An initial-boundary value problem for quasi-linear parabolic systems of higher order

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Abstract. In the paper first the existence of a classical solution of an initial-boundary value problem for the non-linear parabolic system $D_t U - \sum_{|k|=2b} A_k(X, t) D_X^k U = F(X, t, \dots, D_X^\gamma U, \dots)$, where $0 \leq |\gamma| = \gamma_1 + \dots + \gamma_m \leq 2b - 1$, $b > 1$, is proved under the assumption of Hölder continuity of F in a compact set but no other assumptions on the growth of F . Then, some sufficient conditions for the smoothness and uniqueness of solution and for the convergence of successive approximations are given.

These results are obtained by solving an integro-differential system of Hammerstein's type in Banach spaces with the help of fixed point theorems.

1. Introduction. The existence of a solution of initial-boundary value problems for quasi-linear parabolic equations has been studied by many authors. The methods of solutions employed and assumptions imposed on given magnitudes depend on the class of functions in which the expected solution is searched. Some functional methods enable one to prove the existence of a generalized solution provided that the conditions of continuity, measurability and restrictions on the growth of the non-linear part of the equation are satisfied (see e. g. [1], [2], [5], [6], [7], [11]). The existence of a classical solution requires the Hölder continuity and conditions concerning the growth of the non-linear part of the equation in question (see [4], [8], [9]).

The aim of the present paper is to assure the existence of the classical solution of an initial-boundary value problem with the homogeneous conditions for the non-linear system

$$\begin{aligned} \mathcal{L}(X, t; D_X, D_t) U &= \partial U / \partial t - \sum_{k_1 + \dots + k_m = 2b} A_{k_1 \dots k_m}(X, t) \partial^{2b} U / \partial x_1^{k_1} \dots \partial x_m^{k_m} \\ &= F(X, t, \dots, \partial^\gamma U / \partial x_1^{\gamma_1} \dots \partial x_m^{\gamma_m}, \dots), \end{aligned}$$

where $0 \leq |\gamma| = \gamma_1 + \dots + \gamma_m \leq 2b - 1$ and $b \geq 1$ is an integer. The problem is investigated in the cylinder whose axis is parallel to the time axis t and whose base is a bounded domain of the m -dimensional Euclidean space. The coefficients $A_{k_1 \dots k_m}$ are sufficiently smooth matrix functions of X and t and F is Hölder continuous in a compact set. We do not require any other conditions on the growth of F . Applying some results from the

theory of Green's functions of linear parabolic operators (see [3]) the above problem is transformed into an integro-differential system of Hammerstein's type and thus can be studied by means of fixed point theorems. Some sufficient conditions for the smoothness and uniqueness of solution and the convergence of successive approximations are also given.

2. The formulation of problem. We shall start with some notions and notations which will be used throughout this paper.

(a) Ω denotes a bounded domain of the m -dimensional Euclidean space R_m ($m \geq 2$) with the boundary $\partial\Omega$ and with the diameter $\text{diam } \Omega$. For $0 < T < \infty$ we define the cylinder domain $Q = \Omega \times \langle 0, T \rangle \subset R_{m+1}$ and the surface $\Gamma = \partial\Omega \times \langle 0, T \rangle$. The norm of $X = (x_1, \dots, x_m) \in R_m$ is given by $|X| = (\sum_{i=1}^m x_i^2)^{1/2}$. Let $k = (k_1, \dots, k_m)$ be a multiindex whose components k_i are non-negative integers for $i = 1, \dots, m$. The modulus of k is defined by $|k| = \sum_{i=1}^m k_i$. Denote $X^k = x_1^{k_1} \dots x_m^{k_m}$ and $X^2 = |X|^2$. Further, write for an arbitrary integer r , $t(r) = \text{Card}\{k = (k_1, \dots, k_m) : k_i \geq 0 \text{ is an integer for } i = 1, \dots, m \text{ and } |k| = r\}$ and $s = \sum_{r=0}^{2b-1} t(r)$ and let us define the set H as the Cartesian product $Q \times \prod_{i=1}^s \prod_{j=1}^p \{-\infty < u_j^i < \infty\}$ and the set $H(B)$ as the product $Q \times \prod_{i=1}^s \prod_{j=1}^p \{-B \leq u_j^i \leq B\}$, where $p \geq 1$ and $0 < B < \infty$. The symbol D_X^k denotes the differential operator $\partial^{|k|} / \partial x_1^{k_1} \dots \partial x_m^{k_m}$ and for a non-negative integer k_0 , $D_t^{k_0}$ is the differential operator $\partial^{k_0} / \partial t^{k_0}$.

(b) Let $(\mathcal{U}_m^p, \leq)$, resp. $(\mathcal{M}_m^p, \leq)$ ($p \geq 1$ is an integer) be partially ordered sets of all real $(p \times 1)$ -vector functions $U(X) = (u_1(X), \dots, u_p(X))$, resp. of all real $p \times p$ matrix functions $A(X) = (a_{ij}(X))_{i,j=1}^p$ with the range of definition in R_m . The binary relation " $\leq$ " is defined as follows: if $U(X)$ and $V(X) = (v_1(X), \dots, v_p(X))$ belong to $\mathcal{U}_m^p$, then $U \leq V$ iff $u_j(X) \leq v_j(X)$ for every X of their common domain of definition and all $j = 1, \dots, p$. If $A(X)$ and $B(X) = (b_{ij}(X))_{i,j=1}^p$ are matrix functions of $\mathcal{M}_m^p$, then $A \leq B$ iff $a_{ij}(X) \leq b_{ij}(X)$ for every X of their common domain of definition and for all $i, j = 1, \dots, p$.

The $(p \times 1)$ -unit vector and the zero vector are denoted by J and O , respectively. The $p \times p$ matrix whose all elements are equal to 1 and the unit matrix will be denoted by E_1 and E , respectively. (U, V) is the scalar product of $U(X)$ and $V(X)$ at any point X . Finally, we write $|U(X)|^\alpha = (|u_1(X)|^\alpha, \dots, |u_p(X)|^\alpha)$ and $|A(X)|^\alpha = (|a_{ij}(X)|^\alpha)_{i,j=1}^p$ for a real α .

(c) Now we shall define some commonly used classes of vector and matrix functions. A vector function $U(X, \lambda) \in \mathcal{U}_{m_1+m_2}^p$ ($m_1, m_2 \geq 1$) is said

to be *Hölder continuous* with respect to X on $D_1 \subset R_{m_1}$ with the exponent ϱ ($0 < \varrho \leq 1$) and uniformly with respect to $\lambda \in D_2 \subset R_{m_2}$ if it is defined on $D_1 \times D_2$ and there exists a constant $L = L(U) > 0$ such that for any pair of points $X, Y \in D_1$ and every $\lambda \in D_2$ the relation

$$|U(X, \lambda) - U(Y, \lambda)| \leq L|X - Y|^\varrho$$

is true. The set of all such vector functions will be denoted by $H_{\varrho, p}(X, D_1; \lambda, D_2)$. From the above definition it is obvious that if D_1 is a bounded set and $0 < \varrho \leq \beta \leq 1$, then $H_{\beta, p}(X, D_1; \lambda, D_2) \subset H_{\varrho, p}(X, D_1; \lambda, D_2)$.

If $l \geq 0$ is an integer and $D \subset R_m$, then $C_p^l(D)$ is the Banach space of all vector functions $U(X) \in \mathcal{U}_m^p$ defined on D which together with their derivatives $D_X^k U(X)$ up to the order l (including l) are continuous and bounded on D . In this space the norm $\|\cdot\|_l^{D, p}$ is defined by $\max_{j=1, \dots, p} \left\{ \sum_{i=0}^l \sum_{|k|=i} \sup_D |D_X^k u_j(X)| \right\}$. If $0 < \alpha < 1$, then $C_p^{l+\alpha}(D)$ denotes the subspace of all vector functions $U(X) \in C_p^l(D)$ whose derivatives of l -th order satisfy the Hölder condition on D with the exponent α . In this case the norm is defined by the sum

$$\|U\|_l^{D, p} + \max_{j=1, \dots, p} \left\{ \sum_{|k|=l} \sup_{X, Y \in D} |D_X^k u_j(X) - D_X^k u_j(Y)| |X - Y|^\alpha \right\}.$$

We say that the boundary ∂D of $D \subset R_m$ belongs to the class $C^{l+\alpha}$ if to each $\Xi \in \partial D$ there exists $\delta > 0$ such that the part of the boundary ∂D contained in the ball $G = \{X \in R_m: |X - \Xi| \leq \delta\}$ can be explicitly expressed by the equation $x_m = g(x_1, \dots, x_{m-1})$ in the orthonormal coordinate system $X = (Ox_1 \dots x_{m-1} x_m)$. The direction of the axis Ox_m coincides with that of the inner normal to ∂D at the point Ξ and the scalar function g is an element of the class $C_1^{l+\alpha}(G_0)$, where $G_0 = \{X' = (x_1, \dots, x_{m-1}) \in R_{m-1}: |X'| \leq \delta\}$.

