ANNALES
POLONICI MATHEMATICI
XXX (1974)

On the existence and uniqueness of convex solutions of a functional equation in the indeterminate case

by Stefan Czerwik (Katowice)

Abstract. Under assumptions (i)–(iv) it is proved, in Theorem 1, by Helly's theorem, that equation (1) has exactly one convex and continuous solution fulfilling condition (4).

In Theorems 2, 3, 4 it is proved that (under different assumptions) equation (1) possesses at least one convex solution.

1. In the present paper we are concerned with the functional equation

$$\varphi(x) = h(x, \varphi[f(x)]), \tag{1}$$

where $f(x)$ and $h(x, y)$ are known real-valued functions of real variables and $\varphi(x)$ is unknown.

We shall seek convex solution of equation (1) in the indeterminate case:

$$p = |f'(\xi) h_y(\xi, \eta)| = 1, \tag{2}$$

where ξ is the fixed point of the function $f(x)$ in the interval considered and (ξ, η) fulfils the condition $h(\xi, \eta) = \eta$. A theory of the existence and uniqueness of convex solution of equation (1) has been developed in [2], [4], [5], [6] essentially under the $p < 1$.

2. We assume the following hypotheses:

(i) $f(x)$ is defined and continuous in an interval I, $\underset{\xi \in I}{\exists}\, f(\xi) = \xi$

$$0 < \frac{f(x) - \xi}{x - \xi} < 1 \quad \text{for } x \in I,\ x \neq \xi.$$

(ii) $h(x, y)$ is defined and continuous in a domain Ω, $\underset{\eta}{\exists}\, (\xi, \eta) \in \Omega$, $h(\xi, \eta) = \eta$; moreover, for every $x \in I$, the set $\Omega_x \overset{\text{df}}{=} \{y : (x, y) \in \Omega\}$ is a non-empty open interval and $h(f(x), \Omega_{f(x)}) \subset \Omega_x$.

(iii) There exist positive constants α and β such that

$$|h(x, y_1) - h(x, y_2)| \leqslant |y_1 - y_2| \tag{3}$$

for $(x, y_1), (x, y_2) \in \Omega \cap \langle \xi - \alpha, \xi + \alpha \rangle \times \langle \eta - \beta, \eta + \beta \rangle$.

(iv) $f(x)$ is strictly increasing and convex ([1]) in I, $h(x, y)$ is increasing with respect to either variable and convex in two variables in Ω, Ω is a convex domain.

At first we shall prove the following

THEOREM 1. *Let hypotheses* (i)–(iv) *be fulfilled. If, moreover, for every $x \in I$, Ω_x is bounded, then equation* (1) *has exactly one convex and continuous solution φ in I fulfilling the condition*

$$\varphi(\xi) = \eta. \tag{4}$$

Proof. We may assume that ξ is the left end point of I, so that $I = \langle \xi, b)$ (if ξ is the right end point or inside point of I, the proof is analogous). Together with (1) we shall consider the sequence of equations

$$\varphi_n(x) = t_n h\big(x, \varphi_n[f(x)]\big), \quad t_n < 1, \ t_n \to 1 \text{ as } n \to \infty, \ n = 1, 2, \ldots \tag{5}$$

From Theorem 3.4 in [3] for every n there exists exactly one function $\varphi_n(x)$, continuous in I, satisfying equation (5) in I and fulfilling the condition

$$\varphi_n(\xi) = \eta. \tag{6}$$

It is given by the formula

$$\varphi_n(x) = \lim_{v \to \infty} \varphi_{n,v}(x), \tag{7}$$

where

$$\varphi_{n,0}(x) = \eta, \quad \varphi_{n,v+1}(x) = t_n \cdot h\big(x, \varphi_{x,v}[f(x)]\big), \quad v = 0, 1, 2, \ldots$$

In view of (iv), we easily verify that every function $\varphi_{n,v}(x)$ is convex and increasing in I and consequently the function (7) is also convex and increasing in I.

We have also (6) for $n = 1, 2, \ldots$ and, for every n and every $x \in I$, $\varphi_n(x) \in \Omega_{f^{-1}(x)}$. Hence, in view of the boundedness of Ω_x, for every $x \in I$ there exists a number M_x such that $|\varphi_n(x)| \leqslant M_x$. It follows from Helly's theorem (see [7], p. 372) that we can choose from $\{\varphi_n\}$ a subsequence $\{\varphi_{n_k}\}$ convergent in I to a function φ_0. Evidently, φ_0 is also convex and increasing in I. Moreover, we have

$$\varphi_{n_k}(x) = t_{n_k} h\big(x, \varphi_{n_k}[f(x)]\big), \quad k = 1, 2, \ldots \tag{8}$$

From the convexity of φ_0 its continuity in (ξ, b) results. From the convexity of φ_0 in $\langle \xi, b)$ in view of the conditions $\varphi_0(\xi) = \eta$, $\varphi_0(x) \geqslant \eta$ in $\langle \xi, b)$, we infer that φ_0 is continuous in $\langle \xi, b)$.

