COLLOQUIUM MATHEMATICUM

VOL. XXXI 1974 FASC. 2

THE TOPOLOGY OF THE UNIT INTERVAL IS NOT UNIQUELY DETERMINED BY ITS CONTINUOUS SELF MAPS AMONG SET SYSTEMS

BY

JIŘÍ ROSICKÝ (BRNO)

Šneperman [5] and Warndof [6] have proved that the usual topology of the closed unit interval $\langle 0,1\rangle$ is uniquely determined by its continuous self maps. It means that if $\mathfrak{T}$ is an arbitrary topology on $\langle 0,1\rangle$ (i. e., a system of subsets of $\langle 0,1\rangle$ closed under arbitrary unions and finite intersections) such that the system of continuous self maps with respect to $\mathfrak{T}$ is equal to the system of usual continuous self maps, then $\mathfrak{T}$ is the usual topology on $\langle 0,1\rangle$. Their method shows that the same assertion is true even if $\mathfrak{T}$ is a system of subsets of $\langle 0,1\rangle$ closed under finite unions. M. Sekanina has put the problem whether this assertion remains true if we take arbitrary system $\mathfrak{T}$ of subsets of $\langle 0,1\rangle$. The continuity of f is understood naturally: $f^{-1}X\in\mathfrak{T}$ for any $X\in\mathfrak{T}$. We shall show that the answer is negative. Namely, we shall construct for the usual topology on $\langle 0,1\rangle$, and even for any non-discrete completely regular locally connected topology, a sequence of mutually different systems such that their system of continuous self maps is equal to the system of original continuous self maps. A completely regular topology is supposed to be T_1.

BASIC DEFINITIONS

Denote by $\exp A$ the system of all subsets of a set A. Let $\mathfrak{A}\subseteq\exp A$. Put $\overline{\mathfrak{A}}=\{A-X \mid X\in\mathfrak{A}\}$, $\mathfrak{A}^0=\mathfrak{A}\cap\overline{\mathfrak{A}}$ and $\mathfrak{A}|C=\{X\cap C \mid X\in\mathfrak{A}\}$ for any $C\subseteq A$. We remind that $\mathfrak{A}$ is called a *topology* if it is closed under arbitrary unions and finite intersections. The closure and the frontier of $C\subseteq A$ in a topology $\mathfrak{A}$ will be denoted by $\mathrm{Cl}_{\mathfrak{A}}(C)$ and $\mathrm{Fr}_{\mathfrak{A}}(C)$ or briefly (if there is no danger of misunderstanding) by $\mathrm{Cl}(C)$ and $\mathrm{Fr}(C)$. If $\mathfrak{B}$ is a system of subsets of a set B, then $(\mathfrak{A},\mathfrak{B})=\{f\colon A\to B \mid f^{-1}X\in\mathfrak{A}$ for every $X\in\mathfrak{B}\}$ is the system of continuous maps from $\mathfrak{A}$ to $\mathfrak{B}$. We shall extend the notion of connectedness from topological spaces to arbitrary $\mathfrak{A}\subseteq\exp A$. Such an $\mathfrak{A}$ is defined to be *connected* if $\mathfrak{A}^0 \subseteq \{\emptyset, A\}$. Analogously to the topological case it may be easily shown that if $\mathfrak{A}$ is connected, $\mathfrak{B} \subseteq \exp B$, and $f \in (\mathfrak{A}, \mathfrak{B})$, then $\mathfrak{B} | f(A)$ is also connected.

CONSTRUCTION

Let $\mathfrak{A}$ be a topology on the set A and $X \subseteq A$. Let $X^0 = X$, and $X^{n+1} = \mathrm{Cl}_{\mathfrak{A}}(X^n) - X^n$ for every $n \in N$, where $N = \{0, 1, 2, \ldots\}$.

LEMMA 1. *Let* $n \geqslant 2$. *Then* $X^n = \mathrm{Cl}_{\mathfrak{A}}(X^{n-1}) \cap X^{n-2}$.

Proof. If $x \in X^n$, then $x \in \mathrm{Cl}(X^{n-1})$, $x \notin X^{n-1} = \mathrm{Cl}(X^{n-2}) - X^{n-2}$. Further, $\mathrm{Cl}(X^{n-1}) = \mathrm{Cl}\big(\mathrm{Cl}(X^{n-2}) - X^{n-2}\big) \subseteq \mathrm{Cl}(X^{n-2})$, whence $x \in X^{n-2}$. If $x \in \mathrm{Cl}(X^{n-1}) \cap X^{n-2}$, then $x \notin X^{n-1}$. Hence $x \in X^n$.

Put $F_n(\mathfrak{A}) = \{X \subseteq A \,|\, X^{n+1} = \emptyset$ and every component of $\mathfrak{A}|X$ belongs to $\overline{\mathfrak{A}}\}$ for every $n \in N$. Clearly, $F_0(\mathfrak{A}) = \overline{\mathfrak{A}}$ and $F_0(\mathfrak{A}) \subseteq F_1(\mathfrak{A}) \subseteq \ldots \subseteq F_n(\mathfrak{A}) \subseteq \ldots$

PROPOSITION 1. *Let* $\mathfrak{A}$ *be a topology on* A, *and* $\mathfrak{B}$ *on* B. *Then*

$$(\mathfrak{A}, \mathfrak{B}) \subseteq \big(F_n(\mathfrak{A}), F_n(\mathfrak{B})\big) \quad \text{for any } n \in N.$$

Proof. Let $f \in (\mathfrak{A}, \mathfrak{B})$, $n \in N$, $Y \in F_n(\mathfrak{B})$, and $X = f^{-1}Y$. Then $f(X^0) \subseteq Y^0$. Further, $f(X^1) = f\big(\mathrm{Cl}_{\mathfrak{A}}(X) - X\big) \subseteq \mathrm{Cl}_{\mathfrak{B}}(Y) - Y = Y^1$.

