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Periodic analogues of the Euler–Maclaurin and Poisson summation formulas with applications to number theory

by

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1. Introduction. Let $A = \{a_n\}$, $-\infty < n < \infty$, be a sequence of complex numbers with period $k > 0$; thus $a_{n+k} = a_n$ for every integer n. Our first major result is found in Section 3, where we develop a formula for $\sum_{c<n\leqslant d} a_n f(n)$ which is analogous to the Euler–Maclaurin summation formula, where f is a sufficiently smooth, complex-valued function on $[c, d]$. When $A = I \equiv \{1\}$ and $k = 1$, our result reduces to the ordinary Euler–Maclaurin formula. When $A = I_1 \equiv \{(-1)^n\}$ and $k = 2$, we obtain Boole's summation formula. When $a_n = \chi(n)$, a character of modulus k, we obtain results due to Davies and Haselgrove [19], Chowla [17], and Berndt [9]. Another approach to our periodic summation formula has been given by Rosser and Schoenfeld [40].

Appearing in our formulae are certain numbers and functions which generalize the ordinary Bernoulli numbers and functions, respectively, and which reduce to them for $A = I$ and $k = 1$. Properties of these periodic Bernoulli numbers and functions are developed in Sections 2 and 9. These properties are well known in the classical case $A = I$, and many of them have been found in recent years in the case $a_n = \chi(n)$ by Leopoldt [31] and others who use them in connection with various problems in algebraic number theory.

A second major result is obtained in Section 4 where we derive a formula for $\sum_{c\leqslant n\leqslant d} a_n f(n)$ which generalizes the classical Poisson summation formula. We obtain this periodic Poisson formula from the ordinary Poisson formula and then use it to derive a second proof of our generalization of the Euler–Maclaurin formula. The periodic version of the Poisson summation formula was previously obtained by Berndt [9] for $a_n = \chi(n)$

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in a more difficult fashion by appealing to a version of the Voronoï summation formula. See also [5] and [6] by Berndt.

In Section 5, we apply our periodic Poisson formula to prove a reciprocity formula for certain exponential sums which, by specialization, yields the well-known reciprocity formula for generalized Gaussian sums.

In Section 6, we introduce a generalized Lerch function which includes, as special cases, the Riemann zeta-function $\zeta(s)$, the Dirichlet L-series $L(s,\chi)$, the Hurwitz zeta-function $\zeta(s,a)$, and the Lerch function $\varphi(x,a,s)$. For this periodic Lerch function we obtain a functional equation which implies those of $\zeta(s)$, $L(s,\chi)$, $\zeta(s,a)$, and $\varphi(x,a,s)$. To obtain the result, we use the periodic Euler–Maclaurin summation formula.

In Section 7, we introduce periodic theta-functions and apply our periodic Poisson formula to obtain a transformation formula which reduces to that given by Epstein [22] when $A=I$. From this we obtain a second proof of the reciprocity formula for the exponential sums mentioned above.

In Section 8, we give a periodic form of the Lipschitz summation formula. Here the proof is based upon the periodic Poisson formula in a manner similar to that used by Berndt [9] in the case of a primitive character.

Section 10 is devoted to the generalization of some curious identities for trigonometric and hyperbolic functions discovered by Berndt [9] in the case $a_n=\chi(n)$.

The final section is concerned with certain formulas for numerical integration. The formulas give what may be regarded as correction terms as well as explicit formulas for the errors based on the remainder integral for the periodic Euler–Maclaurin formula.

