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THE CLASS OF ALGEBRAS IN WHICH WEAK INDEPENDENCE IS EQUIVALENT TO DIRECT SUMS INDEPENDENCE

BY

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1. Introduction. In 1964, Grätzer [2] defined *weak independence* (we shall call it *G-independence*, following [4]) as an abstraction of the classical (linear) independence in abelian groups. Thus in an abelian group (in fact, in a module over a ring) $\{a_1, \ldots, a_k\}$ is *G-independent* if and only if $n_1 a_1 + \ldots + n_k a_k = 0$ implies that each $n_i a_i = 0$. This is the same as $[a_1, \ldots, a_k] = [a_1] \oplus \ldots \oplus [a_k]$ (direct sum), where [ ] means "subgroup generated by". He then posed the problem of determining the class $\boldsymbol{K}$ of algebras in which $\{a_1, \ldots, a_k\}$ is G-independent if and only if $[a_1, \ldots, a_k] \cong [a_1] \times \ldots \times [a_k]$, assuming that in $\boldsymbol{K}$ there is a nullary operation which determines a one-element subalgebra in every algebra of $\boldsymbol{K}$.

In this paper we observe that, in an abelian group, $\{a_1, \ldots, a_k\}$ is G-independent if and only if $[a_1, \ldots, a_k] \cong [a_1] \times \ldots \times [a_k]$ under a "normal" isomorphism, one which takes $a_i \to (0, \ldots, a_i, \ldots, 0)$. We then show that the class of algebras in which this equivalence holds is precisely the class of semimodules over semirings which, together with Corollary 2, solves problem 56 of [3] (formulated also as P 606 in [2]). We follow the approach developed by Csákány in [1].

2. Definitions and preliminary results. We follow the basic notation and terminology of [3]. In particular, $\mathfrak{F}^{(n)}(\boldsymbol{K})$ denotes the free algebra on n generators over $\boldsymbol{K}$; e_i^n denotes the n-ary polynomial defined by $e_i^n(x_1, \ldots, x_n) = x_i$; $P^{(n)}(\boldsymbol{K})$ denotes the n-ary polynomials over $\boldsymbol{K}$; and if $\mathfrak{A} = \langle A; F\rangle$ is an algebra, $B \subseteq A$, then $\langle [B]; F\rangle$ denotes the subalgebra of $\mathfrak{A}$ generated by B. In this paper $\boldsymbol{K}$ will denote an equational class of algebras each of which has a one-element subalgebra, determined by a nullary operation denoted by 0.

Let $\langle B_i; F\rangle \in \boldsymbol{K}$ for $i \in I$. By $\left\langle \sum_{i\in I} B_i; F\right\rangle$ it is meant the subalgebra of the direct product $\left\langle \prod_{i\in I} B_i; F\right\rangle$ consisting of all those functions which are 0 for all but finitely many $i \in I$.

For $B = \{b_i \mid i \in I\} \subseteq A$, we say that an isomorphism φ from $\langle [B]; F\rangle$ onto $\left\langle \sum_{i \in I} [b_i]; F \right\rangle$ is *normal* if, for every $i \in I$, $\varphi(b_i) = \xi_{b_i}$, defined by $\xi_{b_i}(i) = b_i$, $\xi_{b_i}(j) = 0$ for $i \neq j$. If such φ exists, we write $B \in \mathrm{NDS}$. Note that, under a normal isomorphism, $0 \to \xi_0$ (defined by $\xi_0(i) = 0$ for all $i \in I$).

For $B \subseteq A$, a map $\varphi\colon B \to C$, $\langle C; F\rangle \in \boldsymbol{K}$, is *diminishing* if for any unary polynomials p and q, for every $b \in B$, $p(b) = q(b)$ implies $p(\varphi(b)) = q(\varphi(b))$. B is *G-independent* (with respect to $\boldsymbol{K}$) if and only if for every $\langle C; F\rangle \in \boldsymbol{K}$, every diminishing map from B into C can be extended to a homomorphism of $\langle [B]; F\rangle$ into $\langle C; F\rangle$. In this case we write $B \in \mathrm{GI}(\boldsymbol{K})$, or simply $B \in \mathrm{GI}$. $\mathfrak{R} = \langle R; +, \cdot, 0, 1\rangle$ is a *semiring* if $\langle R; +, 0\rangle$ is a commutative semigroup with identity 0, $\langle R; \cdot, 1\rangle$ is a semigroup with identity 1, and the distributive laws hold. $\mathfrak{M} = \langle M; +, 0, R\rangle$ is a *left unital $\mathfrak{R}$-semimodule* if $\langle M; +, 0\rangle$ is a commutative semigroup with identity 0 (as usual we use the same symbols to denote the zeros of $\mathfrak{M}$ and $\mathfrak{R}$), $\langle R; +, \cdot, 0, 1\rangle$ is a semiring, and there is an operation *rm* defined for $r \in R$, $m \in M$ such that for all $m, n \in M$, $r_1, r_2, r \in R$, we have $r(m+n) = rm + rn$, $(r_1 r_2)m = r_1(r_2 m)$, $(r_1 + r_2)m = r_1 m + r_2 m$, $0m = 0$, $r0 = 0$, $1m = m$.