Let $i \geqslant 2$ and suppose that $f(X^{i-1}) \subseteq Y^{i-1}$, $f(X^{i-2}) \subseteq Y^{i-2}$. Then, by Lemma 1,

$$\begin{aligned} f(X^i) &= f\big(\mathrm{Cl}_{\mathfrak{A}}(X^{i-1}) \cap X^{i-2}\big) \subseteq f\big(\mathrm{Cl}_{\mathfrak{A}}(X^{i-1})\big) \cap f(X^{i-2}) \\ &\subseteq \mathrm{Cl}_{\mathfrak{B}}\big(f(X^{i-1})\big) \cap f(X^{i-2}) \subseteq \mathrm{Cl}_{\mathfrak{B}}(Y^{i-1}) \cap Y^{i-2} = Y^i. \end{aligned}$$

Therefore, $f(X^i) \subseteq Y^i$ for every $i \in N$. Since $Y^{n+1} = \emptyset$, we have $X^{n+1} = \emptyset$.

Let T be a component of $\mathfrak{A}|X$. Thus $f(T)$ is connected and there exists a component U of $\mathfrak{B}|Y$ with $f(T) \subseteq U$. Since $f^{-1}(U) \in \overline{\mathfrak{A}}$ and T is a component of $f^{-1}(U)$, we have $T \in \overline{\mathfrak{A}}$. Thus $X \in F_n(\mathfrak{A})$ and, therefore, $f \in \big(F_n(\mathfrak{A}), F_n(\mathfrak{B})\big)$.

LOCALLY CONNECTED COMPLETELY REGULAR TOPOLOGIES

LEMMA 2. *Let* $\mathfrak{A}$ *be a locally connected topology on* A, $Z \subseteq A$ *and every component of* Z *be closed in* $\mathfrak{A}$. *Then* $Z^1 \subseteq \mathrm{Cl}\big(\mathrm{Fr}(Z) \cap Z\big)$.

Proof. Let $x \in Z^1 = \mathrm{Cl}(Z) - Z$ and let U be a connected neighbourhood of x. Then $U \cap Z \neq \emptyset$. There exists a component T of Z such that $U \cap T \neq \emptyset$. By the supposition, $T \in \overline{\mathfrak{A}}$. Therefore $U \cap T$ is closed in $\mathfrak{A}|U$. Since U is connected, $U \cap T \notin \mathfrak{A}|U$. There exists a $t \in U \cap T$ with $t \in \mathrm{Cl}(U - T)$. Therefore $t \in \mathrm{Fr}(T)$. Since $\mathfrak{A}$ is locally connected and T a component of Z,

$\mathrm{Fr}(T) \subseteq \mathrm{Fr}(Z)$ (see [2]). Hence $t \in \mathrm{Fr}(Z) \cap Z$, i. e., $U \cap \mathrm{Fr}(Z) \cap Z \neq \emptyset$. From the local connectedness of $\mathfrak{A}$ it follows that $x \in \mathrm{Cl}\big(\mathrm{Fr}(Z) \cap Z\big)$.

Corollary 1. *Let $\mathfrak{A}$ be a locally connected topology on A. Then* $\mathfrak{A}^0 = F_n(\mathfrak{A})^0$ *for every* $n \in N$.

Proof. Let $n \in N$. Then $\mathfrak{A} \subseteq F_n(\mathfrak{A})$ implies $\mathfrak{A}^0 \subseteq F_n(\mathfrak{A})^0$. Let $X \in F_n(\mathfrak{A})^0$, $Y = A - X$. By Lemma 2,

$$Y^1 \subseteq \mathrm{Cl}\big(\mathrm{Fr}(Y) \cap Y\big) = \mathrm{Cl}\big(\mathrm{Cl}(X) \cap Y\big) = \mathrm{Cl}(X^1).$$

Further, $Y^1 \cap X^1 = \emptyset$ and, therefore, $Y^1 \subseteq X^2$. We have $X^2 \subseteq \mathrm{Cl}(X^1) \subseteq \mathrm{Cl}(Y)$ and $X^2 \cap Y = \emptyset$. Hence $Y^1 = X^2$, which implies $Y^{i+1} = X^{i+2}$ for all $i \in N$.

In the same way we can prove that $X^{i+1} = Y^{i+2}$ for all $i \in N$. Thereby we get $X, Y \in \bar{\mathfrak{A}}$, i. e. $X \in \mathfrak{A}^0$.