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2. Periodic Bernoulli numbers and functions

DEFINITION 1. The *periodic Bernoulli numbers* $B_n(A)$, $0\leqslant n<\infty$, and *periodic Bernoulli functions* $P_n(x,A)$, $0\leqslant n<\infty$, are defined recursively as follows. Let

$$P_0(x,A)=B_0(A)=\frac{1}{k}\sum_{n=0}^{k-1}a_n, \tag{2.1}$$

$$B_1(A)=\frac{1}{k}\sum_{n=0}^{k-1}\left(n-\frac{1}{2}k\right)a_n=\frac{1}{k}\sum_{n=0}^{k-1}na_n-\frac{1}{2}kB_0(A), \tag{2.2}$$

and for $x\geqslant 0$,

$$P_1(x,A)=B_0(A)x-B_1(A)-\mathscr{A}(x), \tag{2.3}$$

where

$$\mathscr{A}(x)=\sum_{0\leqslant n\leqslant x}a_n.$$

For $n\geqslant 2$ and $x\geqslant 0$, let $B_n(A)$ and $P_n(x,A)$ be defined inductively as follows:

$$B_n(A)=\frac{1}{k}(-1)^{n+1}n!\int_0^k(k-u)P_{n-1}(u,A)\,du \tag{2.4}$$

and

$$P_n(x,A)=\int_0^x P_{n-1}(u,A)\,du+\frac{(-1)^n}{n!}B_n(A). \tag{2.5}$$

PROPOSITION 2.1. *For* $n\geqslant 1$,

$$\int_0^k P_n(x,A)\,dx=0.$$

Proof. For $n=1$ by (2.3),

$$\int_0^k P_1(x,A)\,dx=\tfrac{1}{2}B_0(A)k^2-B_1(A)k-\sum_{0\leqslant n\leqslant k}a_n\int_n^k dx$$
$$=\tfrac{1}{2}B_0(A)k^2-B_1(A)k-k\sum_{n=0}^{k-1}a_n+\sum_{n=0}^{k-1}na_n=0,$$

by (2.1) and (2.2).

For $n\geqslant 2$ by (2.5),

$$\int_0^k P_n(x,A)\,dx=\int_0^k dx\int_0^x P_{n-1}(u,A)\,du+\frac{(-1)^n}{n!}B_n(A)k$$
$$=\int_0^k P_{n-1}(u,A)\,du\int_u^k dx+\frac{(-1)^n}{n!}B_n(A)k=0,$$

by (2.4), and the proof is complete.

PROPOSITION 2.2. *For* $n\geqslant 0$ *and* $x\geqslant 0$, $P_n(x,A)$ *has period* k.

Proof. For $n=1$ and $x\geqslant 0$,

$$P_1(x+k,A)-P_1(x,A)=B_0(A)k-\sum_{x<n\leqslant x+k}a_n=B_0(A)k-\sum_{n=0}^{k-1}a_n=0,$$

by (2.1). Proceeding by induction on n, we have for $n \geqslant 2$ and $x \geqslant 0$ by (2.5),

$$P_n(x+k, A) - P_n(x, A) = \int_x^{x+k} P_{n-1}(u, A)\,du = \int_0^k P_{n-1}(u, A)\,du = 0,$$

upon the use of the induction hypothesis and Proposition 2.1.

We now define the periodic Bernoulli functions $P_n(x, A)$ for all real x by periodicity. Then (2.5) is seen to hold for all x. Observe that for $n \geqslant 2$, $P_n(x, A)$ is continuous for all x. $P_1(x, A)$ is continuous except at those integers n where $a_n \neq 0$, in which cases $P_1(x, A)$ is continuous from the right. Furthermore, it is apparent from (2.5) and (2.3) that

$$(2.6)\qquad \begin{cases} P'_{n+1}(x, A) = P_n(x, A), & n \geqslant 2, \ -\infty < x < \infty, \\ P'_2(x, A) = P_1(x, A), & x \neq n \text{ if } a_n \neq 0, \\ P'_1(x, A) = B_0(A) = P_0(x, A), & x \neq n \text{ if } a_n \neq 0. \end{cases}$$

Lastly, observe that $P_n(x, A)$ is a polynomial on each interval $[N, N+1)$, where N is an integer. The degree of each polynomial is less than or equal to n and exactly equal to n if $B_0(A) \neq 0$.

In the special case $A = I$ and $k = 1$, it is not difficult to prove that $B_1(I) = -1/2 = B_1$ and $P_1(x, I) = x - [x] - 1/2 = P_1(x)$, where B_n and $P_n(x)$ are the nth ordinary Bernoulli numbers and functions, respectively; see Knopp's book [28], pp. 521–523. In general, perhaps the simplest proof that $P_n(x, I) = P_n(x)$ and $B_n(I) = B_n$, $n \geqslant 2$, is given by (4.8) below.