If $\boldsymbol{K}_1$, $\boldsymbol{K}_2$ are equational classes, then $\boldsymbol{K}_1$ is *equivalent to* $\boldsymbol{K}_2$ if and only if, for every natural number n, there is a one-to-one map φ_n from $P^{(n)}(\boldsymbol{K}_1)$ onto $P^{(n)}(\boldsymbol{K}_2)$ such that $\varphi_n(e_i^n) = e_i^n$ and for $p \in P^{(k)}(\boldsymbol{K}_1)$, $q_1, \ldots, q_k \in P^{(n)}(\boldsymbol{K}_1)$, $\varphi_n(p(q_1, \ldots, q_k)) = [\varphi_k(p)](\varphi_n(q_1), \ldots, \varphi_n(q_n))$.

3. Main results.

THEOREM 1. *In the class $\boldsymbol{M}$ of left unital semimodules over semirings,* $\mathrm{GI} = \mathrm{NDS}$.

Proof. Let $\{a_i \mid i \in I\} \subseteq M$, $\mathfrak{M} \in \boldsymbol{M}$, and suppose $\{a_i \mid i \in I\} \in \mathrm{GI}$. Since the map $a_i \to \xi_{a_i}$ is obviously diminishing, it has an extension to a homomorphism φ of $[\{a_i \mid i \in I\}]$ onto $[\{\xi_{a_i} \mid i \in I\}] = \sum_{i \in I} [a_i]$. It is one-to-one. For, if $a, b \in [\{a_i \mid i \in I\}]$, we may suppose

$$a = \sum_{k=1}^{n} r_{i_k} a_{i_k}, \qquad b = \sum_{k=1}^{n} s_{i_k} a_{i_k},$$

so if $\varphi(a) = \varphi(b)$, we have

$$\sum_{k=1}^{n} r_k \xi_{a_{i_k}} = \sum_{k=1}^{n} s_k \xi_{a_{i_k}},$$

so that $r_k a_{i_k} = s_k a_{i_k}$ for every $k = 1, \ldots, n$. Thus

$$a = \sum_{k=1}^{n} r_k a_{i_k} = \sum_{k=1}^{n} s_k a_{i_k} = b,$$

so φ is an isomorphism.

Now let $\{b_i \mid i \in I\} \subseteq \mathfrak{M} \in \boldsymbol{M}$ and suppose $\{b_i \mid i \in I\} \in \mathrm{NDS}$. Let ψ be an isomorphism of $[\{b_i \mid i \in I\}]$ onto $\sum_{i \in I} [b_i]$. Let $b_i \to c_i$, $i \in I$, be a diminishing map, where $c_i \in \mathfrak{N} \in \boldsymbol{M}$. Then φ_i: $rb_i \to rc_i$ defines a homomorphism of $[b_i]$ onto $[c_i]$. Let

$$\varphi = \sum_{i \in I} \varphi_i : \sum_{i \in I} [b_i] \to \sum_{i \in I} [c_i]$$

be the homomorphism defined by $\big(\varphi(f)\big)(i) = \varphi\big(f(i)\big)$. Finally, let

$$\chi : \sum_{i \in I} [c_i] \to \mathfrak{N}$$

be the homomorphism defined by $\chi(f) = \sum_{i \in I'} f(i)$, where $i \in I'$ if and only if $f(i) \neq 0$. $\chi\varphi\psi$ is a homomorphism taking b_i to c_i as required. Thus $\{b_i \mid i \in I\} \in \mathrm{GI}$.