Let $l \geq 0$ and $b \geq 1$ be integers and $0 < \alpha < 1$. The set of all vector functions $U(X, t) = (u_1(X, t), \dots, u_p(X, t)) \in \mathcal{U}_{m+1}^p$ fulfilling the inequality

$$(1) \quad \|U\|_{l+\alpha}^{Q, p} = \max_{j=1, \dots, p} \left\{ \sum_{h=l} \langle\langle D_t^{k_0} D_X^k u_j(X, t) \rangle\rangle_{\alpha, X}^Q + \right. \\ \left. + \sum_{0 < 1+\alpha-h < 2b} \langle\langle D_t^{k_0} D_X^k u_j(X, t) \rangle\rangle_{(1+\alpha-h)/2b}^Q + \sum_{i=0}^l \sum_{h=i} \sup_Q |D_t^{k_0} D_X^k u_j(X, t)| \right\} < \infty,$$

where $h = 2bk_0 + |k|$, will be denoted by $C_{X, t, p}^{l+\alpha, (l+\alpha)/2b}(Q)$. For $0 < \mu < 1$ and $u(X) \in \mathcal{U}_m^1$ the symbol $\langle\langle u \rangle\rangle_{\mu, X}^Q$ means $\sup_{(X, t), (Y, t) \in Q} |u(X, t) - u(Y, t)| |X - Y|^\mu$ and $\langle\langle u \rangle\rangle_{\mu, t}^Q$ means $\sup_{(X, t), (X, t') \in Q} |u(X, t) - u(X, t')| |t - t'|^\mu$.

In the following section we prove some properties explaining the structure of the sets introduced above. By $\tilde{M}$ we shall denote the set

of all matrix functions belonging to $\mathcal{M}_m^p$ whose rows and columns are vector functions of the class M . In the conclusion of that section we shall establish the initial-boundary value problem.

Consider the system of $p \geq 1$ differential equations of $2b$ -th order ($b \geq 1$) with p unknown functions

$$(2) \quad \mathcal{L}(X, t; D_X, D_t) U \equiv D_t U - \sum_{|k|=2b} A_k(X, t) D_X^k U \\ = F(X, t, \dots, D_X^\gamma U, \dots), \quad (X, t) \in Q,$$

where $\gamma = (\gamma_1, \dots, \gamma_m)$ is a multiindex whose components γ_i are positive integers for $i = 1, \dots, m$ and $0 \leq |\gamma| \leq 2b - 1$. The solution of (2) is required to fulfil the initial condition

$$(3) \quad U|_{t=0} = 0, \quad X \in \Omega$$

and the boundary conditions

$$(4) \quad \mathcal{B}_q(X, t, D_X) U|_\Gamma \\ = \sum_{|k|=r_q} (B_k^{(q)}(X, t), D_X^k U) + \sum_{|k| < r_q} (B_k^{(q)}(X, t), D_X^k U)|_\Gamma = 0$$

for $r_q \leq 2b - 1$ and $q = 1, \dots, bp$. $A_k(X, t) = (a_k^{hj}(X, t))_{h,j=1}^p$ is a matrix function of $\mathcal{M}_{m+1}^p$ and $B_k^{(q)}(X, t) = (b_k^{q1}(X, t), \dots, b_k^{qp}(X, t))$ or

$$F(X, t, \dots, D_X^\gamma U, \dots) = (f_1(X, t, \dots, D_X^\gamma U, \dots), \dots, f_p(X, t, \dots, D_X^\gamma U, \dots))$$

is a vector function of $\mathcal{U}_{m+1}^p$ or $\mathcal{U}_{m+1+ps}^p$, respectively. In the following considerations we shall assume that conditions concerning the parabolicity of (2) and smoothness of A_k , $B_k^{(q)}$ and F are satisfied.

3. Some assumptions and theorems. The study of non-linear problem (2), (3), (4) is based on some results of the theory of linear parabolic equations. We shall need the following assumptions (see [10]):

(A) *The system (2) is supposed to be uniformly parabolic in the sense of I. G. Petrovskij, i. e. there is a constant $\delta < 0$ (independent of X, t) such that the roots $a_s(X, t; \Xi)$ of the polynomial*

$$L(X, t; i\Xi, a) = \det \mathcal{L}(X, t; i\Xi, a) = \det \left(\delta_{hj} a - i^{2b} \sum_{|k|=2b} a_k^{hj}(X, t) \Xi^k \right)_{h,j=1}^p = 0$$

($i = \sqrt{-1}$ and δ_{hj} is the Kronecker symbol) satisfy the inequality

$$\operatorname{Re} a_s(X, t; \Xi) \leq -\delta \Xi^{2b} \quad \text{for all } (X, t) \in Q \text{ and } \Xi \in R_m.$$

(B) *Let X be a point of $\partial\Omega$ and let $\nu(X) = (\nu_1(X), \dots, \nu_m(X))$ be the unit vector of the inner normal to $\partial\Omega$ at X and let $\zeta(X)$ be a vector lying in the tangential plane to $\partial\Omega$ at X . Then for any $(X, t) \in \Gamma$ and for a complex number a with the properties $\operatorname{Re} a \geq -\delta_1 \zeta^{2b}$*

and $|a|^2 + \zeta^{4b} > 0$ ($0 < \delta_1 < \delta$, δ_1 is an absolute constant) and for any arbitrary vector $\zeta(X)$ the rows of the matrix

$$\left(i^{r_q} \sum_{|k|=r} b_k^{qj}(X, t) [\zeta(X) + \tau v(X)]^k \right)_{q=1, j=1}^{bp, p} \hat{\mathcal{L}}(X, t; i(\zeta + \tau v), a),$$

where $\hat{\mathcal{L}} = L\mathcal{L}^{-1}$, are linearly independent vector functions of τ with respect to the module of the polynomial $M^+(X, t; \zeta, \tau, a) = \prod_{q=1}^{bp} (\tau - \tau_q^+(X, t; \zeta, a))$; τ_q^+ for $q = 1, \dots, bp$ are the roots of the polynomial $L(X, t; i(\zeta + \tau v), a)$ (in τ) with the positive imaginary part. (This condition is called the uniform supplementary condition.)

(C_{l+a}) Let $l \geq 0$ be an integer and $0 < a < 1$. The coefficients $A_k(X, t)$ belong to $\tilde{C}_{X, t, p}^{l+a, (l+a)/2b}(Q)$ for $|k| = 2b$ and $B_k^{(q)}(X, t)$ belong to the space $C_{X, t, p}^{l+a+2b-r_q, (l+a+2b-r_q)/2b}(\Gamma)$ for $|k| \leq r_q \leq 2b-1$ and $q = 1, \dots, bp$. The boundary $\partial\Omega$ belongs to the class C^{2b+l+a} .

THEOREM 1. (V. A. Solonnikov [10].) Let the requirements (A), (B), (C_{l+a}) be fulfilled and let $\Phi(X, t) = (\varphi_1(X, t), \dots, \varphi_p(X, t))$ be an element of $C_{X, t, p}^{l+a, (l+a)/2b}(Q)$. Moreover, suppose that Φ satisfies the "coordinative conditions" of order l on the boundary $\partial\Omega$ (see [10], p. 86). Then there is one and only one solution $U(X, t) \in C_{X, t, p}^{2b+l+a, (2b+l+a)/2b}(Q)$ of the linear differential equation

$$(5) \quad \mathcal{L}(X, t; D_X, D_t) U = \Phi(X, t)$$

satisfying conditions (3) and (4). The inequality

$$\|U\|_{l+a+2b}^{Q, p} \leq \sum_{j=1}^p \|u_j\|_{l+a+2b}^{Q, 1} \leq C \sum_{j=1}^p \|\varphi_j\|_{l+a}^{Q, 1}$$

is also true.

Estimates of Green's matrices of $\mathcal{L}$ (for the definition see [3]) and their derivatives are stated by

THEOREM 2. (S. D. Ejdel'man and S. D. Ivasišen [3].) Let assumptions (A), (B), (C_{l+a}) ($l \geq 0$) be held. Then there exists Green's matrix $G(X, t; \Xi, \tau) \in \mathcal{M}_{2(m+1)}^p$ of the problem (5), (3), (4). Its derivatives $D_t^{k_0} D_X^k G$ satisfy, for $2bk_0 + |k| \leq 2b+1$ and for $0 \leq \tau < t_0 < t \leq T$, $X, Y, \Xi \in \Omega = \Omega \cup \partial\Omega$ ($r = 1/(2b-1)$), the inequalities:

$$(6) \quad |D_t^{k_0} D_X^k G(X, t; \Xi, \tau)| \leq C(t-\tau)^{-(m+2bk_0+|k|)/2b} \exp\{-c|X-\Xi|^{2br}/(t-\tau)^r\} E_1;$$

$$(7) \quad |D_t^{k_0} D_X^k G(X, t; \Xi, \tau) - D_t^{k_0} D_X^k G(Y, t; \Xi, \tau)| \leq C|X-Y|^a(t-\tau)^{-(m+2bk_0+|k|+a)/2b} \exp\{-c|X^*-\Xi|^{2br}/(t-\tau)^r\} E_1$$

if $2bk_0 + |k| = 2b + l$ and $|X^* - \Xi| = \min(|X - \Xi|, |Y - \Xi|)$;

$$(8) \quad |D_t^{k_0} D_X^k G(X, t; \Xi, \tau) - D_t^{k_0} D_X^k G(X, t_0; \Xi, \tau)| \\ \leq C(t - t_0)^{[2b(1-k_0) + l - |k| + a]/2b} (t_0 - \tau)^{-(m+2b+l+a)/2b} \exp\{-c|X - \Xi|^{2b}/(t - \tau)^r\} E_1$$

if $l < 2bk_0 + |k| \leq 2b + l$. The positive constants C, c are independent of X, Y, t, t_0 and Ξ, τ .

