence A_n is continuous by the closed graph theorem. By assumption we know that $\sup_n \|A_n x'\|_\infty < \infty$ for all $x' \in E'$, so by the principle of uniform boundedness we have that there exists a finite constant K such that

$$P(|\langle x', S_n\rangle| \leqslant K) = 1 \qquad \forall\, n \geqslant 1,\ \forall\, \|x'\| \leqslant 1.$$

Since T' is countable, we can therefore find $\Omega_0 \in \mathscr{F}$, with $P(\Omega_0) = 1$ and

$$|\langle x', S_n(\omega)\rangle| \leqslant K \qquad \forall\, x' \in T',\ \forall\, \omega \in \Omega_0.$$

It is well known that we can find compact subsets K_{nm} of E such that

$$P(S_n \in K_{nm}) \geqslant 1 - 2^{-n-m}$$

because E is a Polish space. Now let

$$\Omega_1 = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \{S_n \in K_{nm}\} \cap \Omega_0;$$

it is then easily seen that $P(\Omega_1) = 1$.

Let us take an $x' \in U'$ and an $\omega \in \Omega_1$. Then by Banach–Dieudonné's theorem (see, for example, [10], 21.10 (1), p. 272) there exists $y'_{nm} \in T'$ such that

$$|\langle x' - y'_{nm}, x\rangle| \leqslant 1 \qquad \forall\, x \in K_{nm},\ \forall\, n, m.$$

Since $\omega \in \Omega_1$, we can find an integer $m \geqslant 1$ with $\omega \in \Omega_0 \cap \{S_n \in K_{nm}\}$ for all $n \geqslant 1$. Hence we have

$$|\langle x', S_n(\omega)\rangle| \leqslant |\langle y'_{nm}, S_n(\omega)\rangle| + |\langle x' - y'_{nm}, S_n(\omega)\rangle| \leqslant K+1$$

since $\omega \in \Omega_0$ and $S_n(\omega) \in K_{nm}$. But this shows that $\|S_n(\omega)\| \leqslant K+1$ for $\omega \in \Omega_1$, and so (3.16) implies (3.13).

Let us then prove that (3.15) implies (3.14) (without the assumption that $\mathbf{E}X_j = 0$). We have already proved that (3.15) implies (3.13). Hence (S_n) is bounded in $L^\infty(E)$, and X_j is integrable for all j. Let $x_j = \mathbf{E}X_j$; then

$$\sum_{j=1}^{\infty} |\langle x', \mathbf{E}X_j\rangle| = \sum_{j=1}^{\infty} |\mathbf{E}\langle x', X_j\rangle| \leqslant \mathbf{E}\Big\{\sum_{j=1}^{\infty} |\langle x', X_j\rangle|\Big\} \leqslant K(x').$$

Hence $\sum_1^\infty x_j$ is weakly unconditionally convergent in E. Now let $\tilde{X}_j = X_j - x_j$; we then have that $\mathbf{E}\tilde{X}_j = 0$ and

$$\sum_{j=1}^{\infty} |\langle x', \tilde{X}_j(\omega)\rangle| \leqslant \sum_{j=1}^{\infty} |\langle x', X_j(\omega)\rangle| + \sum_{j=1}^{\infty} |\langle x', x_j\rangle|$$
$$\leqslant K(x') + \sum_{j=1}^{\infty} |\langle x', X_j(\omega)\rangle|.$$

This shows that $(\tilde{X}_j)$ satisfies (3.15), and so there exists $\Omega_0 \in \mathscr{F}$, with $P(\Omega_0) = 1$ and there exists a constant $K > 0$ with

$$\sum_{j=1}^{\infty} |\langle x', \tilde{X}_j(\omega)\rangle| \leqslant K\|x'\| \quad \forall\, \omega \in \Omega_0,\ \forall\, x' \in E'.$$

It is well known that for a weakly unconditional convergent series $\sum_1^\infty x_n$ we have, for some $L > 0$, that

$$\sum_{j=1}^{\infty} |\langle x', x_j\rangle| \leqslant L\|x'\|.$$

Hence

$$\sum_{j=1}^{\infty} |\langle x', X_j(\omega)\rangle| \leqslant (K+L)\|x'\| \quad \forall\, \omega \in \Omega_0,\ \forall\, x' \in E',$$

and so Theorem 3.6 is proved.

COROLLARY 3.7. *Let (X_j) be a sequence of integrable independent E-valued random variables such that $\mathbf{E}X_j = 0$ for all $j \geqslant 1$, and such that the partial sums:*

$$S_n = \sum_{j=1}^{n} X_j$$

are bounded in $L^\infty(E)$.

If E does not contain a subspace isomorphic to c_0, then $\{S_n\}$ converges in $L^\infty(E)$.

Proof. By Theorem 3.6 we know that for all $\omega \in \Omega_0$, where $P(\Omega_0) = 1$, we have that

$$\sum_{j=1}^{\infty} |\langle x', X_j(\omega)\rangle| \leqslant K\|x'\| \quad \forall\, x' \in E',$$

so by Theorem 5 in [1] we have that $\sum_1^\infty X_j$ converges a.s. and that

$$\Big\|\sum_1^{\infty} X_j(\omega)\Big\| \leqslant K \quad \forall\, \omega \in \Omega_0.$$

Hence by Theorem 3.5 we have that (S_n) converges in $L^\infty(E)$.

4. The comparison principle. Let (X_j) be a sequence of independent E-valued random variables, and let (ξ_j) and (η_j) be sequences of independent scalar valued random variables, we shall then study the series

$$\sum_{j=1}^{\infty} \eta_j X_j$$

in terms of the behaviour of the series

$$\sum_{j=1}^{\infty} \xi_j X_j.$$

First we consider the case where η_j is non-random and $\xi_j = 1$.