THEOREM 2. *Suppose that in* $\boldsymbol{K}$, $\mathrm{GI} \subseteq \mathrm{NDS}$. *Then there exists a unique semiring* $\mathfrak{R}$ *with identity such that* $\boldsymbol{K}$ *is equivalent to the class of all left unital* $\mathfrak{R}$-*semimodules.*

Proof. Let x_1, x_2 be the generators of the free algebra $\mathfrak{F}^{(2)}(\boldsymbol{K})$. Since $\{x_1, x_2\}$ is independent, it is G-independent, hence in NDS. Let φ be the normal isomorphism from $\mathfrak{F}^{(2)}(\boldsymbol{K})$ onto $[x_1] \times [x_2]$. Let $a \in \mathfrak{F}^{(2)}(\boldsymbol{K})$ be such that $\varphi(a) = (x_1, x_2)$, and let q_0 be a polynomial with $a = q_0(x_1, x_2)$. If we denote q_0 by $\oplus$, we get (see [1]) in $\boldsymbol{K}$:

$$x \oplus 0 = x = 0 \oplus x. \tag{1}$$

Now let $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ be the generators of $\mathfrak{F}^{(2n)}(\boldsymbol{K})$, the free algebra over $\boldsymbol{K}$ on $2n$ generators. Since $\{x_1, \ldots, x_n, y_1, \ldots, y_n\} \in \mathrm{NDS}$, the composite of $x_i \to (0, \ldots, x_i, \ldots, 0) \to \big((0, \ldots, x_i, \ldots, 0), (0, \ldots, 0)\big) \to (x_i, 0)$ (and similarly $y_i \to (0, y_i)$) yields an isomorphism $\varphi: \mathfrak{F}^{(2n)}(\boldsymbol{K}) \to \mathfrak{F}^{(n)}(\boldsymbol{K}) \times \mathfrak{F}^{(n)}(\boldsymbol{K})$ which, for any n-ary polynomial p, takes $p(x_1, \ldots, x_n) \to \big(p(x_1, \ldots, x_n), 0\big)$ and $p(y_1, \ldots, y_n) \to \big(0, p(y_1, \ldots, y_n)\big)$. It follows as in [1] that

$$p(x_1 \oplus y_1, \ldots, x_n \oplus y_n) = p(x_1, \ldots, x_n) \oplus p(y_1, \ldots, y_n). \tag{2}$$

From (2), taking $p = \oplus$, we immediately derive associativity and commutativity of $\oplus$. By induction it is easy to obtain the general associative law and the following extension of (2):

$$p(x_1^1 \oplus \ldots \oplus x_1^m, \ldots, x_n^1 \oplus \ldots \oplus x_n^m) = p(x_1^1, \ldots, x_n^1) \oplus \ldots \oplus p(x_1^m, \ldots, x_n^m). \tag{2'}$$

Let R be the set of all unary polynomials in K. Define:

$$(p_1 \oplus p_2)(x) = p_1(x) \oplus p_2(x),$$
$$(p_1 \cdot p_2)(x) = p_1(p_2(x)),$$
$$1(x) = e_1^1(x) = x,$$
$$0(x) = 0.$$

Then $\mathfrak{R} = \langle R; +, \cdot, 0, 1\rangle$ is a semiring with identity (see [1]). We will show that K is equivalent to the class of all left unital $\mathfrak{R}$-semimodules M.

Let $p \in P^{(n)}(K)$. Then, by (2′), p can be written uniquely as

$$p = p_1 e_1^n \oplus \dots \oplus p_n e_n^n, \qquad \text{where } p_i(x) = p(0, \dots, x, \dots, 0).$$

Then φ_n: $P^{(n)}(K) \to P^{(n)}(M)$ is defined by $\varphi_n(p) = p_1 e_1^n + \dots + p_n e_n^n$. Since the $P^{(n)}(K)$ are disjoint, we use the common symbol φ for the φ_n. Note that for unary p, $\varphi(p) = p$, where in $P^{(1)}(M)$ we simply write p for pe_1^1. Also $\varphi(1) = 1$ and $\varphi(\oplus) = +$.