THEOREM 3. (See [3].) *Let assumptions (A), (B), (C_{l+a}) be satisfied and let $\Phi(X, t) \in C_p^0(Q) \cap H_{a,p}(X, \Omega; t, \langle 0, T \rangle)$ be a bounded function in the norm $\|\cdot\|_0^{Q,p}$. Then*

$$(9) \quad U(X, t) = \int_0^t d\tau \int_{\Omega} G(X, t; \Xi, \tau) \Phi(\Xi, \tau) d\Xi$$

is a solution of the linear problem (5), (3), (4).

4. The existence of a solution of the non-linear problem. First of all we shall introduce some useful inequalities. The estimate (6) may be modified to the form

$$(10) \quad |D_t^{k_0} D_X^k G(X, t; \Xi, \tau)| \leq C(t - \tau)^{-\mu} |X - \Xi|^{2b\mu - (m+2bk_0+|k|)} \times \\ \times [|X - \Xi|^{2b}/(t - \tau)]^{(m+2bk_0+|k|-2b\mu)/2b} \exp\{-c[|X - \Xi|^{2b}/(t - \tau)]^r\} E_1 \\ \leq K(t - \tau)^{-\mu} |X - \Xi|^{2b\mu - (m+2bk_0+|k|)} E_1$$

for $0 \leq \tau < t \leq T$ and $X, \Xi \in \bar{\Omega}, \Xi \neq X$ and $\mu \leq (m + 2bk_0 + |k|)/2b$, where K is a positive constant independent of X, Ξ, t and τ .

Note, since $2|x_1 x_2|/(x_1^2 + x_2^2) \leq 1$ for all $x_1, x_2 \in R_1$, thus the inequality $2|x_1 x_2|/(x_1^2 + x_2^2) \leq (1 - K_2^2)/K_2^2$ holds for every $x_1, x_2 \in R_1$ if $(1 - K_2^2)/K_2^2 \geq 1$, i. e. $0 < K_2 \leq 1/\sqrt{2}$. Hence the relation $K_2(|x_1| + |x_2|) \leq \sqrt{x_1^2 + x_2^2}$ is true for all $x_1, x_2 \in R_1$ if K_2 belongs to the interval $(0, 1/\sqrt{2})$. By induction on m one obtains for any $X = (x_1, \dots, x_m) \in R_m$

$$(11) \quad K_m \sum_{i=1}^m |x_i| \leq |X| \leq \sum_{i=1}^m |x_i|,$$

where the constant $K_m \in (0, 1/(\sqrt{2})^{m-1})$ does not depend on X .

From (1) we get for $l = 2b - 1$

$$(12) \quad \|U\|_{2b-1+a}^{Q,p} = \max_{j=1, \dots, p} \left\{ \sum_{k=2b-1} \langle D_X^k u_j(X, t) \rangle_{a,X}^Q + \right. \\ \left. + \sum_{i=0}^{2b-1} \sum_{|k|=i} [\langle D_X^k u_j(X, t) \rangle_{(2b-1+a-|k|)/2b,t}^Q + \sup_Q |D_X^k u_j(X, t)|] \right\}.$$

If $U(X, t) \in C_{X, t, p}^{2b-1+a, (2b-1+a)/2b}(Q)$, then we directly see that $D_X^k U(X, t) \in H_{a, p}(X, \Omega; t, \langle 0, T \rangle)$ for $|k| = 2b-1$ and $D_X^k U(X, t) \in H_{(2b-1+a-|k|)/2b, p}(t, \langle 0, T \rangle; X, \Omega)$ for $|k| = 0, 1, \dots, 2b-1$. In view of the mean value theorem and of (11), also the derivatives $D_X^k U(X, t)$ for $|k| = 0, 1, \dots, 2b-2$ belong to $H_{1, p}(X, \Omega; t, \langle Q, T \rangle)$.

LEMMA 1. Let $|k| = 0, 1, \dots, 2b-1$. If the requirements (A), (B), (C_a) hold, then

$$(13) \quad I_{1, k}(X, Y, t) = \int_0^t d\tau \int_{\Omega} |D_X^k G(X, t; \Xi, \tau) - D_X^k G(Y, t; \Xi, \tau)| d\Xi \\ \leq L_{1, k} |X - Y|^{1+(a-1)[|k|/(2b-1)]} E_1$$

(the expression $[x]$ in the exponent denotes the integer for which $[x] \leq x < [x] + 1$) and

$$(14) \quad I_{2, k}(X, t) = \int_0^t d\tau \int_{\Omega} |D_X^k G(X, t; \Xi, \tau)| d\Xi \leq L_{2, k} E_1,$$

where $L_{1, k}$ and $L_{2, k}$ are positive constants independent of $X, Y \in \Omega$ and $t \in \langle 0, T \rangle$. If conditions (A), (B), (C_{2b-1+a}) are satisfied, then

$$(15) \quad I_{3, k}(X, t, t') = \int_0^t d\tau \int_{\Omega} |D_X^k G(X, t; \Xi, \tau) - D_X^k G(X, t'; \Xi, \tau)| d\Xi + \\ + \int_t^{t'} d\tau \int_{\Omega} |D_X^k G(X, t'; \Xi, \tau)| d\Xi \leq L_{3, k} (t' - t)^{(2b-1+a-|k|)/2b} E_1$$

for $(X, t), (X, t') \in Q$ and $t < t'$, where the constant $L_{3, k} > 0$ does not depend on X, t and t' .

Proof. Let $X, Y = (y_1, \dots, y_m)$ and Ξ be three different points of Ω and $0 \leq \tau < t \leq T$. Then, from the mean value theorem and by (11) and (10),

$$|D_X^k G(X, t; \Xi, \tau) - D_X^k G(Y, t; \Xi, \tau)| \leq \sum_{i=1}^m |x_i - y_i| |D_X^{k(i)} G(X_i^*, t; \Xi, \tau)| \\ \leq (K/K_m) (t - \tau)^{-\mu} |X_i^* - \Xi|^{2b\mu - (m + |k(i)|)} |X - Y| E_1,$$

where $k(i) = (k_1, \dots, k_{i-1}, k_i + 1, k_{i+1}, \dots, k_m)$ and $X_i^* = (y_1, \dots, y_{i-1}, \zeta_i, x_{i+1}, \dots, x_m)$. ζ_i lies between the numbers x_i and y_i and $|X - Y| > |X_i^* - X|$. For $|k| = 0, 1, \dots, 2b-2$ and $|k(i)|/2b \leq \mu < 1$ the last expression in the preceding inequality is an integrable majorant of $|D_X^k G(X, t; \Xi, \tau) - D_X^k G(Y, t; \Xi, \tau)|$ locally independent of the parameters X, t . Thus (13) is true for $|k| = 0, 1, \dots, 2b-2$.

Let now k be a multiindex with the module $|k| = 2b - 1$ and let X, Y, Ξ satisfy the inequality $|\Xi - X| > 2|X - Y|$. Then it is obvious that $|X - Y| < |X_i^* - \Xi|$ and $|X - \Xi| \leq |X - X_i^*| + |X_i^* - \Xi| < 2|X_i^* - \Xi|$. Hence

$$|D_X^k G(X, t; \Xi, \tau) - D_X^k G(Y, t; \Xi, \tau)| \leq (K/K_m)(t - \tau)^{-\mu}(2^{-1}|X - \Xi|)^{2b\mu - (m+2b)}|X - Y|E_1.$$

If we put $S_1 = \{\Xi \in \Omega : |\Xi - X| > 2|X - Y|\}$ and $S_2 = \Omega - S_1$, then we get

$$\begin{aligned} I_{1,k}(X, Y, t) &\leq \left\{ (K/K_m)2^{m+2b-2b\mu} \int_0^t d\tau \int_{S_1} (t - \tau)^{-\mu} |X - Y| |X - \Xi|^{2b\mu - (m+2b)} d\Xi + \right. \\ &\quad \left. + K \int_0^t d\tau \int_{S_2} (t - \tau)^{-\mu} [|X - \Xi|^{2b\mu - (m+2b-1)} + |Y - \Xi|^{2b\mu - (m+2b-1)}] d\Xi \right\} E_1. \end{aligned}$$

For $(2b - 1 + a)/2b \leq \mu < 1$ both integrals converge and the estimate

$$\begin{aligned} I_{1,k}(X, Y, t) &\leq \{L_4|X - Y|^{1+(2b\mu-2b)} + L_5|X - Y|^{2b\mu-(2b-1)} + \\ &\quad + L_6|X - Y|^{2b\mu-(2b-1)}\} E_1 \leq L_7(\text{diam } \Omega)^{2b\mu-(2b-1)-a}|X - Y|^a E_1 \end{aligned}$$

holds, where L_4, L_5, L_6, L_7 are positive constants independent of X, Y and t .

Formula (13) is thus proved. The relations in (14) follow directly from (10).