LEMMA 4.1. *Let $X_1, \ldots, X_n$ be independent E-valued p-integrable random variables, where $1 \leqslant p < \infty$. If $\mathbf{E}X_j = 0$ for all $j = 1, \ldots, n$, we then have that*

$$\Big\{\mathbf{E}\Big\|\sum_{j=1}^{n} a_j X_j\Big\|^p\Big\}^{1/p} \leqslant 2\max_{1\leqslant j\leqslant n}|a_j|\,\Big\{\mathbf{E}\Big\|\sum_{j=1}^{n} X_j\Big\|^p\Big\}^{1/p}$$

for all $a_1, \ldots, a_n \in \mathbf{R}$.

If X_j is symmetric for all $j = 1, \ldots, n$, we then have that

$$\Big\{\mathbf{E}\Big\|\sum_{j=1}^{n} a_j X_j\Big\|^p\Big\}^{1/p} \leqslant \max_{1\leqslant j\leqslant n}|a_j|\,\Big\{\mathbf{E}\Big\|\sum_{j=1}^{n} X_j\Big\|^p\Big\}^{1/p}$$

for all $a_1, \ldots, a_n \in \mathbf{R}$.

Proof. If $\sigma \subseteq \{1, 2, \ldots, n\}$ it then follows from Theorem 2.6 that

$$\mathbf{E}\Big\|\sum_{j\in\sigma} X_j\Big\|^p \leqslant \mathbf{E}\Big\|\sum_{j=1}^{n} X_j\Big\|^p.$$

If $a_j = \pm 1$ for all $j = 1, \ldots, n$, we then may put $\sigma = \{1 \leqslant j \leqslant n \mid a_j = +1\}$ and $\varkappa = \{1 \leqslant j \leqslant n \mid a_j = -1\}$; we then have

$$\Big\{\mathbf{E}\Big\|\sum_{j=1}^{n} a_j X_j\Big\|^p\Big\}^{1/p} \leqslant \Big\{\mathbf{E}\Big\|\sum_{j\in\sigma} X_j\Big\|^p\Big\}^{1/p} + \Big\{\mathbf{E}\Big\|\sum_{j\in\varkappa} X_j\Big\|^p\Big\}^{1/p}$$
$$\leqslant 2\Big\{\mathbf{E}\Big\|\sum_{j=1}^{n} X_j\Big\|^p\Big\}^{1/p}.$$

If $K_n = \{(a_1, \ldots, a_n) \in \mathbf{R}^n \mid |a_1| \leqslant 1, \ldots, |a_n| \leqslant 1\}$, then K_n is a compact convex subset of $\mathbf{R}^n$, whose extreme points are the 2^n points: $(\pm 1, \pm 1, \ldots, \pm 1)$. Thus if $a = (a_1, \ldots, a_n) \in K_n$ then by Carathéodory's theorem a is a convex combination of at most $n+1$ extreme points. That is, we can find $\lambda_m \geqslant 0$ for $m = 1, \ldots, n+1$ with

$$a_j = \sum_{m=1}^{n+1} \lambda_m a_{jm}, \quad \sum_{m=1}^{n+1} \lambda_m = 1,$$

where $a_{jm} = \pm 1$. Hence we find

$$\Big\{\mathbf{E}\Big\|\sum_{j=1}^{n} a_j X_j\Big\|^p\Big\}^{1/p} = \Big\{\mathbf{E}\Big\|\sum_{m=1}^{n+1}\lambda_m\Big(\sum_{j=1}^{n} a_{jm} X_j\Big)\Big\|^p\Big\}^{1/p}$$

$$\leqslant \sum_{m=1}^{n+1}\lambda_m\Big\{\mathbf{E}\Big\|\sum_{j=1}^{n} a_{jm} X_j\Big\|^p\Big\}^{1/p}$$

$$\leqslant 2\Big\{\mathbf{E}\Big\|\sum_{j=1}^{n} X_j\Big\|^p\Big\}^{1/p},$$

from which the first part of the lemma follows easily.

If X_j is symmetric for all $j = 1, \ldots, n$, then we have

$$\mathbf{E}\Big\|\sum_{j=1}^{n} a_j X_j\Big\|^p = \mathbf{E}\Big\|\sum_{j=1}^{n} X_j\Big\|^p$$

whenever $a_j = \pm 1$. The last part of the lemma follows as above.

COROLLARY 4.2. *Let $X_1, \ldots, X_n$ be E-valued, independent, p-integrable random variables with $1 \leqslant p < \infty$ amd $\mathbf{E}X_j = 0$ for all $j = 1, \ldots, n$. If $\eta_1, \ldots, \eta_n$ are real valued, p-integrable random variables, so that $(X_1, \ldots, X_n)$ and $(\eta_1, \ldots, \eta_n)$ are independent, then*

$$2^{-p}\mathbf{E}\big(\min_{1\leqslant j\leqslant n}|\eta_j|^p\big)\mathbf{E}\Big\|\sum_{j=1}^{n} X_j\Big\|^p \leqslant \mathbf{E}\Big\|\sum_{j=1}^{n}\eta_j X_j\Big\|^p \leqslant 2^p\mathbf{E}\big(\max_{1\leqslant j\leqslant n}|\eta_j|^p\big)\mathbf{E}\Big\|\sum_{j=1}^{n} X_j\Big\|^p.$$

Proof. From Lemma 4.1 it follows that

$$2^{-p}\min|a_j|^p\mathbf{E}\Big\|\sum_{j=1}^{n} X_j\Big\|^p \leqslant \mathbf{E}\Big\|\sum_{j=1}^{n} a_j X_j\Big\|^p \leqslant 2^p\max|a_j|^p\mathbf{E}\Big\|\sum_{j=1}^{n} X_j\Big\|^p$$

for all $a_1, \ldots, a_n$ (the first inequality follows from Lemma 4.1 by putting $X_j = a_j X_j$ and $a_j = 1/a_j$). So the corollary follows by integrating with respect to the distribution law of $(\eta_1, \ldots, \eta_n)$.