Corollary 2. *Let $\mathfrak{A}$ be a locally connected topology on A and let $X \in \mathfrak{A}$ be such that $\mathfrak{A}|X$ is connected. Then $F_n(\mathfrak{A})|X$ is connected for every $n \in N$.*

Proof. Let $f\colon X \to A$ be the inclusion mapping. Then $f \in (\mathfrak{A}|X, \mathfrak{A})$. By Proposition 1, $Z \cap X = f^{-1}Z \in F_n(\mathfrak{A}|X)$ for any $Z \in F_n(\mathfrak{A})$. Hence $F_n(\mathfrak{A})|X \subseteq F_n(\mathfrak{A}|X)$. The result follows from Corollary 1 and from the fact that an open subspace of a locally connected space is locally connected.

Let $\mathfrak{I}$ be the usual topology of the closed unit interval $I = \langle 0, 1\rangle$. Let $c, d \in I$ and $c < d$. Put

$$D(c, d) = \left\{c + \frac{d-c}{2i} \,\middle|\, i \in N, i > 0\right\} \cup \left\{c + (d-c)\frac{i}{i+1} \,\middle|\, i \in N, i > 0\right\}.$$

Clearly,

$$D(c, d) \subseteq \langle c, d\rangle, \qquad \big(D(c, d)\big)^1 = \{c, d\}.$$

Lemma 3. *Let $0 < a_1 \leqslant a_2 \leqslant 1$. Then for every $n \in N$ there exists a $D_n \in F_n(\mathfrak{I})$ with the following properties:*

1° $(D_{n+1})^1 = D_n$ *for every* $n \in N$.

2° $D_0 = \{0\}$, $a_1 \in D_1$.

3° *If $S \subseteq I$ is connected, $S \cap D_n \neq \emptyset$ and* $\mathrm{card}(S \cap \langle 0, a_2\rangle) > 1$, *then* $S \cap D_{n+1} \neq \emptyset$ *for every* $n \in N$.

Proof. We use the notation $D_n^i = (D_n)^i$. Sets D_n will be constructed by induction. Put $D_1 = D(0, a_2) \cup \{a_1, a_2\}$. Clearly, $D_1 \in F_1(\mathfrak{I})$, $D_1^1 = \{0\} = D_0$, and D_0, D_1 satisfy 3°. Moreover, every element x of the set

$$\big(D_1 - \mathrm{Cl}(D_1^1)\big) - \{a_2\} = D_1 - \{a_2\}$$

has the immediate predecessor x' in D_1 and the immediate successor x'' in D_1 (with respect to the usual order of real numbers).

Suppose that there exist $D_k \in F_k(\mathfrak{I})$ for every $1 \leqslant k \leqslant n$ such that $D_k^1 = D_{k-1}$, D_{k-1} satisfies 3°, every element $x \in E_k = \big(D_k - \mathrm{Cl}(D_{k+1})\big) - \{a_2\}$

has the immediate predecessor $x' \in D_k$ and the immediate successor $x'' \in D_k$, and $a_2 \in \mathrm{Cl}(D_k)$. Let

$$F = \bigcup_{x \in E_n} [D(x', x) \cup D(x, x'')].$$

It can be easily verified that $F' \subseteq \mathrm{Cl}(D_n)$.

Put $D_{n+1} = F \cup D_{n-1}$. Let T be a component of D_{n+1}. If $T \cap F \neq \emptyset$, then $\mathrm{card}(F) = 1$. If $T \cap F = \emptyset$, then $T \in \overline{\mathfrak{J}}$ by induction.

Let $t \in D^1_{n+1}$. Then $t \in \mathrm{Cl}(D_{n+1}) = \mathrm{Cl}(F) \cup \mathrm{Cl}(D^1_n) \subseteq \mathrm{Cl}(F) \cup \mathrm{Cl}(D_n)$. Since $t \notin D_{n+1} \supseteq F$ and $F^1 \subseteq \mathrm{Cl}(D_n)$, we have $t \in \mathrm{Cl}(D_n)$. Since $t \notin D_{n+1} \supseteq D_{n-1} = \mathrm{Cl}(D_n) - D_n$, we have $t \in D_n$. Conversely, let $t \in D_n$. If $t \in E_n$, we have $t \in D^1_{n+1}$ by construction. Then $\mathrm{Cl}(D_{n-1}) \subseteq \mathrm{Cl}(D_{n+1})$ and $D_n \cap D_{n+1} = \emptyset$. Hence $t \in \mathrm{Cl}(D_{n-1})$ implies $t \in D^1_{n+1}$. If $t = a_2$, then $a_2 \in D^1_{n+1}$ for $a_2 \in \mathrm{Cl}(D_{n-1})$. We have proved that $D^1_{n+1} = D_n$ and, therefore, $D_{n+1} \in F_{n+1}(\mathfrak{J})$.

Let $S \subseteq I$ be connected, $S \cap D_n \neq \emptyset$, and $\mathrm{card}(S \cap \langle 0, a_2 \rangle) > 1$. Now, $S \cap E_n \neq \emptyset$ implies $S \cap F \neq \emptyset$. Let $S \cap \mathrm{Cl}(D_{n-1}) \neq \emptyset$ and suppose that $S \cap D_{n+1} = \emptyset$. Then $S \cap D_{n-1} \neq \emptyset$ and, by the inductive assumption, $S \cap D_{n-1} \neq \emptyset$, a contradiction. If $a_2 \in S \cap D_n$, then $a_2 \in S \cap \mathrm{Cl}(D_{n-1}) \cap D_n \subseteq S \cap D^1_{n-1}$ and, therefore, $S \cap D_{n-1} \neq \emptyset$, i. e. $S \cap D_{n+1} \neq \emptyset$.