To prove the third formula of this lemma we shall again employ the mean value theorem. Then there exists $t^* \in (t, t')$ such that

$$\begin{aligned} |D_X^k G(X, t; \Xi, \tau) - D_X^k G(X, t'; \Xi, \tau)| &= |D_t D_X^k G(X, t^*; \Xi, \tau)|(t' - t) \\ &\leq K(t' - t)(t - \tau)^{-\mu}|X - \Xi|^{2b\mu - (m+2b+|k|)} E_1 \quad (t - \tau < t^* - \tau). \end{aligned}$$

If $S_3 = \{\Xi \in \Omega : |\Xi - X| > (t' - t)^{1/2b}\}$ and $S_4 = \Omega - S_3$, then from the preceding inequality and by (10) one obtains

$$\begin{aligned} I_{3,k} &\leq \int_0^t d\tau \int_{S_3} |D_X^k G(X, t; \Xi, \tau) - D_X^k G(X, t'; \Xi, \tau)| d\Xi + \\ &\quad + \int_t^{t'} d\tau \int_{S_3} |D_X^k G(X, t'; \Xi, \tau)| d\Xi + \\ &\quad + \int_0^t d\tau \int_{S_4} |D_X^k G(X, t; \Xi, \tau)| d\Xi + \int_0^{t'} d\tau \int_{S_4} |D_X^k G(X, t'; \Xi, \tau)| d\Xi \\ &\leq K \left\{ \int_0^t d\tau \int_{S_3} (t' - t)(t - \tau)^{-\mu} |X - \Xi|^{2b\mu - (m+2b+|k|)} d\Xi + \right. \end{aligned}$$

$$\begin{aligned}
& + \int_t^{t'} d\tau \int_{S_3} (t' - \tau)^{-\lambda} |X - \Xi|^{2b\lambda - (m + |k|)} d\Xi + \\
& + \int_0^t d\tau \int_{S_4} (t - \tau)^{-\mu} |X - \Xi|^{2b\mu - (m + |k|)} d\Xi + \\
& + \int_0^{t'} d\tau \int_{S_4} (t' - \tau)^{-\mu} |X - \Xi|^{2b\mu - (m + |k|)} d\Xi \} E_1.
\end{aligned}$$

From this fact for $(2b - 1 + a)/2b \leq \mu < 1$ and $\lambda < |k|/2b$ the estimate

$$\begin{aligned}
I_{3,k}(X, t, t') \leq & \{L_7 T^{1-\mu} (t' - t) (t' - t)^{(2b\mu - 2b - |k|)/2b} + \\
& + L_8 (t' - t)^{(2b\lambda - |k|)/(2b) + 1 - \lambda} + L_9 T^{1-\mu} (t' - t)^{(2b\mu - |k|)/2b} + \\
& + L_{10} T^{1-\mu} (t' - t)^{(2b\mu - |k|)/2b} \} E_1
\end{aligned}$$

is evident. The constants $L_i, i = 7, 8, 9, 10$ do not depend on X, t, t' . Since $\mu - |k|/2b \geq (2b - 1 - |k| + a)/2b$ and $1 - |k|/2b \geq (2b - 1 - |k| + a)/2b$, we have

$$I_{3,k}(X, t, t') \leq T^{1-a/2b} (L_7 + L_8 + L_9 + L_{10}) (t' - t)^{(2b-1+a-|k|)/2b} E_1,$$

which proves formula (15).

LEMMA 2. The set $C_{X,t,p}^{2b-1+a, (2b-1+a)/2b}(Q)$ with the norm $\|\cdot\|_{2b-1+a}^{Q,p}$ defined in (12) constitutes a Banach space.

Proof. It is sufficient to show that the linear space $C_{X,t,1}^{2b-1+a, (2b-1+a)/2b}(Q)$ of real functions $u(X, t)$ with the norm $\|\cdot\|_{2b-1+a}^{Q,1}$ is a Banach space. Indeed, if $U_n(X, t) = (u_1^n(X, t), \dots, u_p^n(X, t))$ for $n = 1, 2, \dots$ is a Cauchy sequence of elements belonging to $C_{X,t,p}^{2b-1+a, (2b-1+a)/2b}(Q)$ in the norm $\|\cdot\|_{2b-1+a}^{Q,p}$, then $\{u_j^n(X, t)\}_{n=1}^\infty$ is a Cauchy sequence in the sense of the norm $\|\cdot\|_{2b-1+a}^{Q,1}$, for $j = 1, \dots, p$. Consequently, we can find p real functions $u_j^0(X, t) \in C_{X,t,1}^{2b-1+a, (2b-1+a)/2b}(Q)$ such that $\lim_{n \rightarrow \infty} \|u_j^n - u_j^0\|_{2b-1+a}^{Q,1} = 0$ ($j = 1, \dots, p$). Hence $\lim_{n \rightarrow \infty} \|U_n - U_0\|_{2b-1+a}^{Q,p} = 0$, where $U_0(X, t) = (u_1^0(X, t), \dots, u_p^0(X, t))$.

Now let $\{u_n(X, t)\}_{n=1}^\infty$ be a Cauchy sequence of real functions belonging to the space $C_{X,t,1}^{2b-1+a, (2b-1+a)/2b}(Q)$, i. e. to each $\varepsilon > 0$ there is $N(\varepsilon)$ such that for all $l, n > N(\varepsilon)$, $\|u_l - u_n\|_{2b-1+a}^{Q,1} < \varepsilon$. Then the inequality $\|u_n\|_{2b-1+a}^{Q,1} \leq L$ holds for all $n = 1, 2, \dots$ and positive constant L . Hence and by (12) for $p = 1$ the uniform boundedness of derivatives $\{D_X^k u_n(X, t)\}_{n=1}^\infty$ on Q follows ($|k| = 0, 1, \dots, 2b-1$). Furthermore, the equicontinuity of $\{D_X^k u_n(X, t)\}_{n=1}^\infty$ follows from the well-known relations: $D_X^k u_n(X, t) \in H_{1,1}(X, \Omega; t, \langle 0, T \rangle)$ for $|k| = 0, 1, \dots, 2b-2$, $D_X^k u_n(X, t) \in H_{a,1}(X, \Omega; t, \langle 0, T \rangle)$ for $|k| = 2b-1$ and $D_X^k u_n(X, t) \in H_{(2b-1+a-|k|)/2b,1}(t, \langle 0, T \rangle; X, \Omega)$ for $|k| = 0, 1, \dots, 2b-1$. If the Ascoli theorem is repeatedly t -times applied to $\{D_X^k u_n(X, t)\}_{n=1}^\infty$, then we find a subsequence $\{u_{n_s}(X, t)\}_{s=1}^\infty$ of $\{u_n(X, t)\}_{n=1}^\infty$, which together with

derivatives $\{D_X^k u_{n_s}(X, t)\}_{s=1}^\infty$ for $|k| = 0, 1, \dots, 2b-1$ uniformly converges on Q as $s \rightarrow \infty$. Denote the uniform limit $\lim_{s \rightarrow \infty} u_{n_s}(X, t)$ by $u_0(X, t)$. Then $\lim_{s \rightarrow \infty} D_X^k u_{n_s}(X, t) = D_X^k u_0(X, t)$ for $|k| = 0, 1, \dots, 2b-1$ uniformly on Q . Letting $s \rightarrow \infty$ in the relations

$$\begin{aligned} |D_X^k u_{n_s}(X, t) - D_X^k u_{n_s}(Y, t)| &\leq L |X - Y|^a, \quad |k| = 2b-1; \\ |D_X^k u_{n_s}(X, t) - D_X^k u_{n_s}(X, t')| &\leq L |t - t'|^{(2b-1+a-|k|)/2b}, \\ |D_X^k u_{n_s}(X, t)| &\leq L \quad \text{for } |k| = 0, 1, \dots, 2b-1 \end{aligned}$$

($X, Y \in \Omega$; $t, t' \in \langle 0, T \rangle$) we have $u_0(X, t) \in C_{X, t, 1}^{2b-1+a, (2b-1+a)/2b}(Q)$. It remains to show that the sequence $\{u_n(X, t)\}_{n=1}^\infty$ converges to $u_0(X, t)$ in the norm $\|\cdot\|_{2b-1+a}^{Q, 1}$. Since $\{u_{n_s}(X, t)\}_{s=1}^\infty$ is a Cauchy sequence in that norm, thus to each $\varepsilon > 0$ there is $N(\varepsilon) > 0$ such that for all $l, s > N(\varepsilon)$ and $(X, t), (Y, t), (X, t') \in Q$:

$$\begin{aligned} |D_X^k u_{n_s}(X, t) - D_X^k u_{n_l}(X, t) - D_X^k u_{n_s}(Y, t) + D_X^k u_{n_l}(Y, t)| \\ \leq \varepsilon |X - Y|^a \quad \text{if } |k| = 2b-1; \\ |D_X^k u_{n_s}(X, t) - D_X^k u_{n_l}(X, t) - D_X^k u_{n_s}(X, t') + D_X^k u_{n_l}(X, t')| \\ \leq \varepsilon |t - t'|^{(2b-1+a-|k|)/2b}, \\ |D_X^k u_{n_s}(X, t) - D_X^k u_{n_l}(X, t)| < \varepsilon \quad \text{if } |k| = 0, 1, \dots, 2b-1. \end{aligned}$$

Letting $l \rightarrow \infty$ in these inequalities, we get $\lim_{s \rightarrow \infty} \|u_{n_s} - u_0\|_{2b-1+a}^{Q, 1} = 0$.

Hence and by the inequality

$$\|u_n - u_0\|_{2b-1+a}^{Q, 1} \leq \|u_n - u_{n_s}\|_{2b-1+a}^{Q, 1} + \|u_{n_s} - u_0\|_{2b-1+a}^{Q, 1}$$

we may conclude the proof.

Remark 1. (a) If we replace the norm $\|\cdot\|_{2b-1+a}^{Q, p}$ by another equivalent norm, then the result of Lemma 2 remains correct.