THEOREM 4.3. *Let (η_j) and (ξ_j) be sequences of independent, real valued, p-integrable random variables with $1 \leqslant p < \infty$, and let (X_j) be a sequence of E-valued, independent p-integrable random variables such that*

(4.1) $$N = \sup_j |\eta_j| \in L^p(\mathbf{R}),$$

(4.2) $$a = \inf_j \mathbf{E}|\xi_j| > 0,$$

(4.3) *(X_j) and (ξ_j) are independent, and (X_j) and (η_j) are independent,*

(4.4) $$\mathbf{E}(\eta_j X_j) = \mathbf{E}(\xi_j X_j) = 0 \quad \forall j \geqslant 1.$$

Then we have

$$\mathbf{E}\Big\|\sum_{j=1}^{n}\eta_j X_j\Big\|^p \leqslant \Big(\frac{8}{a}\Big)^p\mathbf{E}(N^p)\mathbf{E}\Big\|\sum_{j=1}^{n}\xi_j X_j\Big\|^p \quad \forall n.$$

Remark. In my original version I assumed that $P(\inf_j|\xi_j| > 0) > 0$ instead of (4.2). The strengthening of the theorem is based on an idea of B. Maurey (oral communication).

Proof. Let (ε_j) be a Bernoulli sequence, which is independent of $((X_j), (\xi_j), (\eta_j))$. By Corollary 4.2, with $X_j = \eta_j X_j$ and $\eta_j = \varepsilon_j$, we have

$$\mathbf{E}\Big\|\sum_{j=1}^{n}\eta_j X_j\Big\|^p \leqslant 2^p\mathbf{E}\Big\|\sum_{j=1}^{n}\eta_j\varepsilon_j X_j\Big\|^p.$$

Since $\varepsilon_j X_j$ is symmetric and (η_j) is independent of $(\varepsilon_j X_j)$, we find by Corollary 4.2 that

$$\mathbf{E}\Big\|\sum_{j=1}^{n}\eta_j X_j\Big\|^p \leqslant 2^p\mathbf{E}\Big\|\sum_{j=1}^{n}\eta_j\varepsilon_j X_j\Big\|^p \leqslant 4^p\mathbf{E}(N^p)\mathbf{E}\Big\|\sum_{j=1}^{n}\varepsilon_j X_j\Big\|^p.$$

Let $a_j = \mathbf{E}|\xi_j|$; we then have by Lemma 4.1

$$\mathbf{E}\Big\|\sum_{j=1}^{n}\eta_j X_j\Big\|^p \leqslant 4^p\mathbf{E}(N^p)\max_{1\leqslant j\leqslant n}\Big(\frac{1}{a_j}\Big)^p\mathbf{E}\Big\|\sum_{j=1}^{n} a_j\varepsilon_j X_j\Big\|^p$$

$$\leqslant \Big(\frac{4}{a}\Big)^p\mathbf{E}(N^p)\mathbf{E}\Big\|\sum_{j=1}^{n} a_j\varepsilon_j X_j\Big\|^p.$$

Let $\mathscr{F}$ be the σ-algebra generated by $(\varepsilon_1, \ldots, \varepsilon_n, X_1, \ldots, X_n)$, and define

$$\varepsilon_j^* = \begin{cases} -\varepsilon_j & \text{if } \xi_j < 0, \\ \varepsilon_j & \text{if } \xi_j \geqslant 0. \end{cases}$$

Then we have

$$\Big\|\sum_{j=1}^{n} a_j\varepsilon_j X_j\Big\|^p = \Big\|\mathbf{E}\Big(\sum_{j=1}^{n}|\xi_j|\varepsilon_j X_j\Big|\mathscr{F}\Big)\Big\|^p \leqslant \mathbf{E}\Big(\Big\|\sum_{j=1}^{n}|\xi_j|\varepsilon_j X_j\Big\|^p\Big|\mathscr{F}\Big)$$

$$= \mathbf{E}\Big(\Big\|\sum_{j=1}^{n}\xi_j\varepsilon_j^* X_j\Big\|^p\Big|\mathscr{F}\Big)$$

and so we find

$$\mathbf{E}\Big\|\sum_{j=1}^{n}\eta_j X_j\Big\|^p \leqslant \Big(\frac{4}{a}\Big)^p\mathbf{E}(N^p)\mathbf{E}\Big\|\sum_{j=1}^{n}\varepsilon_j^*\xi_j X_j\Big\|^p.$$

It is easily checked that (ε_j^*) and $((\xi_j), (X_j))$ are independent, and so by Corollary 4.2 we find

$$\mathbf{E}\Big\|\sum_{j=1}^{n}\eta_j X_j\Big\|^p \leqslant \Big(\frac{8}{a}\Big)^p \mathbf{E}(N^p)\mathbf{E}\Big\|\sum_{j=1}^{n}\xi_j X_j\Big\|^p.$$

COROLLARY 4.4. *Let (ξ_j), (η_j) and (X_j) be a sequence of real (respectively E-valued), independent p-integrable random variables satisfying* (4.1)–(4.4). *If*

$$S_n = \sum_{j=1}^{n}\xi_j X_j, \quad T_n = \sum_{j=1}^{n}\eta_j X_j,$$

we then have

(4.5) *If (S_n) is bounded in $L^p(E)$, then so is (T_n).*

(4.6) *If (S_n) converges in $L^p(E)$, then so does (T_n).*

(4.7) *If (S_n) is bounded in $L^p(E)$, and if $\eta_j \xrightarrow[j\to\infty]{} 0$ a.s., then (T_n) converges in $L^p(E)$.*

Proof. Immediate consequence of Theorem 4.3.