It remains to prove that every element $x \in E_{n+1}$ has the immediate predecessor (successor) x' $(x'') \in D_{n+1}$ and $a_2 \in \mathrm{Cl}(D_{n+1})$. The second assertion follows from $a_2 \in \mathrm{Cl}(D_{n-1})$. Let

$$x \in E_{n+1} = \big(D_{n+1} - \mathrm{Cl}(D_n)\big) - \{a_2\} = (F \cup D_{n-1}) - \mathrm{Cl}(D_n) \subseteq F$$
$$\text{for } D_{n-1} = D^1_n \subseteq \mathrm{Cl}(D_n).$$

By construction x has the immediate predecessor (successor) x' (x'') in D_{n+1}.

THEOREM 1. *Let $\mathfrak{A}$ be a locally connected topology on A and let $\mathfrak{B}$ be a completely regular topology on B. Then*

$$\big(F_n(\mathfrak{A}), F_n(\mathfrak{B})\big) = (\mathfrak{A}, \mathfrak{B}) \quad \textit{for every } n \in N.$$

Proof. Let $n \in N$. By Proposition 1, $(\mathfrak{A}, \mathfrak{B}) \subseteq \big(F_n(\mathfrak{A}), F_n(\mathfrak{B})\big)$. Suppose that there exists $f \in \big(F_n(\mathfrak{A}), F_n(\mathfrak{B})\big)$ such that $f \notin (\mathfrak{A}, \mathfrak{B})$. Then we can find a zero-set $X \in \overline{\mathfrak{B}}$ such that $Z = f^{-1}(X) \notin \overline{\mathfrak{A}}$. There exists a $b \in Z^1$. Further, $X = h^{-1}(0)$ for some $h \in (\mathfrak{B}, \mathfrak{J})$. Put $g = hf$. Clearly, $g \in \big(F_n(\mathfrak{A}), F_n(\mathfrak{J})\big)$. Let K be the component of $\mathfrak{J} | g(A)$ containing $g(b)$. Let U be a connected open neighbourhood of b in $\mathfrak{A}$. By Corollary 2, $F_n(\mathfrak{A}) | U$ is connected and, therefore, $F_n(\mathfrak{J}) | g(U)$ is also connected. Since $\overline{\mathfrak{J}} \subseteq F_n(\mathfrak{J})$, $\mathfrak{J} | g(U)$ is connected. We have $0 \in g(U)$ for $U \cap Z \neq \emptyset$. Hence $0 \in K$ and, therefore, $\mathrm{Cl}(K) = \langle 0, a_2 \rangle$ for some $a_2 \in I$. We have $0 < g(b) \leqslant a_2 \leqslant 1$. Take $D_k \in F_k(\mathfrak{J})$ from Lemma 3 for $k \leqslant n$ and $a_1 = g(b)$. Put $C = g^{-1}(D_n)$.

(α) $\mathrm{Fr}\big(g^{-1}(D_k)\big) \cap g^{-1}(D_k) \subseteq C^{n-k}$ for every $k \leqslant n$.

We have $\mathrm{Fr}(g^{-1}(D_n)) \cap g^{-1}(D_n) \subseteq g^{-1}(D_n) = C = C^0$. Let us take $t \in \mathrm{Fr}(g^{-1}(D_{n-1})) \cap g^{-1}(D_{n-1})$. Since $g(t) \in D_{n-1}$, we have $g(t) \notin D_n$, i. e. $t \notin C$.

Let U be a connected open neighbourhood of t. Analogously as above, it can be shown that $g(U)$ is connected in $\mathfrak{J}$. Therefore $g(U) \subseteq \langle 0, a_2 \rangle$. Since $t \in \mathrm{Fr}(g^{-1}(D_{n-1}))$, we have $U \cap g^{-1}(D_{n-1}) \neq \emptyset \neq U - g^{-1}(D_{n-1})$. Hence $\mathrm{card}\, g(U) > 1$. Since $g(U) \cap D_{n-1} \neq \emptyset$, it follows from 3° of Lemma 3 that $g(U) \cap D_n \neq \emptyset$, i. e. $U \cap C \neq \emptyset$. As $\mathfrak{A}$ is locally connected, $t \in \mathrm{Cl}(C)$. We have proved that $t \in C^1$, i. e. $\mathrm{Fr}(g^{-1}(D_{n-1})) \cap g^{-1}(D_{n-1}) \subseteq C^1$.