([1]) g is convex in $\Omega \subset \boldsymbol{R}^n$, if for every $\hat{x}$, $\hat{y} \in \Omega$ and $\lambda \in \langle 0, 1 \rangle$

$$g[\lambda \hat{x} + (1-\lambda)\hat{y}] \leqslant \lambda g(\hat{x}) + (1-\lambda) g(\hat{y}).$$

Since Ω is open, there exists an interval $I_0 = \langle \xi, a\rangle \subset I$ such that $I_0 \times \langle \eta, \varphi_0(a)\rangle \subset \Omega$. From (i) it follows that $f(x) < x$ and because φ_0 is increasing in I, for every $x \in I_0$ we have $\varphi_0[f(x)] \in \Omega_x$.

Passing the limit as $n \to \infty$, according to (8), we obtain

$$\varphi_0(x) = h\big(x, \varphi_0[f(x)]\big), \qquad x \in I_0,$$

i.e. $\varphi_0(x)$ is a convex solution of equation (1) in I_0. The function $\varphi_0(x)$, $x \in I_0$ has a unique extension φ onto the whole interval I and the solution φ is continuous in I (cf. [3], p. 70, Theorem 3.2). It is easily verified (as in [5]) that is also convex in I.

The uniqueness of such solutions follows from Theorem 1 in [1]. This completes the proof.

3. We shall prove the following

THEOREM 2. *Let hypotheses* (i), (ii), (iv) *be fulfilled. If, moreover, there exists an* $h_y(\xi, \eta)$ *and*

$$|f'(\xi)\cdot h_y(\xi, \eta)| = 1,$$

where ξ *is the left end point of* I *and* Ω *is bounded, then equation* (1) *possesses at least one convex solution in* I *fulfilling condition* (4).

Proof. It follows from [2] that for every n there exists a function $\varphi_n(x)$, convex, continuous and increasing in I, satisfying equation (5) in I and fulfilling condition (6).

Similarly as in Theorem 1, making use of Helly's theorem, we can prove that there exists a convex solution of equation (1) fulfilling condition (4), which was to be proved.

Suppose that:

(v) There exist positive constants a, β, s, k, l such that:

$$|f(x) - f(x_1)| \leqslant s\,|x - x_1| \quad \text{ for } x, x_1 \in \langle \xi - a, \xi + a\rangle \cap I,$$

$$|h(x, y) - h(x_1, y_1)| \leqslant k\,|x - x_1| + l\,|y - y_1|$$

$$\text{for } (x, x_1) \in \langle \xi - a, \xi + a\rangle, (y, y_1) \in \langle \eta - \beta, \eta + \beta\rangle,$$

$$(x, y), (x_1, y_1) \in \Omega,\ s\cdot l = 1.$$

(vi) Let $\xi = \eta = 0$ and $h(x, y) = Ax + By + o(|x| + |y|)$, $(x, y) \to (0, 0)$ and $|h(x, y_1) - h(x, y_2)| \leqslant L\,|y_1 - y_2|$ in a neighbourhood of $(0, 0)$, $L\cdot f'(0) = 1$.

Then we have

THEOREM 3. *Let hypotheses* (i), (ii), (iv), (v) *be fulfilled. If, moreover,* Ω *is bounded, then equation* (1) *possesses at least one convex solution in* I *fulfilling condition* (4).

THEOREM 4. *If hypotheses* (i), (ii), (iv), (vi) *are fulfilled, then equation* (1) *possesses at least one convex solution in* I *fulfilling condition* $\varphi(0) = 0$.

The proof of Theorems 3 and 4 in view of [6] and [4] respectively does not differ from that given in Theorems 1 and 2, and is therefore omitted.

Remark. In [4] it is proved that φ_n is continuous and convex in I but it is easily seen that φ_n is also increasing in I.

References

[1] D. Czaja-Pośpiech and M. Kuczma, *Continuous solutions of some functional equations in the indeterminate case*, Ann. Polon. Math. 24 (1970), p. 9–20.

[2] Z. Kominek and J. Matkowski, *On the existence of a convex solution of the functional equation*, $\varphi(x) = h(x, \varphi[f(x)])$, this volume, p. 1–4.

[3] M. Kuczma, *Functional equations in a single variable*, Monografie Matematyczne 46, Warszawa 1968.

[4] — *Special solutions of a functional equation*, Ann. Polon. Math. 27 (1972), p. 29–32.

[5] J. Matkowski, *On some properties of solutions of a functional equation*, Prace Matematyczne, Uniwersytet Śląski w Katowicach I (1969), p. 79–82.

[6] — *On the solution of a functional equation fulfilling a Lipschitz condition*, Ann. Polon. Math. (to appear).

[7] R. Sikorski, *Funkcje rzeczywiste*, t. 1, Monografie Matematyczne 35, Warszawa 1958.

Reçu par la Rédaction le 4. 3. 1971