5. Boundedness and convergence. It is well known that if $E = \mathbf{R}$ (or even if E is a Hilbert space) then boundedness of the partial sums

$$S_n = \sum_{j=1}^{n} X_j,$$

where (X_j) is a symmetric sequence, implies a.s. convergence of (S_n). Now from Proposition 2.8 it follows that a Banach space E has this property if and only if

$$B(E) = C(E).$$

We shall in this section study the class of Banach spaces with this property.

THEOREM 5.1. *If $(\Omega, \mathscr{F}, P)$ is a probability space on which we can define a Bernouilli sequence then the following statements are equivalent:*

(5.1) $B(E) \nsubseteq c_0(E)$,

(5.2) $B(E) \neq C(E)$,

(5.3) $\exists\, 1 \leqslant p < \infty$, *such that $L^p(\Omega, \mathscr{F}, P, E)$ contains a closed subspace isomorphic to c_0*,

(5.4) $\forall\, 1 \leqslant p < \infty$, *$L^p(\Omega, \mathscr{F}, P, E)$ contains a closed subspace isomorphic to c_0.*

Remark. I conjecture that (5.1)–(5.4) are equivalent to

(5.5) *E contains a closed subspace isomorphic to c_0.*

Notice that (5.5) implies (5.2), as $(e_n) \in B(E)$, but $(e_n) \notin c_0(E)$ if e_n is the nth unit vector in c_0. Notice also that Corollary 3.7 gives a partial answer to the converse implication.

Proof of Theorem 5.1. (5.4) implies (5.3) for trivial reasons. Suppose that (5.3) holds, and let (ε_j) be a Bernoulli sequence defined on $(\Omega, \mathscr{F}, P)$. By assumption there exist a $1 \leqslant p < \infty$ and $f_j \in L^p(E)$ such that

$$a \leqslant \|f_j\|_p \leqslant b \quad \forall j \geqslant 1, \quad \text{where } 0 < a \leqslant b < \infty,$$

$$\Big\|\sum_{j=1}^{n}\alpha_j f_j\Big\|_p \leqslant K \max_{1\leqslant j\leqslant n}|\alpha_j| \quad \forall n \geqslant 1,\ \forall \alpha_1, \ldots, \alpha_n \in \mathbf{R}$$

where K is a finite constant. On the probability space $(\Omega\times\Omega, \mathscr{F}\otimes\mathscr{F}, P\times P) = (\Omega', \mathscr{F}', P')$ we define

$$f_j'(\omega_1, \omega_2) = \varepsilon_j(\omega_1) f_j(\omega_2) \quad \text{for} \quad (\omega_1, \omega_2) \in \Omega'.$$

It is then easily seen that

$$a \leqslant \|f_j'\|_p \leqslant b \quad \forall j \geqslant 1, \tag{5.6}$$

$$\Big\|\sum_{j=1}^{n}\alpha_j f_j'\Big\|_p \leqslant K \max_{1\leqslant j\leqslant n}|\alpha_j| \quad \forall n \geqslant 1,\ \forall \alpha_1, \ldots, \alpha_n \in \mathbf{R}, \tag{5.7}$$

and that (f_j') is a symmetric sequence. From (5.7) it follows that $g_n' = \sum_{j=1}^{n} f_j'$ s bounded in $L^p(E)$. So by Proposition 2.8 we have that $(f_j'(\omega')) \in B(E)$ for a.a. $\omega' \in \Omega'$. Now if $(f_j'(\omega')) \in C(E)$ a.s., we then know that (g_n') converges a.s. to a random variable g' by Proposition 2.8. So by Fatou's lemma we have

$$\mathbf{E}\|g'\|^p \leqslant \liminf_{n\leqslant\infty} \mathbf{E}\|g_n'\|^p \leqslant K^p,$$

i.e. $g' \in L^p(E)$. Since

$$P(\sup_n \|g_n'\| \geqslant t) \leqslant 2P(\|g'\| \geqslant t),$$

we have that $\sup_n \|g_n'\| \in L^p(\mathbf{R})$. Thus by Lebesgue's theorem of dominated convergence we have that (g_n') converges in $L^p(E)$, but this contradicts the fact that $\|f_j\|_p \geqslant a$ for all $j \geqslant 1$. Hence we have

$$(f_j'(\omega')) \in B(E) \quad \text{for a.a. } \omega' \in \Omega',$$

$$(f_j'(\omega')) \notin C(E) \quad \text{for all } \omega' \in A' \quad \text{where} \quad P(A') > 0.$$

That is, $B(E) \neq C(E)$. And so (5.3) implies (5.2).