Let $0 \leqslant k < n-1$. Suppose that $\mathrm{Fr}(g^{-1}(D_i)) \cap g^{-1}(D_i) \subseteq C^{n-i}$ for every $k < i \leqslant n$. Let $t \in \mathrm{Fr}(g^{-1}(D_k)) \cap g^{-1}(D_k)$. Let U be a connected open neighbourhood of t. Then $g(U)$ is connected, $g(U) \subseteq \langle 0, a_2 \rangle$, $\mathrm{card}\, g(U) > 1$, and $g(U) \cap D_k \neq \emptyset$. By Lemma 3, $g(U) \cap D_{k+1} \neq \emptyset$. Therefore $t \in \mathrm{Cl}(g^{-1}(D_{k+1}))$. Since $t \in g^{-1}(D_k)$, $t \in (g^{-1}(D_{k+1}))^1$. Further, $D_{k+1} \in F_{k+1}(\mathfrak{J}) \subseteq F_n(\mathfrak{J})$. Since $g \in (F_n(\mathfrak{A}), F_n(\mathfrak{J}))$, we have $g^{-1}(D_{k+1}) \in F_n(\mathfrak{A})$. By Lemma 2,

$$t \in \mathrm{Cl}\left[\mathrm{Fr}(g^{-1}(D_{k+1})) \cap g^{-1}(D_{k+1})\right].$$

By the inductive assumption, $t \in \mathrm{Cl}(C^{n-k-1})$. In view of Lemma 1,

$$D_k = D_{k+2}^2 = \mathrm{Cl}(D_{k+2}^1) \cap D_{k+2} = \mathrm{Cl}(D_{k+1}) \cap D_{k+2}$$

and, therefore,

$$g^{-1}(D_k) = g^{-1}(\mathrm{Cl}(D_{k+1})) \cap g^{-1}(D_{k+2}).$$

Hence $t \in g^{-1}(D_{k+2})$. Since $t \in \mathrm{Cl}(g^{-1}(D_{k+1}))$ and $g^{-1}(D_{k+1}) \cap g^{-1}(D_{k+2}) = \emptyset$, we get $t \in \mathrm{Fr}(g^{-1}(D_{k+2}))$. Thus

$$t \in \mathrm{Fr}(g^{-1}(D_{k+2})) \cap g^{-1}(D_{k+2}) \subseteq C^{n-k-2}$$

by the inductive assumption. Therefore $t \in \mathrm{Cl}(C^{n-k-1}) \cap C^{n-k-2}$ and, by Lemma 1, $t \in C^{n-k}$. The proof of (α) is completed.

By Lemma 3 and (α) for $k = 0$ we have

$$\mathrm{Fr}(Z) \cap Z = \mathrm{Fr}(g^{-1}(0)) \cap g^{-1}(0) = \mathrm{Fr}(g^{-1}(D_0)) \cap g^{-1}(D_0) \subseteq C^n.$$

By Lemma 2, $Z^1 \subseteq \mathrm{Cl}(\mathrm{Fr}(Z) \cap Z)$. Hence $b \in \mathrm{Cl}(C^n) = C^n$ for $C^n \in F_n(\mathfrak{A})$. Further, $g(b) = a_1 \in D_1$ and $b \in g^{-1}(D_1)$. Consequently, $b \in \mathrm{Cl}(Z)$ and $Z \cap g^{-1}(D_1) = \emptyset$ implies

$$b \in \mathrm{Fr}(g^{-1}(D_1)) \cap g^{-1}(D_1) \subseteq C^{n-1}.$$

A contradiction to $C^n \cap C^{n-1} = \emptyset$. The proof is completed.

PROPOSITION 2. *Let $\mathfrak{A} \neq \exp A$ be a locally connected completely regular topology on A. Then $F_n(\mathfrak{A}) \subsetneqq F_{n+1}(\mathfrak{A})$ for every $n \in N$.*

Proof. Let $n \in N$. There exists a zero-set $X \in \overline{\mathfrak{A}}$ such that $X \notin \mathfrak{A}$, i. e. there exists an $x \in \mathrm{Fr}(X)$. Further, $X = h^{-1}(0)$ for some $h \in (\mathfrak{A}, \mathfrak{J})$. Let K be the component of $h(A)$ containing 0. Then $\mathrm{Cl}(K) = \langle 0, a \rangle$

for some $a \in I$. Let U be a connected open neighbourhood of x. Then $h(U)$ is connected, $0 \in h(U)$, and $\operatorname{card} h(U) > 1$. Therefore $0 < a$. Take $D_{n+1} \in F_{n+1}(\mathfrak{I})$ from Lemma 3 for $a_1 = a_2 = a$. Let $C = h^{-1}(D_{n+1}) \in F_{n+1}(\mathfrak{A})$. In the same way as ($\alpha$) in the proof of Theorem 1 it can be shown that

$$\emptyset \neq \operatorname{Fr}(X) \cap X = \operatorname{Fr}\big(g^{-1}(D_0)\big) \cap g^{-1}(D_0) \subseteq C^{n+1}.$$

Therefore, $C \notin F_n(\mathfrak{A})$.

Remarks. 1. The minimal T_1-topology $\mathfrak{A} = \{X \mid A - X \text{ is finite}\} \cup \{\emptyset\}$ on A is an example of a locally connected T_1-topology which is uniquely determined by continuous self maps among set systems (see [4]).

2. $(\mathfrak{I}, \mathfrak{I}) \subsetneqq \big(\bigcup_{n \in N} F_n(\mathfrak{I}),\ \bigcup_{n \in N} F_n(\mathfrak{I})\big)$.