(b) Let $l_1 < l_2$ be non-negative integers and $0 < \beta < a < 1$. Then the following relations are valid:

$$C_{X, t, p}^{l_2+a, (l_2+a)/2b}(Q) \subset C_{X, t, p}^{l_1+a, (l_1+a)/2b}(Q) \subset C_{X, t, p}^{l_1+\beta, (l_1+\beta)/2b}(Q).$$

The analogous relations are satisfied for $\tilde{C}_{X, t, p}^{l+a, (l+a)/2b}(Q)$.

LEMMA 3. Let conditions (A), (B), (C_{2b-1+a}) be fulfilled and let the vector function $F(X, t, \dots, D_X^\gamma U, \dots)$ ($|\gamma| = 0, 1, \dots, 2b-1$) be continuous and bounded in the norm $\|\cdot\|_0^{H, p}$. Then the operator

$$(16) \quad AU(X, t) = \int_0^t d\tau \int_\Omega G(X, t; \Xi, \tau) F[\Xi, \tau, \dots, D_X^\gamma U(\Xi, \tau), \dots] d\Xi, \quad (X, t) \in Q$$

maps the space $C_{X, t, p}^{2b-1+a, (2b-1+a)/2b}(Q)$ into the ball $S = \{U \in \mathcal{U}_m^p : \|U\|_{2b-1+a}^{Q, p} \leq B\}$, where $B > 0$ is a sufficiently large constant.

Proof. Let $U(X, t) \in C_{X, t, p}^{2b-1+a, (2b-1+a)/2b}(Q)$ and $\|F\|_0^{H, p} \leq M$, $M > 0$. Then, by (16), for $(X, t), (Y, t), (X, t') \in Q$, $t < t'$ we have:

$$\begin{aligned} |D_X^k A U(X, t) - D_X^k A U(Y, t)| &\leq M I_{1, k}(X; Y, t) J, \quad |k| = 2b-1; \\ |D_X^k A U(X, t)| &\leq M I_{2, k}(X, t) J; \\ |D_X^k A U(X, t) - D_X^k A U(X, t')| &\leq M I_{3, k}(X, t, t') J, \quad |k| = 0, 1, \dots, 2b-1. \end{aligned}$$

By Lemma 1 we may assert that there exists a constant $B > 0$ for which $\|A U\|_{2b-1+a}^{Q, p} \leq B$.

Now we formulate the existence theorem:

THEOREM 4. Assume that the requirements (A), (B), (C_{2b-1+a}) are satisfied. Let the right-hand side F of equation (2) be a continuous and bounded vector function in the norm $\|\cdot\|_0^{H(B_0), p}$, where $B_0 \geq B$ (B is the constant of Lemma 3) and let the Hölder condition

$$(17) \quad |F(X, t, \dots, U^\gamma, \dots) - F(Y, t, \dots, V^\gamma, \dots)| \leq \{q_0 |X - Y|^\beta + \sum_{i=0}^{2b-1} \sum_{|\gamma|=i} (Q^\gamma |U^\gamma - V^\gamma|^{\beta_\gamma})\} J$$

hold for $(X, t, \dots, U^\gamma, \dots), (Y, t, \dots, V^\gamma, \dots) \in H(B_0)$. Q^γ is a constant vector $(q_1^\gamma, \dots, q_p^\gamma)$ with positive components q_j^γ for $j = 1, \dots, p$ and q_0 is a positive real number. Further, let $0 < \beta \leq \alpha$, $\beta \leq \beta_\gamma \leq 1$ for $|\gamma| = 0, 1, \dots, 2b-2$ and $\beta_\gamma \alpha \geq \beta$ for $|\gamma| = 2b-1$. Then the problem (2), (3), (4) has at least one solution $U(X, t) \in C_{X, t, p}^{2b-1+a, (2b-1+a)/2b}(Q)$ satisfying the inequality $\|U\|_{2b-1+a}^{Q, p} \leq B_0$.

Proof. Let P denote the Banach space of vector functions $U(X, t) \in C_{X, t, p}^{2b-1+a, (2b-1+a)/2b}(Q)$ with the norm $\|U\|_{2b-1+a}^{Q, p}$ defined by (12). In view of Lemma 3 the operator A of (16) maps the closed and convex set $S_0 = \{U \in P : \|U\|_{2b-1+a}^{Q, p} \leq B_0\}$ into itself ($AS_0 \subset S_0$). Let $(X, t), (Y, t)$ and (X, t') , $t < t'$ be points of Q and $V(X, t) \in S_0$. In virtue of (17) for $F_V(X, t) = F[X, t, \dots, D_X^\gamma V(X, t), \dots]$,

$$\begin{aligned} |F_V(X, t) - F_V(Y, t)| &\leq \{q_0 |X - Y|^\beta + \sum_{i=0}^{2b-1} \sum_{|\gamma|=i} (Q^\gamma |D_X^\gamma V(X, t) - D_X^\gamma V(Y, t)|^{\beta_\gamma})\} J \\ &\leq \{q_0 |X - Y|^\beta + \sum_{i=0}^{2b-2} \sum_{|\gamma|=i} \left(\sum_{j=1}^p q_j^\gamma L^{\beta_\gamma} |X - Y|^{\beta_\gamma} \right) + \sum_{|\gamma|=2b-1} \left(\sum_{j=1}^p q_j^\gamma B_0^{\beta_\gamma} |X - Y|^{\alpha \beta_\gamma} \right)\} J, \end{aligned}$$

where L is a positive constant independent of V . Consequently, the vector function $F_V(X, t)$ is Hölder continuous with respect to X on Ω with exponent β and uniformly with respect to $t \in \langle 0, T \rangle$, i. e. $F_V(X, t) \in H_{\beta, p}(X, \Omega; t, \langle 0, T \rangle)$ for every $V(X, t) \in S_0$. According to Theorem 3, the problem

(2), (3), (4) is reduced to solving the operator equation $AU = U$ on S_0 . The existence theorem for the given problem may be proved by the Schauder fixed point theorem.

First we prove the continuity of the operator A on the set S_0 . Let $U_n(X, t) = (u_1^n(X, t), \dots, u_p^n(X, t))$ for $n = 1, 2, \dots$ be a sequence in S_0 tending to $U_0(X, t) = (u_1^0(X, t), \dots, u_p^0(X, t))$ ($U_0 \in S_0$), i. e. $\|U_n - U_0\|_{2b-1+\alpha}^{Q,p} \rightarrow 0$ as $n \rightarrow \infty$. By (17) we get

$$\begin{aligned} & |D_X^k AU_n(X, t) - D_X^k AU_0(X, t) - D_X^k AU_n(Y, t) + D_X^k AU_0(Y, t)| \\ & \leq I_{1,k}(X, Y, t) \sup_Q \sum_{i=0}^{2b-1} \sum_{|\gamma|=i} (Q^\gamma, |D_X^\gamma U_n(\Xi, \tau) - D_X^\gamma U_0(\Xi, \tau)|^{\beta_\gamma}) J, |k| = 2b-1; \\ & |D_X^k AU_n(X, t) - D_X^k AU_0(X, t)| \\ & \leq I_{2,k}(X, t) \sup_Q \sum_{i=0}^{2b-1} \sum_{|\gamma|=i} (Q^\gamma, |D_X^\gamma U_n(\Xi, \tau) - D_X^\gamma U_0(\Xi, \tau)|^{\beta_\gamma}) J, \\ & |D_X^k AU_n(X, t) - D_X^k AU_0(X, t) - D_X^k AU_n(X, t') + D_X^k AU_0(X, t')| \\ & \leq I_{3,k}(X, t, t') \sup_Q \sum_{i=0}^{2b-1} \sum_{|\gamma|=i} (Q^\gamma, |D_X^\gamma U_n(\Xi, \tau) - D_X^\gamma U_0(\Xi, \tau)|^{\beta_\gamma}) J \end{aligned}$$

for $|k| = 0, 1, \dots, 2b-1$. From this estimates and Lemma 1, $\lim_{n \rightarrow \infty} \|AU_n - AU_0\|_{2b-1+\alpha}^{Q,p} = 0$. The operator A is thus continuous.