For later use, we note that if $x \geqslant -1$ then

$$\begin{aligned} P_1(x, A) &= P_1(x+k, A) = B_0(A)(x+k) - B_1(A) - \sum_{0 \leqslant n \leqslant x+k} a_n \\ &= B_0(A)x + kB_0(A) - B_1(A) - \sum_{0 \leqslant n \leqslant k-1} a_n - \sum_{k-1 < n \leqslant k+x} a_n \\ &= B_0(A)x - B_1(A) - \sum_{-1 < m \leqslant x} a_m, \end{aligned}$$

where we set $m = n-k$ and used the fact that $a_{m+k} = a_m$. Hence,

$$(2.7)\qquad P_1(x, A) = \begin{cases} B_0(A)x - B_1(A), & \text{if } -1 \leqslant x < 0, \\ B_0(A)x - B_1(A) - a_0, & \text{if } 0 \leqslant x < 1, \\ B_0(A)x - B_1(A) - a_0 - a_1, & \text{if } 1 \leqslant x < 2. \end{cases}$$

In particular,

$$(2.8)\qquad P_1(0, A) = -B_1(A) - a_0.$$

Furthermore, for a positive integer m, we have

$$P_1(m-, A) = B_0(A)m - B_1(A) - \sum_{0 \leqslant n \leqslant m-1} a_n = P_1(m, A) + a_m.$$

By periodicity, the above is valid for all integers m. Hence,

$$(2.9)\qquad P_1(m-, A) = P_1(m, A) + a_m,$$

$$(2.10)\qquad P_1(m+, A) = P_1(m, A),$$

and

$$(2.11)\qquad \tfrac{1}{2}\{P_1(m-, A) + P_1(m+, A)\} = P_1(m, A) + \tfrac{1}{2}a_m.$$

Finally, (2.5) and (2.1) give

$$(2.12)\qquad P_n(0, A) = \frac{(-1)^n}{n!} B_n(A) \quad \text{if} \quad n \geqslant 2 \text{ or } n = 0.$$

3. The periodic Euler–Maclaurin summation formula

THEOREM 3.1. *Let $f \in C^{(r)}[c, d]$ where $r \geqslant 1$ and c and d are real. Then for each real number θ,*

$$(3.1)\qquad \begin{aligned} \sum_{c < n+\theta \leqslant d} a_n f(n+\theta) &= B_0(A) \int_c^d f(x)\,dx + \\ &+ \sum_{j=1}^{r} (-1)^j \{P_j(d-\theta, A) f^{(j-1)}(d) - P_j(c-\theta, A) f^{(j-1)}(c)\} + E_r(\theta, A), \end{aligned}$$

where

$$E_r(\theta, A) = (-1)^{r+1} \int_c^d P_r(x-\theta, A) f^{(r)}(x)\,dx.$$

Proof. For simplicity, we begin by assuming that $c \geqslant 0$ and $\theta = 0$. With the help of (2.3) and an integration by parts, we find that

$$\begin{aligned} E_1(0, A) &= \int_c^d P_1(x, A) f'(x)\,dx \\ &= \{B_0(A)d - B_1(A)\}f(d) - \{B_0(A)c - B_1(A)\}f(c) - \\ &\qquad - B_0(A)\int_c^d f(x)\,dx - \sum_{0 \leqslant n \leqslant d} a_n \int_{\max(c,n)}^d f'(x)\,dx. \end{aligned}$$

The sum on the right side above is

$$\begin{aligned} &\sum_{0 \leqslant n \leqslant d} a_n f(d) - \Big\{\sum_{0 \leqslant n \leqslant c} a_n f(c) + \sum_{c < n \leqslant d} a_n f(n)\Big\} \\ &\qquad = f(d)\mathscr{A}(d) - f(c)\mathscr{A}(c) - \sum_{c < n \leqslant d} a_n f(n). \end{aligned}$$