Indeed,

$$f \in \big(\bigcup_{n \in N} F_n(\mathfrak{I}),\ \bigcup_{n \in N} F_n(\mathfrak{I})\big) - (\mathfrak{I}, \mathfrak{I}),$$

where

$$f(x) = \frac{1}{2}\left(1 + \sin\frac{1}{x}\right) \quad \text{for } x \in I,\ x \neq 0 \text{ and } f(0) = 0.$$

We are going to show that $F_n(\mathfrak{A})$ do not exhaust all systems of subsets of A with continuous self maps the same as for a non-discrete locally connected metric topology $\mathfrak{A}$ on A. Let $\mathfrak{A}$ be a topology on A. Define

$$\tilde{F}(\mathfrak{A}) = \left\{h^{-1}\left\{\frac{1}{n} \;\middle|\; n \in N,\ n \neq 0\right\} \;\middle|\; h \in (\mathfrak{A}, \mathfrak{I})\right\}.$$

PROPOSITION 3. *Let $\mathfrak{A}$ be a locally connected topology on A and $\mathfrak{B}$ a completely regular topology on B. Then* $\big(\tilde{F}(\mathfrak{A}), \tilde{F}(\mathfrak{B})\big) = (\mathfrak{A}, \mathfrak{B})$.

Proof. By the definition, $(\mathfrak{A}, \mathfrak{B}) \subseteq \big(\tilde{F}(\mathfrak{A}), \tilde{F}(\mathfrak{B})\big)$. Since $\tilde{F}(\mathfrak{A}) \subseteq F_1(\mathfrak{A})$ and $D_1 \in \tilde{F}(\mathfrak{I})$ for D_1 from Lemma 3, the proof of Proposition 3 follows from that of Theorem 1.

PROPOSITION 4. *Let $\mathfrak{A} \neq \exp A$ be a locally connected metric topology on A. Then* $F_0(\mathfrak{A}) \subsetneqq \tilde{F}(\mathfrak{A}) \subsetneqq F_1(\mathfrak{A})$.

Proof. Every closed set of a metric space is a zero-set and, therefore, $F_0(\mathfrak{A}) \subseteq \tilde{F}(\mathfrak{A})$. Now $F_0(\mathfrak{A}) \neq \tilde{F}(\mathfrak{A})$ can be proved analogously as Proposition 2, because $D_1 \in \tilde{F}(\mathfrak{I})$.

We claim that $\tilde{F}(\mathfrak{A}) \neq F_1(\mathfrak{A})$. Let $x \in A$ be non-isolated in A and U_1 a connected neighbourhood of x. By regularity and local connectedness of $\mathfrak{A}$ there exists a connected open neighbourhood U of x such that $\operatorname{Cl}(U) \subsetneqq U_1$. Since U_1 is connected, there exists

$$y \in \operatorname{Cl}(U) \cap \operatorname{Cl}\big(A - (U \cup \{y\})\big).$$

Hence we have a sequence $\{y_i \mid i \in N\} \subseteq A-(U \cup \{y\})$ converging to y. Clearly

$$X = \{x\} \cup \{y_i \mid i \in N\} \in F_1(\mathfrak{A}).$$

Suppose that $X \in \tilde{F}(\mathfrak{A})$. Then there exists an $h \in (\mathfrak{A}, \mathfrak{I})$ with

$$X = h^{-1}\left\{\frac{1}{n} \,\middle|\, 0 \neq n \in N\right\}.$$

Evidently, $h(y) = 0 \in \mathrm{Cl}\big(h(U)\big)$. Since $h(U)$ is connected, $U \cap X$ is infinite, a contradiction.

For the $\mathfrak{I}$-pattern of this proof I am indebted to M. Sekanina.

MORE ABOUT $F_1(\mathfrak{A})$

LEMMA 4. *Let $\mathfrak{A}$ be a topology on A and $\mathfrak{A}^0 = F_1(\mathfrak{A})^0$. Then every component of $\mathfrak{A}$ is open.*

Proof. Let $T \subseteq A$ be a component of A. Since $T \in \overline{\mathfrak{A}}$, we have $(A-T)^2 = \emptyset$. Clearly, any component of $A-T$ is a component of A and, therefore, is closed in $\mathfrak{A}$. Hence $A-T \in F_1(\mathfrak{A})$, i. e. $T \in F_1(\mathfrak{A})^0 = \mathfrak{A}^0 \subseteq \mathfrak{A}$.

PROPOSITION 5. *Let $\mathfrak{A}$ be topology on A. Consider the following statements:*

(i) *$\mathfrak{A}$ is locally connected.*

(ii) $(\mathfrak{A}|X)^0 = F_1(\mathfrak{A}|X)^0$ *for every* $X \in \mathfrak{A}$.

(iii) $(\mathfrak{A}|X, \mathfrak{A}|X) = \big(F_1(\mathfrak{A}|X), F_1(\mathfrak{A}|X)\big)$ *for every* $X \in \mathfrak{A}$.

Then (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i).

All statements are equivalent if $\mathfrak{A}$ is completely regular.

Proof. By Corollary 1, (i) $\Rightarrow$ (ii). A topology $\mathfrak{A}$, for which every component of any $X \in \mathfrak{A}$ is open, is locally connected (see [2]) and, therefore, (ii) $\Rightarrow$ (i) by Lemma 4. Further, (iii) $\Rightarrow$ (ii) (see [4]). If A is completely regular, (i) $\Rightarrow$ (iii) according to Theorem 1.