Now suppose that $B(E) \neq C(E)$. And let (x_j) be a sequence in $B(E) \setminus C(E)$. Since $\sum_1^n \varepsilon_j x_j$ is not a Cauchy sequence in $L^1(E)$, we can find $a > 0$ and a sequence $1 = n_1 < n_2 < \ldots$ such that

$$\mathbf{E}\Big\| \sum_{n_k \leqslant j < n_{k+1}} \varepsilon_j x_j \Big\| \geqslant a \qquad \forall\, k = 1, 2, \ldots$$

Now let

$$X_k = \sum_{n_k \leqslant j < n_{k+1}} \varepsilon_j x_j \quad \text{for} \quad k = 1, 2, \ldots$$

Then $X_1, X_2, \ldots$ are independent symmetric random variables. Furthermore, if $M(\omega) = \sup_n \|\sum_1^n \varepsilon_j(\omega) x_j\|$, then by Theorem 3.1 we have

$$M(\omega) < \infty \text{ a.s. and } \mathbf{E}M < \infty,$$

$$\mathbf{E}\Big\| \sum_{v=1}^{k} X_v \Big\| = \mathbf{E}\Big\| \sum_{1 \leqslant j < n_{k+1}} \varepsilon_j x_j \Big\| \leqslant \mathbf{E}M < \infty,$$

$$\mathbf{E}\|X_v\| \geqslant a \qquad \forall\, v \geqslant 1,$$

$$\|X_v(\omega)\| \leqslant 2M(\omega) \qquad \forall\, v \geqslant 1.$$

This implies that $P(X_v \nrightarrow 0) > 0$. And from Proposition 2.8 it follows that $(X_v(\omega)) \in B(E)$ a.s. Hence $B(E) \nsubseteq c_0(E)$. That is, (5.2) implies (5.1).

Now suppose that $B(E) \nsubseteq c_0(E)$. Then we can find $(x_j) \in B(E)$ such that $\|x_j\| = 1$ for all $j = 1, 2, \ldots$ Now let (ε_j) be a Bernoulli sequence defined on $(\Omega, \mathscr{F}, P)$ and put

$$X_j = \varepsilon_j x_j.$$

Then by Lemma 4.1 and Theorem 2.6 we have

$$|a_j| = (\mathbf{E}\|a_j X_j\|^p)^{1/p} \leqslant \Big(\mathbf{E}\Big\| \sum_{v=1}^{n} a_v X_v \Big\|^p\Big)^{1/p} \leqslant \max_{1 \leqslant v \leqslant n} |a_v| \Big(\mathbf{E}\Big\| \sum_{v=1}^{n} \varepsilon_v x_v \Big\|^p\Big)^{1/p}$$

for $n \geqslant j \geqslant 1$ and all $a_1, \ldots, a_n \in \mathbf{R}$. Hence if

$$K = \sup_n \Big(\mathbf{E}\Big\| \sum_{v=1}^{n} \varepsilon_v x_v \Big\|^p\Big)^{1/p}$$

then

$$\max_{1 \leqslant j \leqslant n} |a_j| \leqslant \Big\| \sum_{j=1}^{n} a_j X_j \Big\|^p \leqslant K \max_{1 \leqslant j \leqslant n} |a_j|.$$

And by Corollary 4.4 we have that $\sum_1^\infty a_j X_j$ exist for all $(a_j) \in c_0$. Then the inequalities above show that the map

$$(a_j) \curvearrowright \sum_1^\infty a_j X_j$$

is an isomorphism of c_0 into a closed subspace of $L^p(E)$ for all $p \in [1, \infty)$. That is, (5.1) implies (5,4). and so the theorem is proved.

Theorem 5.2. *If E has the Radon–Nikodym property, then $B(E) = C(E)$.*

Remark. Later we shall see that if $E = L^p(S, \Sigma, \mu)$ for some measure space (S, Σ, μ) and some $1 \leqslant p < \infty$, then $B(E) = C(E)$. Now it is well known that $L^1[0, 1]$ does not have the Radon–Nikodym property. Thus $B(E) = C(E)$ does not imply the Radon–Nikodym property.

Proof of Theorem 5.2. Immediate consequence of Theorem 6 in [2].

We shall now show that if the partial sums, $S_n = \sum_1^n X_j$, are a.s. bounded and relatively compact, then the series $\sum_1^\infty X_j$ converges a.s. Before we state the theorem, we observe the following application of Chatterji's results on vector-valued martingales (see [2]; compare also Theorem 4.1 in [8]):

Theorem 5.3. *Let (X_n) be independent integrable E-valued random variables, and let F' be a subset of E' such that F' separates points in E. If there exists $S \in L^1(E)$ such that*

$$\Big\langle x', \sum_{j=1}^{n} X_j \Big\rangle \xrightarrow[n\to\infty]{} \langle x', S\rangle \text{ in probability}$$

for all $x' \in F'$, and $\mathbf{E}X_j = 0$ for all $j \geqslant 1$, then the series

$$\sum_1^\infty X_j$$

converges to S a.s. and in $L^1(E)$,

Proof. By Corollary 3.3 we have that $\langle x', S_n\rangle \xrightarrow[n\to\infty]{} \langle x', S\rangle$ in $L^1(\mathbf{R})$ for all $x' \in F'$, where S_n is the partial sum

$$S_n = \sum_{j=1}^{n} X_j.$$

Hence $\mathbf{E}S = 0$ as F' separates E.

Now let $\mathscr{F}_n$ be the σ-algebra spanned by $\{X_1, \ldots, X_n\}$; then for $x' \in F'$ we have

$$\int_A \langle x', S_n \rangle dP = \int_A \langle x', S \rangle dP \quad \forall A \in \mathscr{F}_n$$

since $\langle x', S - S_n \rangle = \sum_{j=n+1}^{\infty} \langle x', X_j \rangle$ is independent of $\mathscr{F}_n$ and has mean 0. Now since S_n and S both belong to $L^1(E)$ and F' separates points of E, we have that

$$\int_A S_n dP = \int_A S\, dP \quad \forall A \in \mathscr{F}_n.$$

That is, $S_n = \mathbf{E}(S | \mathscr{F}_n)$. Thus by Theorem 1 (a) in [2] we have that (S_n) converges in $L^1(E)$, and so by Theorem 2.5, (S_n) converges a.s. Also the limit of (S_n) is of course equal to S as F' separates points of E.

