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Addendum and Corrigendum to the paper "Mean value theorems for a class of arithmetic functions"

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by

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The proof of Theorem 2 of our paper *Mean value theorems for a class of arithmetic functions*, requires correction. In the proof of (6.12) and again in the end argument on p. 256 we have assumed tacitly that C_0 is an absolute constant whereas, in fact, C_0 may depend on L. We are grateful to Dr. R. C. Vaughan for drawing our attention to the error.

Happily, this error may be repaired; in the process we shall even simplify the argument and arrive at a slightly better result.

We set out in Section 6 from Lemma 3, as before, but use (6.1) in place of (6.4) to obtain (instead of (6.5))

$$G(t,z)\log t = (\varkappa+1)T(t,z)-\varkappa T(t/z,z)+O\big(L/W(z)\big). \tag{6.5$'$}$$

Then, arguing as in the paper, we derive the 'reduction' formula

$$\frac{T(\xi,z)}{\log^{\varkappa+1}\xi} = \frac{T(w,z)}{\log^{\varkappa+1}w}-\varkappa\int_w^\xi \frac{T(t/z,z)}{t\log^{\varkappa+2}t}\,dt+O\left(\frac{1}{W(z)}\,\frac{L}{\log^{\varkappa+1}w}\right), \tag{6.6$'$}$$

$$2\leqslant w\leqslant \xi.$$

Now modify the definition of the remainder term $R(\xi,z)$ by

$$T(\xi,z) = \tfrac{1}{2}\sigma_\varkappa(2\tau_0)\frac{\log z}{W(z)}+R(\xi,z),\qquad \tau_0=\frac{\log\xi}{\log z}, \tag{6.7$'$}$$

whence (6.6′) leads, in place of (6.10), to

$$\frac{R(\xi,z)}{\log^{\varkappa+1}\xi} = \frac{R(w,z)}{\log^{\varkappa+1}w}-\varkappa\int_w^\xi \frac{R(t/z,z)}{t\log^{\varkappa+2}t}\,dt+O\left(\frac{1}{W(z)}\,\frac{L}{\log^{\varkappa+1}w}\right), \tag{6.10$'$}$$

$$z\leqslant w\leqslant \xi.$$

Our object will be to prove, not (6.9), but

$$R(\xi,z)\ll \frac{L}{W(z)}\,\tau_0^{2\varkappa+1}\log(\tau_0+1)\quad\text{ if }\quad \xi>z. \tag{6.9$'$}$$

As in the paper, we argue by induction on the range of τ_0; we begin by proving, for a suitable positive constant B_3, that

$$|R(t, z)| \leqslant \frac{B_3 L}{W(z)}, \quad 1 < t \leqslant z \tag{6.12'}$$

in place of (6.12). We have, for $1 < t \leqslant z$, by Lemma 3, Corollary, and (6.1), that

$$T(t, z) = T(t) = \frac{1}{\varkappa+1} G(t)\log t + O\left(\frac{L}{W(t)}\right)$$

$$= \frac{e^{-\gamma\varkappa}}{\Gamma(\varkappa+2)} \frac{\log t}{W(t)} + O\left(\frac{L}{W(t)}\right)$$

by Theorem 1, provided in the first instance that $\log t \geqslant B_1 L$; hence

$$T(t, z) = \frac{\log z}{W(z)} \cdot \frac{e^{-\gamma\varkappa}}{\Gamma(\varkappa+2)} \left(\frac{\log t}{\log z}\right) \frac{W(z)}{W(t)} + O\left(\frac{L}{W(t)}\right), \quad e^{B_1 L} \leqslant t \leqslant z,$$

$$= \frac{\log z}{W(z)} \cdot \frac{e^{-\gamma\varkappa}}{\Gamma(\varkappa+2)} \left(\frac{\log t}{\log z}\right)^{\varkappa+1} \left\{1 + O\left(\frac{L}{\log t}\right)\right\} + O\left(\frac{L}{W(t)}\right)$$

$$= \tfrac{1}{2}\sigma_\varkappa\left(2\frac{\log t}{\log z}\right) \frac{\log z}{W(z)} + O\left(\frac{L}{W(z)}\right), \quad e^{B_1 L} \leqslant t \leqslant z,$$

by (2.7) and (5.4). Comparing this with (6.7′), we see that (6.12′) is true if $e^{B_1 L} \leqslant t \leqslant z$. Next, if $1 < t < e^{B_1 L}$,

$$T(t, z) = T(t) = \int_1^t G(u) \frac{du}{u} \leqslant \int_1^t \frac{1}{W(u)} \frac{du}{u} \leqslant \frac{\log t}{W(z)} < \frac{B_1 L}{W(z)}$$

by (3.7) and (4.3); hence, in view of (6.7′) and (5.4), (6.12′) is true also if $1 < t < e^{B_1 L}$, and so for the entire range $1 < t \leqslant z$.