We now prove the relative compactness of AS_0 . Put $\|F\|_0^{H(B_0),p} \leq M$, $M > 0$ and consider a sequence $V_n(X, t) = (v_1^n(X, t), \dots, v_p^n(X, t))$ of elements of AS_0 . There is a sequence $U_n(X, t) = (u_1^n(X, t), \dots, u_p^n(X, t))$ in the space P for which $V_n(X, t) = AU_n(X, t)$ ($n = 1, 2, \dots$). Since $\|V_n\|_{2b-1+\alpha}^{Q,p} \leq B_0$, thus the sequence of derivatives $\{D_X^k V_n(X, t)\}_{n=1}^\infty$ for $|k| = 0, 1, \dots, 2b-1$ is uniformly bounded on Q . From the inequality

$$\begin{aligned} |D_X^k V_n(X, t) - D_X^k V_n(Y, t')| & \leq |D_X^k AU_n(X, t) - D_X^k AU_n(Y, t)| + \\ & + |D_X^k AU_n(Y, t) - D_X^k AU_n(Y, t')| \leq M(I_{1,k} + I_{3,k})J, \end{aligned}$$

in view of Lemma 1 we obtain the equicontinuity of $\{D_X^k V_n(X, t)\}_{n=1}^\infty$ on Q for every k with $|k| = 0, 1, \dots, 2b-1$. Then we find a subsequence $\{V_{n_l}(X, t)\}_{l=1}^\infty = \{AU_{n_l}(X, t)\}_{l=1}^\infty$ of $\{V_n(X, t)\}_{n=1}^\infty$ tending to a bounded vector function $V_0(X, t) = (v_1^0(X, t), \dots, v_p^0(X, t))$, for which $\|D_X^k V_{n_l} - D_X^k V_0\|_0^{Q,p} \rightarrow 0$ as $l \rightarrow \infty$ ($|k| = 0, 1, \dots, 2b-1$). Moreover, letting $l \rightarrow \infty$ in the relations

$$\begin{aligned} |D_X^k V_0(X, t) - D_X^k V_0(Y, t)| & \leq |D_X^k V_0(X, t) - D_X^k V_{n_l}(X, t)| + \\ & + MI_{1,k}(X, Y, t)J + |D_X^k V_{n_l}(Y, t) - D_X^k V_0(Y, t)| \quad \text{for } |k| = 2b-1; \\ |D_X^k V_0(X, t)| & \leq |D_X^k V_0(X, t) - D_X^k V_{n_l}(X, t)| + MI_{2,k}(X, t)J, \\ |D_X^k V_0(X, t) - D_X^k V_0(X, t')| & \leq |D_X^k V_0(X, t) - D_X^k V_{n_l}(X, t)| + \\ & + MI_{3,k}(X, t, t')J + |D_X^k V_{n_l}(X, t') - D_X^k V_0(X, t')| \quad \text{for } |k| = 0, 1, \dots, 2b-1 \end{aligned}$$

we see that $V_0(X, t)$ belongs to the class $C_{X,t,p}^{2b-1+\alpha, (2b-1+\alpha)/2b}(Q)$. We have constructed the sequence $\{V_{n_l}(X, t)\}_{l=1}^{\infty}$ and the element $V_0(X, t)$ for which it is now necessary to establish: $\|V_{n_l} - V_0\|_{2b-1+\alpha}^{Q,p} \rightarrow 0$ as $l \rightarrow \infty$. For this purpose the following consideration will be helpful. Let S_0^* be the completion of AS_0 in the norm

$$(18) \quad \|U\|_{2b-1}^{Q,p,t} = \max_{j=1, \dots, p} \left(\sum_{i=0}^{2b-1} \sum_{|k|=i} \sup_Q |D_X^k u_j(X, t)| \right).$$

It is obvious that $V_0(X, t) \in S_0^*$. We shall show that S_0^* is a subset of $C_{X,t,p}^{2b-1+\alpha, (2b-1+\alpha)/2b}(Q)$. Indeed, let $V(X, t) = (v_1(X, t), \dots, v_p(X, t))$ be an arbitrary vector function of S_0^* . There is a sequence $\{W_n(X, t)\}_{n=1}^{\infty} \subset AS_0$ such that $\|W_n - V\|_{2b-1}^{Q,p,t} \rightarrow 0$ as $n \rightarrow \infty$. Similarly, like in the estimates (13), (14), (15), letting $n \rightarrow \infty$ we get

$$\begin{aligned} |D_X^k V(X, t) - D_X^k V(Y, t)| &\leq |D_X^k V(X, t) - D_X^k W_n(X, t)| + \\ &\quad + MI_{1,k}(X, Y, t)J + |D_X^k W_n(Y, t) - D_X^k V(Y, t)| \\ &\leq ML_{1,k}^* |X - Y|^{\alpha} |X - Y|^{2b\mu - (2b-1) - \alpha} E_1 J, \end{aligned}$$

where $|k| = 2b - 1$ and

$$\begin{aligned} |D_X^k V(X, t)| &\leq |D_X^k V(X, t) - D_X^k W_n(X, t)| + MI_{2,k}(X, t)J \leq ML_{2,k} E_1 J, \\ |D_X^k V(X, t) - D_X^k V(X, t')| &\leq |D_X^k V(X, t) - D_X^k W_n(X, t)| + MI_{3,k}(X, t, t')J + \\ &\quad + |D_X^k W_n(X, t') - D_X^k V(X, t')| \\ &\leq ML_{3,k}^* (t' - t)^{(2b-1+\alpha-|k|)/2b} [(t' - t)^{2b\mu - (2b-1) - \alpha}/2b + (t' - t)^{(1-\alpha)/2b}] E_1 J \end{aligned}$$

for $k = 0, 1, \dots, 2b - 1$, where $L_{1,k}^* > 0$ and $L_{3,k}^* > 0$ are constants. μ is a fixed real number of the interval $\langle (2b - 1 + \alpha)/2b, 1 \rangle$. Hence the expected assertion follows and besides $\lim_{X \rightarrow Y} |D_X^k v_j(X, t) - D_X^k v_j(Y, t)| / |X - Y|^{\alpha} = 0$ uniformly with respect to $t \in \langle 0, T \rangle$ for $|k| = 2b - 1$ and

$$\lim_{t \rightarrow t' -} |D_X^k v_j(X, t) - D_X^k v_j(X, t')| / (t' - t)^{(2b-1+\alpha-|k|)/2b} = 0$$

uniformly with respect to $X \in \Omega$ for $|k| = 0, 1, \dots, 2b - 1$ and for every $j = 1, \dots, p$.

For brevity let us use the notations: $\langle u \rangle_{\mu, X}^Q = |u(X, t) - u(Y, t)| / |X - Y|^{\mu}$ and $\langle u \rangle_{\mu, t}^Q = |u(X, t) - u(X, t')| / |t - t'|^{\mu}$. By the preceding limits, to each $\varepsilon > 0$ we find $\delta > 0$ such that for every $j = 1, \dots, p$; $l = 1, 2, \dots$ ($V_{n_l}(X, t) - V_0(X, t) \in S_0^*$ for $l = 1, 2, \dots$),

$$\begin{aligned} \langle D_X^k v_j^{n_l} - D_X^k v_j^0 \rangle_{\alpha, X}^Q &< \varepsilon, \quad \text{if } |k| = 2b - 1 \text{ and } 0 < |X - Y| < \delta, \\ \langle D_X^k v_j^{n_l} - D_X^k v_j^0 \rangle_{(2b-1+\alpha-|k|)/2b, t}^Q &< \varepsilon, \\ &\text{if } |k| = 0, 1, \dots, 2b - 1 \text{ and } 0 < t' - t < \delta. \end{aligned}$$

Since $\|V_{n_l}(X, t) - V_0(X, t)\|_{2b-1}^{Q,p,t} \rightarrow 0$ as $l \rightarrow \infty$, thus there exists $N(\varepsilon) > 0$ such that for $l > N(\varepsilon)$ and $|X - Y| \geq \delta$

$$\begin{aligned} \langle D_X^k v_j^{n_l} - D_X^k v_j^0 \rangle_{a,X}^Q &\leq \delta^{-a} \max_{j=1, \dots, p} \left\{ \sup_{(X,t) \in Q} |D_X^k v_j^{n_l}(X, t) - D_X^k v_j^0(X, t)| + \right. \\ &\quad \left. + \sup_{(Y,t) \in Q} |D_X^k v_j^{n_l}(Y, t) - D_X^k v_j^0(Y, t)| \right\} < \varepsilon, \quad |k| = 2b-1 \end{aligned}$$

and for $t' - t \geq \delta$

$$\begin{aligned} \langle D_X^k v_j^{n_l} - D_X^k v_j^0 \rangle_{(2b-1+a-|k|)/2b,t}^Q &\leq \delta^{-(2b-1+a-|k|)/2b} \max_{j=1, \dots, p} \left\{ \sup_{(X,t) \in Q} |D_X^k v_j^{n_l}(X, t) - D_X^k v_j^0(X, t)| + \right. \\ &\quad \left. + \sup_{(X,t') \in Q} |D_X^k v_j^{n_l}(X, t') - D_X^k v_j^0(X, t')| \right\} < \varepsilon, \end{aligned}$$

where $|k| = 0, 1, \dots, 2b-1$. These considerations allow us to write

$$\begin{aligned} \|V_{n_l} - V_0\|_{2b-1+a}^{Q,p} &\leq \max_{j=1, \dots, p} \left\{ \sum_{i=0}^{2b-1} \sum_{|k|=i} \sup_Q |D_X^k v_j^{n_l}(X, t) - D_X^k v_j^0(X, t)| + \right. \\ &\quad + \sum_{|k|=2b-1} \max_{\substack{(X,t), (Y,t) \in Q \\ 0 < |X-Y| < \delta}} \langle D_X^k v_j^{n_l} - D_X^k v_j^0 \rangle_{a,X}^Q; \\ &\quad \sup_{\substack{(X,t), (Y,t) \in Q \\ |X-Y| \geq \delta}} \langle D_X^k v_j^{n_l} - D_X^k v_j^0 \rangle_{a,X}^Q + \\ &\quad + \sum_{i=1}^{2b-1} \sum_{|k|=i} \max_{\substack{(X,t), (X,t') \in Q \\ 0 < t'-t < \delta}} \langle D_X^k v_j^{n_l} - D_X^k v_j^0 \rangle_{(2b-1+a-|k|)/2b,t}^Q; \\ &\quad \left. \sup_{\substack{(X,t), (X,t') \in Q \\ t'-t \geq \delta}} \langle D_X^k v_j^{n_l} - D_X^k v_j^0 \rangle_{(2b-1+a-|k|)/2b,t}^Q \right\} < \varepsilon[s + t(2b-1) + s] \end{aligned}$$

for $l > N(\varepsilon)$, whence $\lim_{l \rightarrow \infty} \|V_{n_l} - V_0\|_{2b-1+a}^{Q,p} = 0$. (For the definition of s and $t(r)$ see Section 2a.) This concludes the proof of Theorem 4.