THEOREM 5.4. *Let τ be a locally convex Hausdorff topology on E which is weaker than the norm topology, and such that the closed unit ball in E is τ-closed. Let $x = (x_j) \in E^\infty$, and put*

$$F(\omega) = \Big\{ \sum_{j=1}^{n} \varepsilon_j(\omega) x_j \Big| \; n \geqslant 1 \Big\} \quad \text{for} \quad \omega \in \Omega,$$

where (ε_j) is a Bernoulli sequence.

If E is separable and $F(\omega)$ is relatively τ-compact for a.a. ω, then $x \in C(E)$.

Proof. Let us put $S_n = \sum_1^n \varepsilon_j x_j$, and let F' be the dual of (E, τ). Then $F' \subseteq E'$ as τ is weaker than $\|\cdot\|$.

Now, as E is separable, we can find a countable set $T \subseteq F'$ such that T separates E.

As $(\langle x' x_n \rangle) \in B(\mathbf{R}) = C(\mathbf{R})$, we have that

$$\Phi(x', \omega) = \lim_{n\to\infty} \langle x', S_n(\omega) \rangle$$

exists for all $x' \in T$ and all $\omega \notin N_0$ where $P(N_0) = 0$. By assumption there exists a null set $N_1 \supseteq N_0$ such that $\{S_n(\omega) | \; n \geqslant 1\}$ is relatively τ-compact for all $\omega \notin N_1$, and so if $\omega \notin N_1$ we can find a sequence (depending on ω): $n_1 < n_2 < \ldots$ such that for some $S(\omega) \in E$ we have $\tau\text{-}\lim_{j\to\infty} S_{n_j}(\omega) = S(\omega)$. Since $T \subseteq F'$, we find that

$$\langle x', S(\omega) \rangle = \lim_{j\to\infty} \langle x', S_{n_j}(\omega) \rangle = \Phi(x', \omega)$$

for all $x' \in T$. This shows that for $\omega \notin N_1$ we have that $\{S_n(\omega)\}$ has one and only one τ-limit point as T separates E. That is,

$$S(\omega) = \tau\text{-}\lim_{n\to\infty} S_n(\omega)$$

exists for all $\omega \notin N_1$.

Since $(E, \|\cdot\|)$ is a Polish space, we have that (E, τ) is a standard space, and the Borel sets of (E, τ) are equal to Borel sets of $(E, \|\cdot\|)$ (see, for example, Proposition III.1.7 and Corollary III.2.6 in [6]). Hence, by Theorem IV.2.4 in [6], we have that S is an E-valued random variable.

The property that the unit ball is τ-closed is easily seen to be équivalent to

(5.8) $$\|x\| = \sup\{|\langle x', x \rangle| : \; x' \in F', \; \|x'\| \leqslant 1\} \quad \forall x \in E.$$

Now since E is separable, we can find a countable set $B' \subseteq \{x' \in F' | \; \|x'\| \leqslant 1\}$, with the property

(5.9) $$\|x\| = \sup\{|\langle x', x \rangle| : \; x' \in B'\} \quad \forall x \in E.$$

Now let $B' = \{x_1', x_2', \ldots\}$, and define

$$S_j^k = \sum_{v=1}^{j} \varepsilon_j \langle x_k', x_v \rangle = \langle x_k', S_j \rangle,$$

$$S^k = \sum_{v=1}^{\infty} \varepsilon_j \langle x_k', x_v \rangle = \langle x_k', S \rangle,$$

$$\tilde{S}_j^k = \sum_{v=1}^{j} \varepsilon_j \langle x_k', x_v \rangle - \sum_{v=j+1}^{\infty} \varepsilon_j \langle x_k', x_v \rangle = 2S_j^k - S^k.$$

(Notice that the sums converge a.s. since x_n' is τ-continuous.) Now let $t > 0$, and define

$$T = \inf\{j | \; \|S_j\| > t\} \quad (\inf(\emptyset) = \infty).$$

We then have

$$P(\sup_j \|S_j\| > t) = P(T < \infty) = \sum_{j=1}^{\infty} P(T = j).$$

If $T = j$ we then have, for some $k \geqslant 1$, that $|S_j^k| > t$, and since $S_j^k = \frac{1}{2}\tilde{S}_j^k + \frac{1}{2}S^k$ we have that either

$$\sup_k |S^k| > t$$

or

$$\sup_k |\tilde{S}_j^k| > t.$$

Hence we have

$$P(\sup_j \|S_j\| > t) \leqslant \sum_{j=1}^{\infty} \{P(T = j, \sup_k |\tilde{S}_j^k| > t) + P(T = j, \|S\| > t)\}.$$

Since ε_j is symmetric for all j, we have that the two families, $S_1, \ldots, S_j, S^1, S^2, \ldots$ and $S_1, \ldots, S_j, \tilde{S}_j^1, \tilde{S}_j^2, \ldots$ are equidistributed for all $j = 1, \ldots,$ hence

$$P(T = j, \sup_k |\tilde{S}_j^k| > t) = P(T = j, \|S\| > t)$$

and so

$$P(\sup_j \|S_j\| > t) \leqslant 2P(\|S\| > t) \quad \forall t > 0.$$

Hence $\sup_j \|S_j\| < \infty$ a.s., and so $(x_j) \in B(E)$, and $M = \sup_j \|S_j\|$ is integrable. Now, since the unit ball is τ-closed, we have that $\|S\| \leqslant M$, and so $S \in L^1(E)$, and the theorem then follows from Theorem 5.3.