LEMMA 5. *Let $\mathfrak{A}$ be a metric continuum on A and $Z \in F_1(\mathfrak{A}) - \overline{\mathfrak{A}}$. Then*

$$Z^1 \cap \mathrm{Cl}\big(\mathrm{Fr}(Z) \cap Z\big) \neq \emptyset.$$

Proof. Let $x \in Z^1$. There exists a sequence $x_1, x_2, \ldots$ of points of Z converging to x. Let T_n be the component of Z containing x_n. All T_n are subcontinua of $\mathfrak{A}$. Therefore

$$\begin{aligned} H &= \operatorname*{Ls}_{n\to\infty} T_n \\ &= \{x \in A \mid \text{every neighbourhood of } x \text{ intersects infinitely many } T_n\} \end{aligned}$$

is a continuum (see [2], § 47, II, Theorem 6). Note that $H = (Z^1 \cap H) \cup (Z \cap H)$. Suppose that $Z \cap H \neq \emptyset$. Let T be a component of $Z \cap H$.

Since $Z \cap H \in F_1(\mathfrak{A}|H)$, we have $T \in \overline{\mathfrak{A}}|H$. By [2], § 47, III, Theorem 2, we have $\mathrm{Cl}_{\mathfrak{A}|H}(T) \cap \mathrm{Fr}_{\mathfrak{A}|H}(Z \cap H) \neq \emptyset$, i. e.

$$T \cap (Z^1 \cap H) = T \cap \mathrm{Cl}_{\mathfrak{A}|H}(Z^1 \cap H) \neq \emptyset,$$

which contradicts to $Z \cap Z^1 = \emptyset$. We have proved that $Z \cap H = \emptyset$, i. e. $H \subseteq Z^1$.

Further, $\mathrm{Cl}_{\mathfrak{A}}(T_n) \cap \mathrm{Fr}_{\mathfrak{A}}(Z) \neq \emptyset$ (see [2]), i. e. $T_n \cap \mathrm{Fr}_{\mathfrak{A}}(Z) \neq \emptyset$ for all n. Let $t_n \in T_n \cap \mathrm{Fr}_{\mathfrak{A}}(Z)$. Then $t_n \in \mathrm{Fr}_{\mathfrak{A}}(Z) \cap Z$ for all n. Since $\mathfrak{A}$ is compact, there exists an accumulation point of $\{t_n \mid n = 1, 2, \ldots\}$. Hence $\mathrm{Cl}_{\mathfrak{A}}(\{t_n \mid n = 1, 2, \ldots\}) \cap H \neq \emptyset$, i. e. $\mathrm{Cl}_{\mathfrak{A}}(\mathrm{Fr}_{\mathfrak{A}}(Z) \cap Z) \cap Z^1 \neq \emptyset$.

Corollary 3. *Let $\mathfrak{A}$ be a metric continuum on A. Then $F_1(\mathfrak{A})$ is connected.*

Proof. Let us take $X \in F_1(\mathfrak{A})$, $\emptyset \neq X \neq A$ and $Y = A - X$. Since $\mathrm{Cl}(\mathrm{Fr}(Y) \cap Y) = \mathrm{Cl}(X^1) = X^1$, we infer by Lemma 5 that $X^1 \cap Y^1 \neq \emptyset$, a contradiction.

Corollary 4. *Let $\mathfrak{A}$ be a metric continuum on A such that the set A_1 of all points of A in which $\mathfrak{A}$ is not locally connected forms a locally connected continuum. Then $(\mathfrak{A}, \mathfrak{A}) = (F_1(\mathfrak{A}), F_1(\mathfrak{A}))$.*

Proof. Let f, X and Z be as in the proof of Theorem 1. Since A_1 and $A_2 = A - A_1$ are locally connected, it follows from Theorem 1 that $Z \cap A_1 \in \overline{\mathfrak{A}}|A_1 \subseteq \overline{\mathfrak{A}}$ and $Z \cap A_2 \in \overline{\mathfrak{A}}|A_2$. Therefore $Z^1 \subseteq A_1$ and, by Lemma 5,

$$Z^1 \cap \mathrm{Cl}(\mathrm{Fr}(Z) \cap Z \cap A_2) = Z^1 \cap \mathrm{Cl}(\mathrm{Fr}(Z) \cap Z) \neq \emptyset.$$

The rest of the proof is the same as the last section of the proof of Theorem 1.

Problem. Does $(\mathfrak{A}, \mathfrak{A}) = (F_1(\mathfrak{A}), F_1(\mathfrak{A}))$ hold for any metric continuum $\mathfrak{A}$? (**P 904**)

An example of a continuum satisfying assumptions of Corollary 4:

$$\{(0, y) \mid -1 \leqslant y \leqslant 1\} \cup \left\{(x, y) \mid 0 < x \leqslant 1, y = \sin\frac{1}{x}\right\}.$$

Compactness is here essential as is shown by the following example of a connected metric topology $\mathfrak{A}$ with $(\mathfrak{A}, \mathfrak{A}) \neq (F_1(\mathfrak{A}), F_1(\mathfrak{A}))$:

Example. Let

$$A = \{(0, 0)\} \cup \left\{(x, y) \mid 0 < x \leqslant 1, y = \sin\frac{1}{x}\right\}$$

with the topology $\mathfrak{A}$ induced by the topology of the Euclidean plane. Let

$$B = \{(0, y) \mid -1 \leqslant y \leqslant 1\}$$

and let g_1 be the projection of A onto B. Clearly, g_1 is continuous.