Remark 2. Let the assumptions of Theorem 4 will be fulfilled. Then in view of Remark 1b for $0 < \varrho \leq a$, the solution $U(X, t)$ of (2), (3), (4) belongs to the class $C_{X,t,p}^{2b-1+\varrho, (2b-1+\varrho)/2b}(Q)$, too.

THEOREM 5. Let conditions (A), (B), (C_{2b-1+a}) be valid and let the vector function F of (2) be bounded in $\|\cdot\|_0^{H(B_0), p}$, where $B_0 \geq B$ (B is a constant of Lemma 3) and

$$\begin{aligned} (19) \quad |F(X, t, \dots, U^r, \dots) - F(Y, t', \dots, V^r, \dots)| \\ \leq \{q_0 |X - Y|^\beta + p_0 |t' - t|^{\beta/2b} + \sum_{i=0}^{2b-1} \sum_{|r|=i} (Q^r, |U^r - V^r|^{\beta_r})\} J \end{aligned}$$

for $(X, t, \dots, U^r, \dots), (Y, t', \dots, V^r, \dots) \in H(B_0)$, $Q^r = (q_1^r, \dots, q_p^r)$, $q_j^r > 0$ ($j = 1, \dots, p$) and $q_0, p_0 > 0$. Further, let ϱ be a number of the interval

$(0, 1)$ and let the exponents β, β_γ satisfy the inequalities $\varrho < \beta \leq \alpha, \beta \leq \beta_\gamma \leq 1$ for $|\gamma| = 0, 1, \dots, 2b-2$ and $\beta_\gamma \varrho \geq \beta$ for $|\gamma| = 2b-1$. Then the problem (2), (3), (4) has at least one solution belonging to $C_{X,t,p}^{2b+e, (2b+e)/2b}(Q)$.

Proof. In view of Theorem 4 and Remark 2 the problem (2), (3), (4) has at least one solution $U(X, t)$ belonging to the class $C_{X,t,p}^{2b-1+e, (2b-1+e)/2b}(Q)$. $U(X, t)$ satisfies the integro-differential equation $U = AU$ on Q . We have to prove that every term of the sum

$$\begin{aligned} \|U\|_{2b+e}^{Q,p} = \max_{j=1, \dots, p} \left\{ \sum_{|k|=2b} \langle D_X^k u_j \rangle_{e,X}^Q + \langle D_t u_j \rangle_{e,X}^Q + \langle D_t u_j \rangle_{e/2b,t}^Q + \right. \\ \left. + \sum_{i=1}^{2b} \sum_{|k|=i} \langle D_X^k u_j \rangle_{(2b+e-|k|)/2b,t}^Q + \sum_{i=0}^{2b} \sum_{|k|=i} \sup_Q |D_X^k u_j(X, t)| + \sup_Q |D_t u_j(X, t)| \right\} \end{aligned}$$

is finite. Let $\|F\|_0^{H(b_0),p} \leq M$ and $(X, t), (Y, t), (X, t') \in Q, t < t'$.

(a) We start with the boundedness of $D_X^k U(X, t)$ ($|k| = 0, 1, \dots, 2b$) and $D_t U(X, t)$. It is sufficient to prove the boundedness of $D_X^k U(X, t)$ for $|k| = 2b$ and $D_t U(X, t)$ on Q . If we employ Green's formula and the estimate (6) to the first integral in the inequality

$$\begin{aligned} |D_X^k U(X, t)| \leq M \int_0^t d\tau \left| \int_\Omega D_X^k G(X, t; \Xi, \tau) d\Xi \right| J + \\ + \int_0^t d\tau \int_\Omega |D_X^k G(X, t; \Xi, \tau)| |F[\Xi, \tau, \dots, D_X^\gamma U(\Xi, \tau), \dots] - \\ - F[X, \tau, \dots, D_X^\gamma U(X, \tau), \dots]| d\Xi \end{aligned}$$

and the estimates (10) and (19) to the second integral then we obtain

$$\begin{aligned} |D_X^k U(X, t)| \leq pC \left\{ M \int_0^t d\tau \int_{\partial\Omega} (t-\tau)^{-(m+2b-1)/2b} \exp[-c|X-\Xi|^{2b\gamma}/(t-\tau)^\gamma] d\Xi + \right. \\ + \int_0^t d\tau \int_\Omega (t-\tau)^{-\mu} |X-\Xi|^{2b\mu-m-2b} [q_0 |X-\Xi|^\beta + \\ + \sum_{i=0}^{2b-1} \sum_{|\gamma|=1} (Q^\gamma, |D_X^\gamma U(X, \tau) - D_X^\gamma U(\Xi, \tau)|^{\beta_\gamma})] d\Xi \Big\} J \\ \leq pC \left\{ ML_1 + \int_0^t d\tau \int_\Omega (t-\tau)^{-\mu} |X-\Xi|^{2b\mu-m-2b} [q_0 |X-\Xi|^\beta + \right. \\ + L^{\beta_\gamma} \sum_{i=0}^{2b-2} \sum_{|\gamma|=i} (Q^\gamma, |X-\Xi|^{\beta_\gamma} J) + L^{\beta_\gamma} \sum_{|\gamma|=2b-1} (Q^\gamma, |X-\Xi|^{\beta_\gamma} J) \Big\} d\Xi \Big\} J \\ \leq pC \left\{ ML_1 + L_2 \int_0^t d\tau \int_\Omega (t-\tau)^{-\mu} |X-\Xi|^{2b\mu-(m+2b-\beta)} d\Xi \right\} J, \end{aligned}$$

where L, L_1, L_2, μ are positive constants. Equation (2) yields

$$|D_t U(X, t)| \leq |F[X, t, \dots, D_X^\nu U(X, t), \dots]| + \sum_{|k|=2b} |A_k(X, t)| |D_X^k U(X, t)|.$$

By these estimates the boundedness of the derivatives in question is proved.

(b) For the difference $|D_X^k U(X, t) - D_X^k U(Y, t)|$ with $|k| = 2b$ one obtains

$$\begin{aligned} |D_X^k U(X, t) - D_X^k U(Y, t)| &\leq M \left| \int_0^t d\tau \int_{\Omega} [D_X^k G(X, t; \Xi, \tau) - \right. \\ &\quad \left. - D_X^k G(Y, t; \Xi, \tau)] d\Xi \right| J + \int_0^t d\tau \int_{\Omega} |D_X^k G(X, t; \Xi, \tau) - D_X^k G(Y, t; \Xi, \tau)| \times \\ &\quad \times |F[\Xi, \tau, \dots, D_X^\nu U(\Xi, \tau), \dots] - F[X^*, \tau, \dots, D_X^\nu U(X^*, \tau), \dots]| d\Xi, \end{aligned}$$

where $|X^* - \Xi| = \min(|X - \Xi|, |Y - \Xi|)$. Employing again Green's formula and the mean value theorem to the first integral and the estimates (7) (with $\alpha = \varrho$) and (19) to the second integral we get after easy calculations

$$\begin{aligned} |D_X^k U(X, t) - D_X^k U(Y, t)| &\leq pC \left\{ M |X - Y| \int_0^t d\tau \int_{\partial\Omega} (t - \tau)^{-(m+2b)/2b} \exp[-c|X^{**} - \Xi|^{2br}/(t - \tau)^r] d\Xi + \right. \\ &\quad \left. + L_3 |X - Y|^{\varrho} \int_0^t d\tau \int_{\Omega} (t - \tau)^{-\mu} |X^* - \Xi|^{2b\mu - (m+2b+\varrho-\beta)} d\Xi \right\} J, \end{aligned}$$

where X^{**} is a point of int Ω and L_3 is a positive constant. Both integrals converge to a bounded function uniformly with respect to X^{**}, X^* and t if $2b - (\beta - \varrho)/2b \leq \mu < 1$. Thus $D_X^k U(X, t) \in H_{\varrho, p}(X, \Omega; t, \langle 0, T \rangle)$ for $|k| = 2b$. From equation (2) we get

$$\begin{aligned} |D_t U(X, t) - D_t U(Y, t)| &\leq |F[X, t, \dots, D_X^\nu U(X, t), \dots] - F[Y, t, \dots, \\ &\quad \dots, D_X^\nu U(Y, t), \dots]| + \sum_{|k|=2b} |A_k(X, t)| |D_X^k U(X, t) - D_X^k U(Y, t)| + \\ &\quad + \sum_{|k|=2b} |A_k(X, t) - A_k(Y, t)| |D_X^k U(Y, t)|. \end{aligned}$$

Hence, according to conditions (C₀) and (19), $D_t U(X, t) \in H_{\varrho, p}(X, \Omega; t, \langle 0, T \rangle)$.