COROLLARY 5.5. *Let τ be a locally convex Hausdorff topology on E such that τ is weaker than the norm topology, and the unit ball of E is τ-closed. Let (X_n) be a symmetric sequence of E-valued random variables and put*

$$F(\omega) = \Big\{\sum_{j=1}^{n} X_j(\omega) \mid n \geqslant 1\Big\}.$$

If E is separable and $F(\omega)$ is relatively τ-compact for almost all $\omega \in \Omega$, then $\sum_1^\infty X_j$ converges a.s.

COROLLARY 5.6. *Let S be a compact metric space and let $X_0(t, \omega)$, $X_1(t, \omega)$, $X_2(t, \omega), \ldots$ be stochastic processes with time set S. Suppose that*

(5.10) $X_n(t, \cdot)$ *is symmetric for all $t \in S$, and $n \geqslant 0$;*

(5.11) $X_n(\cdot, \omega)$ *is continuous and real for almost all $\omega \in \Omega$, and all $n \geqslant 0$;*

(5.12) *The processes $\{X_n(t) \mid t \in S\}$ for $n = 1, 2, \ldots$ are independent;*

(5.13) $X_0(t) = \sum_{n=1}^{\infty} X_n(t)$ *a.s. for all $t \in S$.*

Then the series $\sum_1^\infty X_n(t, \omega)$ converges uniformly in $t \in S$ for almost all $\omega \in \Omega$.

6. Weak convergence. If (X_n) is a sequence of E-valued random variables and τ is a topology on E that is weaker than the norm-topology, we say that X_n *converges τ-weakly* to μ, where μ is a probability measure on $(E, \mathscr{B}(E))$, if we have

$$\lim_{n \to \infty} \mathbf{E} f(X_n) = \int_E f(x) \mu(dx)$$

for all τ-continuous, bounded functions f from E into $\mathbf{R}$. And we say that (X_n) is τ-tight if for all $\varepsilon > 0$ there exists a τ-compact set $K \subseteq E$ with

$$P(X_n \notin K) < \varepsilon \quad \forall n \geqslant 1.$$

LEMMA 6.1. *Let Φ be a map from $F' \times \Omega$ into $\mathbf{R}$, where F' is a linear subspace of E'. Suppose that E is separable and μ is a probability measure on $(E, \mathscr{B}(E))$ such that*

(6.1) $\Phi(x', \cdot)$ *is measurable for all $x' \in F'$;*

(6.2) $P(\Phi(x') \in A) = \mu(x \in E \mid \langle x', x \rangle \in A) \quad \forall x' \in F, \forall A \in \mathscr{B}(\mathbf{R})$.

Then there exists an E-valued random variable X such that for $x' \in F'$ we have

(6.3) $\Phi(x', \omega) = \langle x', X(\omega) \rangle$ *for a.a. $\omega \in \Omega$.*

Proof. Since E is separable, there exists a countable subset $T' \subseteq F'$ such that T' is sequentially $\sigma(E', E)$-dense in F' (i.e., for all $x' \in F'$ exists a sequence $(t_n') \subseteq T'$ with $t_n' \xrightarrow[n\to\infty]{} x'$ in $\sigma(E', E)$).

Now let $T' = \{x_1', x_2', \ldots\}$ be an enumeration of T and define

$$\Psi(\omega) = \big(\Phi(x_j', \omega)\big)_{j=1}^{\infty}.$$

Then $\Psi(\cdot)$ is a measurable map from Ω into $\mathbf{R}^\infty$. Distribution of Ψ is given by

$$P_\Psi(A) = \mu\big(x \mid (\langle x_j', x \rangle)_{j=1}^{\infty} \in A\big) \quad \forall A \in \mathscr{B}(\mathbf{R}^\infty).$$

Now let $R_0 \subseteq \mathbf{R}^\infty$ be the set

$$R_0 = \{(t_j) \in \mathbf{R}^\infty \mid \exists x \in E \text{ with } t_j = \langle x_j', x \rangle \ \forall j \geqslant 1\}.$$

Since R_0 is a continuous image of E, we have that R_0 is an analytic space in the sense of [6] Chapter III.1, and obviously $P_\Psi(R_0) = 1$. (Notice that R_0 is universally measurable.) Now consider the set

$$A = \{(t, x) \in R_0 \times E \mid t_j = \langle x_j', x \rangle \ \forall j \geqslant 1\}.$$

A is then a closed subset of $R_0 \times E$, and so A is analytic. By definition of R_0 we know that the projection of A onto R_0 is equal to R_0. Thus by Theorem III.9.6 of [6] exists a universally measurable map $\varphi\colon R_0 \to E$ whose graph is contained in A, i.e.

$$\langle x_j', \varphi(t) \rangle = t_j \quad \forall t = (t_j) \in R_0.$$

Now let us define

$$X(\omega) = \begin{cases} \varphi(\Psi(\omega)) & \text{if } \omega \in \Psi^{-1}(R_0), \\ 0 & \text{if } \omega \notin \Psi^{-1}(R_0). \end{cases}$$

Then X is a P-measurable map from Ω into E, and as E is separable we have that X is an E-valued random variable.