φ is 1-1. If $\varphi(p) = \varphi(q)$, then $p_1 e_1^n + \dots + p_n e_n^n = q_1 e_1^n + \dots + q_n e_n^n$. Taking $x_i = x$, $x_j = 0$ for $j \neq i$, we get $p_i(x) = q_i(x)$ in M. Since $p_i = q_i$ s an identity in M, it must hold in $\mathfrak{R}$ regarded as a semimodule over tself. Thus

$$p_1 e_1^n \oplus \dots \oplus p_n e_n^n = q_1 e_1^n \oplus \dots \oplus q_n e_n^n$$

holds in K, i. e., $p = q$ in K.

φ is onto. It is clear that in unital semimodules over $\mathfrak{R}$ every polynomial p' has a representation of the form $p_1 e_1^n + \dots + p_n e_n^n$, $p_i \in R$, so $\varphi(p_1 e_1^n \oplus \dots \oplus p_n e_n^n) = p'$.

φ is an equivalence. Let

$$p \in P^{(k)}(K), \qquad q^1, \dots, q^k \in P^{(n)}(K),$$
$$p = p_1 e_1^k \oplus \dots \oplus p_k e_k^k, \qquad q^j = q_1^j e_1^n \oplus \dots \oplus q_n^j e_n^n,$$

$$\begin{aligned}
\varphi[p(q^1, \dots, q^k)] &= \varphi[p(q_1^1 e_1^n \oplus \dots \oplus q_n^1 e_n^n, \dots, q_1^k e_1^n \oplus \dots \oplus q_n^k e_n^n)] \\
&= \varphi[p_1(q_1^1 e_1^n \oplus \dots \oplus q_n^1 e_n^n) \oplus \dots \oplus p_k(q_1^k e_1^n \oplus \dots \oplus q_n^k e_n^n)] \\
&= \varphi[(p_1 q_1^1 e_1^n \oplus \dots \oplus p_1 q_n^1 e_n^n) \oplus \dots \oplus (p_k q_1^k e_1^n \oplus \dots \oplus p_k q_n^k e_n^n)] \\
&= \varphi[(p_1 q_1^1 \oplus \dots \oplus p_k q_1^k) e_1^n \oplus \dots \oplus (p_1 q_n^1 \oplus \dots \oplus p_k q_n^k) e_n^n] \\
&= (p_1 q_1^1 \oplus \dots \oplus p_k q_1^k) e_1^n + \dots + (p_1 q_n^1 \oplus \dots \oplus p_k q_n^k) e_n^n \\
&= (p_1 q_1^1 e_1^n + \dots + p_k q_1^k e_1^n) + \dots + (p_1 q_n^1 e_n^n + \dots + p_k q_n^k e_n^n) \\
&= p_1(q_1^1 e_1^n + \dots + q_n^1 e_n^n) + \dots + p_k(q_1^k e_1^n + \dots + q_n^k e_n^n) \\
&= [\varphi(p)](\varphi(q^1), \dots, \varphi(q^k)).
\end{aligned}$$

That $\mathfrak{R}$ is unique follows from the properties of $\oplus$, as in [1].

COROLLARY 1. *If in* $\boldsymbol{K}$, $\mathrm{GI} \subseteq \mathrm{NDS}$, *then* $\mathrm{GI} = \mathrm{NDS}$.

COROLLARY 2. *If in* $\boldsymbol{K}$, $\mathrm{GI} \subseteq \mathrm{NDS}$, *and if for every* $\mathfrak{A} \in \boldsymbol{K}$, *any two congruences in* $\mathfrak{A}$ *permute, then there exists a unique ring* $\mathfrak{R}$ *with* 1 *such that* $\boldsymbol{K}$ *is equivalent to the class of all left unital modules over* $\mathfrak{R}$.

Proof. By a result of Mal'cev [5], there is a ternary polynomial symbol p such that in $\boldsymbol{K}$, $x_1 = p(x_1, x_2, x_2)$ and $x_2 = p(x_1, x_1, x_2)$. Then we have $p(0, x, 0) = -x$, since $x \oplus p(0, x, 0) = p(x, 0, 0) \oplus p(0, x, 0) = p(x, x, 0) = 0$.

COROLLARY 3. *In* $\boldsymbol{K}$, $\mathfrak{F}^{(n)}(\boldsymbol{K})$ *is normally isomorphic to* $\mathfrak{F}^{(1)}(\boldsymbol{K}) \times \ldots \times \mathfrak{F}^{(1)}(\boldsymbol{K})$ *if and only if* $\boldsymbol{K}$ *is equivalent to the class of left unital semimodules over some semiring.*

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