Define $g_2 \in I^B$ as follows:

$$g_2(0, y) = \begin{cases} y+1 & \text{for } -\frac{1}{2} \geqslant y \geqslant -1, \\ \frac{1}{2} & \text{for } -\frac{1}{2} \leqslant y \leqslant \frac{1}{2}, \\ y & \text{otherwise.} \end{cases}$$

Clearly, g_2 is continuous and therefore $g = g_2 g_1 \in (\mathfrak{A}, \mathfrak{J})$. Let $f(t) = g(t)$ for $t \in A - \{(0, 0)\}$ and $f(0, 0) = 1$. Note that $f \in I^A$ and $f \notin (\mathfrak{A}, \mathfrak{J})$, because

$$(0, 0) \in \mathrm{Cl}\left(f^{-1}\left(\frac{1}{2}\right)\right) - f^{-1}\left(\frac{1}{2}\right).$$

We claim that $f \in \big(F_1(\mathfrak{A}), F_1(\mathfrak{J})\big)$. Let $X \in F_1(\mathfrak{J})$. If $\frac{1}{2}, 1 \in X$ or $\frac{1}{2}, 1 \notin X$, then $f^{-1}(X) = g^{-1}(X) \in F_1(\mathfrak{A})$. Let $\frac{1}{2} \in X$, $1 \notin X$. Then $f^{-1}(X) = g^{-1}(X) - \{(0, 0)\}$. Further, $\mathrm{Cl}\big(f^{-1}(X)\big) = \mathrm{Cl}\big(g^{-1}(X)\big)$. Hence

$$\big(f^{-1}(X)\big)^1 = \big(g^{-1}(X)\big)^1 \cup \{(0, 0)\} \in \bar{\mathfrak{A}}.$$

Since $1 \notin X$, every component of $f^{-1}(X)$ is closed in $\mathfrak{A}$. Therefore $f^{-1}(X) \in F_1(\mathfrak{A})$. Let $\frac{1}{2} \notin X$, $1 \in X$. Then $f^{-1}(X) = g^{-1}(X) \cup \{(0, 0)\}$. $\frac{1}{2} \notin X$ implies $(0, 0) \notin \mathrm{Cl}\big(g^{-1}(X)\big)$. Therefore $\big(f^{-1}(X)\big)^1 = \big(g^{-1}(X)\big)^1 \in \bar{\mathfrak{A}}$. Since $\frac{1}{2} \notin X$, every component of $f^{-1}(X)$ is closed in $\mathfrak{A}$. Therefore $f^{-1}(X) \in F_1(\mathfrak{A})$.

Since A contains an arc, $(\mathfrak{A}, \mathfrak{A}) \neq \big(F_1(\mathfrak{A}), F_1(\mathfrak{A})\big)$.

Note. This paper is essentially a continuation of the study of realizations of subcategories of the category of topological spaces in the category $\mathscr{S}^-$, begun in [1] and [4]. Objects of the category $\mathscr{S}^-$ are pairs $(A, \mathfrak{A})$, where $\mathfrak{A} \subseteq \exp A$ and $(\mathfrak{A}, \mathfrak{B})$ is the set of morphisms from $(A, \mathfrak{A})$ to $(B, \mathfrak{B})$. Clearly, the category of topological spaces and continuous maps is a full subcategory of $\mathscr{S}^-$. A realization F of a full subcategory $\mathscr{L}$ of $\mathscr{S}^-$ into $\mathscr{S}^-$ is a full embedding $F\colon \mathscr{L} \to \mathscr{S}^-$ preserving supports of objects and maps, i. e. $F\square' = \square$, where $\square\colon \mathscr{L} \to \mathrm{Ens}$ and $\square'\colon \mathscr{S}^- \to \mathrm{Ens}$ are the forgetful functors into the category of sets (cf. [3]).

Theorem 1 implies that F_n is a realization from the category of locally connected completely regular spaces into $\mathscr{S}^-$ for any n. By Proposition 2 restrictions of these realizations to any full subcategory of these spaces containing a non-discrete space are mutually different. Hence the category of locally connected metric spaces has at least countably many realizations in $\mathscr{S}^-$ while the category of metric spaces has only two realizations in $\mathscr{S}^-$ (see [4]).

REFERENCES

[1] J. Chvalina and M. Sekanina, *Realizations of closure spaces by set-systems,* Proceedings of the 3rd Prague Topological Symposium, Prague 1972.

[2] K. Kuratowski, *Topology,* Warszawa 1957.

[3] A. Pultr, *On selecting of morphisms among all mappings between underlying sets of objects in concrete categories and realizations of these,* Commentationes Mathematicae Universitatis Carolinae 8 (1) (1967), p. 53-83.

[4] J. Rosický and M. Sekanina, *Realizations of topologies by set systems,* Proceedings of the Colloquium on Topology, Keszthely 1972.

[5] Л. В. Шнеперман, *Полугруппы непрерывных преобразований топологических пространств,* Сибирский математический журнал 6 (1965), p. 221-229.

[6] J. C. Warndof, *Topologies uniquely determined by their continuous self maps,* Fundamenta Mathematicae 56 (1969), p. 25-43.

Reçu par la Rédaction le 17. 12. 1972;
en version modifiée le 1. 10. 1973