If $\omega \in \Psi^{-1}(R_0)$ we then have

$$\langle x', X(\omega)\rangle = \langle x', \varphi(\Psi(\omega))\rangle = \Phi(x', \omega) \quad \forall\, x' \in T',$$

and since $P(\Psi^{-1}(R_0)) = 1$, we have that

$$\langle x', X(\omega)\rangle = \Phi(x', \omega) \text{ a.s.} \quad \forall x' \in T'.$$

If $x' \in F'$ and $t_n' \in T'$ such that $t_n' \to x'$ in $\sigma(E', E)$, then

$$\langle t_n', X(\omega)\rangle \xrightarrow[n\to\infty]{} \langle x', X(\omega)\rangle \quad \text{for all } \omega$$

and an easy argument shows that

$$\Phi(t_n', \cdot) \xrightarrow[n\to\infty]{} \Phi(x', \cdot) \text{ in probability}$$

and so we have

$$\Phi(x', \omega) = \langle x', X(\omega)\rangle \text{ a.s.} \quad \forall\, x' \in F'$$

which proves Lemma 6.1.

THEOREM 6.2. *Let $X_1, X_2, \ldots$ be independent, symmetric, E-valued random variables and let τ be a locally convex Hausdorff topology on E, which is weaker than the norm-topology, and such that the unit ball of E is τ-closed.*

Let $S_n = \sum_1^\infty X_j$. If E is separable then the following statements are equivalent:

(6.4) *(S_n) converges a.s.;*

(6.5) *(S_n) converges τ-weakly;*

(6.6) *(S_n) is τ-tight;*

(6.7) *(S_n) has a subsequence which converges τ-weakly.*

Proof. It is evident that (6.4) implies (6.5) and (6.6), and that (6.5) implies (6.7), and (6.6) implies (6.7). Hence the only non-trivial implication is that (6.7) implies (6.4).

So let (n_j) be a sequence of integers such that (S_{n_j}) converges τ-weakly. Let $F' = (E, \tau)'$; then $F' \subseteq E'$, and since

$$\langle x', S_n\rangle = \sum_{j=1}^n \langle x', X_j\rangle,$$

we find, from Theorem 17.B, p. 251, in [12], that

$$\lim_{n\to\infty} \langle x', S_n(\omega)\rangle = \Phi(x', \omega)$$

exists for a.a. $\omega \in \Omega$ and all $x' \in F'$. And if μ is the τ-weak limit of (S_{n_j}) then

$$P(\Phi(x') \in A) = \mu(x| \langle x', x\rangle \in A) \quad \forall\, x \in' F'.$$

Hence, by Lemma 6.1, there exists a random variable $S(\omega)$ such that for all $x' \in F'$ we have

$$\langle x', S_n(\omega)\rangle \xrightarrow[n\to\infty]{} \langle x', S(\omega)\rangle \text{ a.s.}$$

Now let T' be a countable subset of $A' = F' \cap \{x' \in E' |\ \|x'\| \leqslant 1\}$ which is sequentially $\sigma(E', E)$-dense in A'. Then there exists $\Omega_0 \in \mathscr{F}$, with $P(\Omega_0) = 1$, and such that

$$S_n(\omega) \xrightarrow[n\to\infty]{} S(\omega) \text{ in } \sigma(E, T') \quad \forall \omega \in \Omega_0.$$

Now it is easily seen that $\sigma(E, T') = \tau_0$ is a locally convex Hausdorff topology on E (see (5.8)), such that the unit ball of E is τ_0-closed, and τ_0 is weaker than the norm-topology. Hence, by Corollary 5.5, we have that (S_n) converges a.s. and so the theorem is proved.

Remark. Compare this with Theorem 4.1 in [8]. Ito and Nisio also shows in [8] that $\|\cdot\|$-weak convergence of $\sum_1^\infty X_n$ implies a.s. convergence whenever (X_n) is a sequence of independent E-valued random variables.

Added in proof. S. Kwapień has recently proved that my conjecture (5.5) is true (oral communication).

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Received October 2, 1973 (740)

STUDIA MATHEMATICA, T. LII. (1974)

On Banach spaces containing c_0

A supplement to the paper by J. Hoffmann-Jørgensen "Sums of independent Banach space valued random variables"

by

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Abstract. It is proved that a Banach space E does not contain subspaces isomorphic to c_0 if and only if the almost surely boundedness of sums of independent, symmetric E-valued random variables implies the almost surely convergence of the sums.

We shall prove the following result conjectured by Hoffmann-Jørgensen in the preceding paper [2].

THEOREM. *For every Banach space E the following conditions are equivalent:*

(i) *E does not contain subspaces isomorphic to the space c_0 of all scalar-valued sequences convergent to zero,*

(ii) *$L_1(E)$ does not contain subspaces isomorphic to c_0,*

(iii) *the almost surely boundedness of sums of independent, symmetric, E-valued random variables implies the almost surely convergence of the sums.*

It has been proved in [2] that to prove the theorem it is enough to establish the following

PROPOSITION. *Let E be a Banach space. Let (ε_i) be a Bernouilli sequence on a probability space $(\Omega, \mathscr{B}, P)$, i.e., a sequence of independent random variables such that $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}$ for all i. Let (x_i) be a sequence in E, such that $\inf_i \|x_i\| > 0$ and $P\big(\sup_n \big\|\sum_{i=1}^n \varepsilon_i x_i\big\| = \infty\big) = 0$. Then E contains a subspace isomorphic to c_0.*

Proof. Let $\mathscr{B}_0$ denote the σ-subfield of $\mathscr{B}$ generated by all ε_i's. Then it is easy to see that for $B \in \mathscr{B}_0$

$$\lim_i P\big(B \cap (\varepsilon_i = 1)\big) = \lim_i P\big(B \cap (\varepsilon_i = -1)\big) = 2^{-1} P(B). \tag{1}$$

Pick $M < \infty$ so that $P(A) > 2^{-1}$ where

$$A = \Big\{\omega \in \Omega\colon \sup_n \Big\|\sum_{i=1}^n \varepsilon_i(\omega) x_i\Big\| < M\Big\}.$$