(c) For $|k| = 1, \dots, 2b$ we have

$$\begin{aligned}
 (20) \quad & |D_X^k U(X, t) - D_X^k U(X, t')| \\
 & \leq M \left\{ \int_0^t d\tau \left| \int_{\Omega} [D_X^k G(X, t; \Xi, \tau) - D_X^k G(X, t'; \Xi, \tau)] d\Xi \right| + \right. \\
 & \quad \left. + \int_t^{t'} d\tau \left| \int_{\Omega} D_X^k G(X, t'; \Xi, \tau) d\Xi \right| \right\} J + \\
 & + \int_0^t d\tau \int_{\Omega} |D_X^k G(X, t; \Xi, \tau) - D_X^k G(X, t'; \Xi, \tau)| \times \\
 & \times |F[\Xi, \tau, \dots, D_X^{\gamma} U(\Xi, \tau), \dots] - F[X, \tau, \dots, D_X^{\gamma} U(X, \tau), \dots]| d\Xi + \\
 & + \int_t^{t'} d\tau \int_{\Omega} |D_X^k G(X, t'; \Xi, \tau)| \times \\
 & \times |F[\Xi, \tau, \dots, D_X^{\gamma} U(\Xi, \tau), \dots] - F[X, \tau, \dots, D_X^{\gamma} U(X, \tau), \dots]| d\Xi.
 \end{aligned}$$

The final estimate will be constructed as follows:

The first and the second integral (denote them by I_4, I_5) satisfy on account of Green's formula and the mean value theorem and (6), the inequalities:

$$\begin{aligned}
 I_4 & \leq (t' - t) \int_0^t d\tau \int_{\partial\Omega} |D_t D_X^{\tilde{k}} G(X, t^*; \Xi, \tau)| d\Xi \\
 & \leq C \left\{ (t' - t) \int_0^t d\tau \int_{\partial\Omega} (t^* - \tau)^{-(m+2b+|k|-1)/2b} \exp[-c|X - \Xi|^{2br}/(t^* - \tau)^r] d\Xi \right\} E_1,
 \end{aligned}$$

where $\tilde{k} = (\tilde{k}_1, \dots, \tilde{k}_m)$, $|\tilde{k}| = |k| - 1$ and $t < t^* < t'$,

$$I_5 \leq C \left\{ \int_t^{t'} d\tau \int_{\partial\Omega} (t' - \tau)^{-(m+|k|-1)/2b} \exp[-c|X - \Xi|^{2br}/(t' - \tau)^r] d\Xi \right\} E_1.$$

Since $\Xi \neq X$, both integrals are bounded functions of their variables and so $I_4, I_5 \in \tilde{H}_{1,p}(t, \langle 0, T \rangle; X, \Omega)$. Inequalities (8) (with $\alpha = \varrho$), (19) and (10) yield the following estimates for the third and the fourth integral of (20) (denote them by I_6, I_7):

$$\begin{aligned}
 I_6 & \leq \left\{ pCL_4 (t' - t)^{(2b-|k|+\varrho)/2b} \int_0^t d\tau \int_{\Omega} (t - \tau)^{-\mu} |X - \Xi|^{2b\mu - (m+2b+\varrho-\beta)} d\Xi \right\} J, \\
 I_7 & \leq \left\{ pCL_5 \int_t^{t'} d\tau \int_{\Omega} (t' - \tau)^{-\mu} |X - \Xi|^{2b\mu - (m+|k|-\beta)} d\Xi \right\} J,
 \end{aligned}$$

where L_4, L_5 are positive constants. Hence for $(2b - \beta + \varrho)/2b \leq \mu < 1$,

$$I_6 \in H_{(2b-|k|+\varrho)/2b,p}(t, \langle 0, T \rangle; X, \Omega)$$

and for $(|k| - \beta/2b) \leq \mu \leq (|k| - \varrho)/2b$,

$$I_7 \in H_{(2b-|k|+\varrho)/2b,p}(t, \langle 0, T \rangle; X, \Omega).$$

Similarly as in part (b) we have:

$$\begin{aligned}
 & |D_t U(X, t) - D_t U(X, t')| \\
 & \leq |F[X, t, \dots, D_X^\gamma U(X, t), \dots] - F[X, t', \dots, D_X^\gamma U(X, t'), \dots]| + \\
 & \quad + \sum_{|k|=2b} |A_k(X, t)| |D_X^k U(X, t) - D_X^k U(X, t')| + \\
 & \quad + \sum_{|k|=2b} |A_k(X, t) - A_k(X, t')| |D_X^k U(X, t')|.
 \end{aligned}$$

The majorant of the first term is

$$\left\{ p_0(t' - t)^{\beta/2b} + L_6^{\beta\gamma} \sum_{i=0}^{2b-1} \sum_{|\gamma|=i} (Q^\gamma, |t' - t|^{(2b-1+\alpha-|\gamma|)\beta\gamma/2b} J) \right\} J, \quad L_6 > 0.$$

The estimates of the remaining expressions follow from (20) and (C₀). Then, $D_t U(X, t)$ belongs to $H_{\epsilon/2b, p}(t, \langle 0, T \rangle; X, \Omega)$. The proof of the theorem is finished.

Remark 3. If all assumptions of Theorem 5 are valid and, moreover, F satisfies the "coordinative condition" of order $2b-1$ on $\partial\Omega$ for every $V(X, t) \in C_{X, t, p}^{2b-1+\alpha, (2b-1+\alpha)/2b}(Q)$, then there is one and only one solution of the problem (2), (3), (4) belonging to $C_{X, t, p}^{2b+\alpha, (2b+\alpha)/2b}(Q)$.

This statement follows from Theorems 1 and 5.

THEOREM 6. Let assumptions (A), (B), (C_{2b-1+\alpha}) be satisfied and let F be a vector function continuous and bounded in the norm $\|\cdot\|_0^{H(B_0), p}$, $B_0 \geq B$ ($B > 0$ is the constant of Lemma 3) fulfilling on $H(B_0)$

$$\begin{aligned}
 (21) \quad & |F(X, t, \dots, U^\gamma, \dots) - F(X, t, \dots, V^\gamma, \dots)| \\
 & \leq \sum_{i=0}^{2b-1} \sum_{|\gamma|=i} (|R^\gamma(X, t)|, |U^\gamma - V^\gamma|) J.
 \end{aligned}$$

The vector function

$$R^\gamma(X, t) = (r_1^\gamma(X, t), \dots, r_p^\gamma(X, t)) \quad \text{for } |\gamma| = 0, 1, \dots, 2b-1$$

is continuous and bounded in the norm $\|\cdot\|_0^{Q, p}$. Further, let

$$Rp^2 \left\{ \sum_{|k|=2b-1} L_{1,k} + \sum_{i=0}^{2b-1} \sum_{|k|=i} (L_{2,k} + L_{3,k}) \right\} < 1,$$

where $R = \max_{\substack{|\gamma|=0, 1, \dots, 2b-1 \\ j=1, \dots, p}} [\sup_Q r_j^\gamma(X, t)]$ and $L_{1,k}, L_{2,k}, L_{3,k}$ are constants

occurring in (13), (14), (15). Then the integro-differential equation $U = AU$ has one and only one solution in the class $C_{X, t, p}^{2b-1+\alpha, (2b-1+\alpha)/2b}(Q)$ and, furthermore, the successive approximations $U_n = AU_{n-1}$, $n = 1, 2, \dots$ tend to this unique solution in the norm $\|\cdot\|_{2b-1+\alpha}^{Q, p}$ for any $U_0 \in C_{X, t, p}^{2b-1+\alpha, (2b-1+\alpha)/2b}(Q)$.

Proof. Consider the Banach space P and the ball $S_0 \subset P$ with radius B_0 defined in the proof of Theorem 4. Then, by Lemma 1 and assumption (21), for $U(X, t), V(X, t) \in S_0$ we have

$$\begin{aligned} & |D_X^k A U(X, t) - D_X^k A V(X, t) - D_X^k A U(Y, t) + D_X^k A V(Y, t)| \\ & \leq R p \|U - V\|_{2b-1}^{Q,p,t} I_{1,k}(X, Y, t) J \leq R p^2 L_{1,k} \|U - V\|_{2b-1+\alpha}^{Q,p} |X - Y|^\alpha J, \\ & |k| = 2b - 1; \\ & |D_X^k A U(X, t) - D_X^k A V(X, t)| \\ & \leq R p \|U - V\|_{2b-1}^{Q,p,t} I_{2,k}(X, t) J \leq R p^2 L_{2,k} \|U - V\|_{2b-1+\alpha}^{Q,p}, \\ & |D_X^k A U(X, t) - D_X^k A V(X, t) - D_X^k A U(X, t') + D_X^k A V(X, t')| \\ & \leq R p \|U - V\|_{2b-1}^{Q,p,t} I_{3,k}(X, t, t') J \leq R p^2 L_{3,k} \|U - V\|_{2b-1+\alpha}^{Q,p} (t' - t)^{(2b-1+\alpha-|k|)/2b} J \end{aligned}$$

for $|k| = 0, 1, \dots, 2b - 1$. Here $\|\cdot\|_{2b-1}^{Q,p,t}$ is the norm defined in (18). Hence we obtain the relation

$$\|AU - AV\|_{2b-1}^{Q,p} \leq R p^2 \left[\sum_{|k|=2b-1} L_{1,k} + \sum_{i=0}^{2b-1} \sum_{|k|=i} (L_{2,k} + L_{3,k}) \right] \|U - V\|_{2b-1+\alpha}^{Q,p},$$

which proves the contraction of the operator A on S_0 . The assertion is evident from the Banach theorem on the contractive mapping.

Remark 4. If F satisfies together with the assumptions of Theorem 6 also condition (17), then the sequence $\{AU_{n-1}\}_{n=1}^\infty$ converges to the solution of problem (2), (3), (4) in the norm $\|\cdot\|_{2b-1+\alpha}^{Q,p}$.

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