Hence,

$$E_1(0, A) = \{B_0(A)d - B_1(A) - \mathscr{A}(d)\}f(d) - $$
$$- \{B_0(A)c - B_1(A) - \mathscr{A}(c)\}f(c) - B_0(A)\int_c^d f(x)\,dx + \sum_{c<n\leqslant d} a_n f(n)$$
$$= P_1(d, A)f(d) - P_1(c, A)f(c) - B_0(A)\int_c^d f(x)\,dx + \sum_{c<n\leqslant d} a_n f(n).$$

Rearranging the above, we obtain

$$\sum_{c<n\leqslant d} a_n f(n) = B_0(A)\int_c^d f(x)\,dx - P_1(d, A)f(d) + P_1(c, A)f(c) + E_1(0, A),$$

which is (3.1) for $r = 1$, $c \geqslant 0$ and $\theta = 0$.

For general c and θ, let N be a positive integer chosen large enough so that $c_1 = c - \theta + kN \geqslant 0$. Furthermore, let $d_1 = d - \theta + kN$ and $f_1(x) = f(x + \theta - kN)$ so that $f_1 \in C^{(r)}[c_1, d_1]$. By applying the preceding case to f_1 on $[c_1, d_1]$ and using the periodicity of $\{a_n\}$ and $P_1(x, A)$, we obtain

$$(3.2)\qquad \sum_{c<n+\theta\leqslant d} a_n f(n+\theta)$$
$$= B_0(A)\int_c^d f(x)\,dx - P_1(d-\theta, A)f(d) + P_1(c-\theta, A)f(c) + E_1(\theta, A),$$

which is (3.1) for $r = 1$.

Integrating by parts with the help of (2.6), we obtain for $1 \leqslant j \leqslant r-1$

$$(3.3)\qquad E_j(\theta, A) = (-1)^{j+1}\int_c^d P'_{j+1}(x-\theta, A)f^{(j)}(x)\,dx$$
$$= (-1)^{j+1}\{P_{j+1}(d-\theta, A)f^{(j)}(d) - P_{j+1}(c-\theta, A)f^{(j)}(c)\} + E_{j+1}(\theta, A).$$

From (3.2) and an easy inductive argument with the aid of (3.3), we easily obtain (3.1) for all $r \geqslant 1$.

COROLLARY 3.2. *Let $f \in C^{(r)}[c, \infty)$ where $r \geqslant 1$ and c and θ are real. Suppose also that*

$$\int_c^\infty P_r(x-\theta, A)f^{(r)}(x)\,dx$$

converges. Then there exists a constant $C_r(\theta, A)$ such that for all $d \geqslant c$,

$$(3.4)\qquad \sum_{c<n+\theta\leqslant d} a_n f(n+\theta)$$
$$= B_0(A)\int_c^d f(x)\,dx + C_r(\theta, A) + \sum_{j=1}^r (-1)^j P_j(d-\theta, A)f^{(j-1)}(d) - F_r(d; \theta, A),$$

where

$$(3.5)\qquad F_r(d; \theta, A) = (-1)^{r+1}\int_d^\infty P_r(x-\theta, A)f^{(r)}(x)\,dx.$$

Furthermore, if $1 \leqslant m \leqslant r$ and if $f^{(m-1)}(x), f^{(m)}(x), \ldots, f^{(r-1)}(x)$ all tend to 0 as $x \to \infty$, then

$$(3.6)\qquad C_{m-1}(\theta, A) = C_m(\theta, A) = \ldots = C_r(\theta, A).$$

Proof. We obtain (3.4) from (3.1) on defining

$$(3.7)\qquad C_r(\theta, A) = -\sum_{j=1}^r (-1)^j P_j(c-\theta, A)f^{(j-1)}(c) + F_r(c; \theta, A).$$