We derive (6.9′) from (6.10′) by showing that, for all integers $\nu \geqslant 2$,

$$\frac{|R(\xi, z)|}{\log^{\varkappa+1}\xi} \leqslant \frac{BL}{W(z)\log^{\varkappa+1} z} (\nu-1)^\varkappa \log\nu \quad \text{if} \quad \nu - 1 < \tau_0 \leqslant \nu; \tag{6.11'}$$

it is clear that (6.9′) follows if (6.11′) holds for all integers $\nu \geqslant 2$, since $\log\xi = \tau_0 \log z$. Begin by taking $\nu = 2$; then $z < \xi \leqslant z^2$ and (6.10′) with

$w = z$ implies, in view of (6.12′), that

$$\frac{|R(\xi, z)|}{\log^{\varkappa+1}\xi} \leqslant \frac{B_3 L}{W(z)} \left\{\frac{1}{\log^{\varkappa+1} z} + \varkappa \int_z^\xi \frac{dt}{t\log^{\varkappa+2} t} + \frac{B_4}{B_3} \frac{1}{\log^{\varkappa+1} z}\right\}$$

$$\leqslant \frac{B_3 L}{W(z)\log^{\varkappa+1} z} \left\{1 + \frac{\varkappa}{\varkappa+1} + \frac{B_4}{B_3}\right\}$$

where B_4 is the O-constant in (6.10′). Taking $B_3 \geqslant (\varkappa+1)B_4$, as we may, we obtain

$$\frac{|R(\xi, z)|}{\log^{\varkappa+1}\xi} \leqslant \frac{2B_3}{W(z)\log^{\varkappa+1} z}, \quad 1 < \tau_0 \leqslant 2,$$

which confirms (6.11′) with $\nu = 2$ if we take $B = 2B_3/\log 2$. Suppose now that $\nu \geqslant 2$, and (6.11′) holds. Let ξ satisfy

$$z^\nu < \xi \leqslant z^{\nu+1}$$

and take $w = z^\nu$ in (6.10′). Then, by (6.11′),

$$\frac{|R(\xi, z)|}{\log^{\varkappa+1}\xi}$$

$$\leqslant \frac{BL}{W(z)\log^{\varkappa+1} z} \left\{(\nu-1)^\varkappa \log\nu + \varkappa(\nu-1)^\varkappa \log\nu \int_{z^\nu}^\xi \frac{\log^{\varkappa+1}(t/z)}{t\log^{\varkappa+2} t} dt + \frac{B_4}{B\nu^{\varkappa+1}}\right\}$$

$$\leqslant \frac{BL}{W(z)\log^{\varkappa+1} z} \left\{(\nu-1)^\varkappa \log\nu + \frac{\varkappa}{\nu}(\nu-1)^\varkappa \log\nu + \frac{\log 2}{2(\varkappa+1)\nu^{\varkappa+1}}\right\}$$

$$\leqslant \frac{BL\log\nu}{W(z)\log^{\varkappa+1} z} \left\{(\nu-1)^\varkappa \left(1 + \frac{\varkappa}{\nu}\right) + \frac{1}{2\nu}\right\}$$

$$\leqslant \frac{BL}{W(z)} \frac{\nu^\varkappa \log(\nu+1)}{\log^{\varkappa+1} z},$$

which confirms the truth of (6.11′) with $\nu+1$ in place of ν.

This completes the proof of (6.11′), and hence of (6.9′).

If we now take $t = x$ in (6.5′) and suppose that $x > z$ we obtain, by (6.7′) and (6.9′),

$$G(x, z)\log x = \{(\varkappa+1)\sigma_\varkappa(2\tau) - \varkappa\sigma_\varkappa(2\tau-2)\}\tfrac{1}{2} \cdot \frac{\log z}{W(z)} +$$

$$+ O\left(\frac{L}{W(z)} \tau^{2\varkappa+1}\log(\tau+1)\right)$$

$$= \sigma_\varkappa(2\tau)\frac{\log x}{W(z)} + O\left(\frac{L}{W(z)} \tau^{2\varkappa+1}\log(\tau+1)\right)$$

by (¹) (5.5), so that

$$W(z)G(x, z) = \sigma_\varkappa(2\tau) + O\left(\frac{L}{\log x}\tau^{2\varkappa+1}\log(\tau+1)\right)$$

$$= \sigma_\varkappa(2\tau) + O\left(\frac{L}{\log z}\tau^{2\varkappa}\log(\tau+1)\right),$$

a result that is slightly better than Theorem 2.

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(¹) There is a misprint in (5.5), which should read

$$\sigma_\varkappa(u) = (\varkappa+1)\frac{\bar{\sigma}_\varkappa(u)}{u} - \varkappa\frac{\bar{\sigma}_\varkappa(u-2)}{u}, \quad u > 2.$$

There is another misprint on p. 250, line following (3.7), where $T(z)$ should be replaced by $T(x)$.