Furthermore, replacing j by $l-1$ in (3.3), where $m \leqslant l \leqslant r$, and then letting $d \to \infty$, we get with the use of (3.5)

$$(3.8)\qquad F_{l-1}(c; \theta, A) = -(-1)^l P_l(c-\theta, A)f^{(l-1)}(c) + F_l(c; \theta, A),$$

provided that either F_l or F_{l-1} exists. As F_r exists by hypothesis, so do, in turn, $F_{r-1}, F_{r-2}, \ldots, F_{m-1}$. Consequently, for $m \leqslant l \leqslant r$ we obtain from (3.7) and (3.8)

$$C_r(\theta, A) - C_{l-1}(\theta, A)$$
$$= -\sum_{j=l}^r (-1)^j P_j(c-\theta, A)f^{(j-1)}(c) + F_r(c; \theta, A) - F_{l-1}(c; \theta, A)$$
$$= \sum_{j=l}^r \{F_{j-1}(c; \theta, A) - F_j(c; \theta, A)\} + F_r(c; \theta, A) - F_{l-1}(c; \theta, A) = 0,$$

and the proof is complete.

In actual cases, the constant $C_r(\theta, A)$ is usually approximated by using (3.4) for a moderately large value of d chosen so that $F_r(d; \theta, A)$ is rather small.

It may be noted that although the proofs of (3.1) and (3.4) used the periodicity of $P_1(x, A)$, they did not use the periodicity of $P_n(x, A)$ for $n \geqslant 2$. Hence the results hold for any set of functions $P_n^*(x)$ such that $P_1^*(x) = P_1(x, A)$ and $\frac{d}{dx}P_{n+1}^*(x) = P_n^*(x)$ for all $n \geqslant 1$. For the case $k = 1$, a slightly altered form of (3.1) has been given by L. K. Hua [26], p. 79.

COROLLARY 3.3. *Let $f \in C^{(r)}[Mk, Nk]$, where $r \geqslant 1$, M and N are integers, and $0 \leqslant \theta \leqslant 1$. Then*

$$(3.9)\quad \sum_{Mk \leqslant n < Nk} a_n f(n+\theta) = B_0(A)\int_{Mk}^{Nk} f(x)\,dx + \{B_0(A)\theta + B_1(A)\}\times$$
$$\times\{f(Nk)-f(Mk)\} + \sum_{j=2}^{r}(-1)^j P_j(-\theta, A)\{f^{(j-1)}(Nk) - f^{(j-1)}(Mk)\} + E_r(\theta, A),$$

where

$$E_r(\theta, A) = (-1)^{r+1}\int_{Mk}^{Nk} P_r(x-\theta, A) f^{(r)}(x)\,dx.$$

Proof. We first let $0 < \theta \leqslant 1$. In this case, if c and d are integers, we have

$$\sum_{c < n+\theta \leqslant d} a_n f(n+\theta) = \sum_{c \leqslant n < d} a_n f(n+\theta) = \sum_{Mk \leqslant n < Nk} a_n f(n+\theta)$$

on putting $c = Mk$ and $d = Nk$. Also, for $j \geqslant 1$, the periodic ity of P gives

$$P_j(Nk-\theta, A) = P_j(-\theta, A) = P_j(Mk-\theta, A).$$

Furthermore, since $0 < \theta \leqslant 1$, (2.7) gives

$$P_1(-\theta, A) = -B_0(A)\theta - B_1(A).$$

The result now follows from (3.1) provided that $0 < \theta \leqslant 1$.

The integrand of $E_r(\theta, A)$ is bounded for bounded θ and x; and, as $\theta \to 0+$, the integrand tends uniformly to $P_r(x, A) f^{(r)}(x)$ except, possibly, in neighborhoods of integral values of x. Hence, $E_r(\theta, A)$ tends to $E_r(0, A)$ as $\theta \to 0+$. Since the rest of the terms in (3.9) are also right-continuous at $\theta = 0$, it follows that (3.9) holds for $\theta = 0$ as well.

COROLLARY 3.4. *Let f be as in Corollary 3.3. Then*

$$(3.10)\quad \sum_{Mk \leqslant n < Nk} a_n f(n)$$
$$= B_0(A)\int_{Mk}^{Nk} f(x)\,dx + \sum_{j=1}^{r}\frac{B_j(A)}{j!}\{f^{(j-1)}(Nk) - f^{(j-1)}(Mk)\} + E_r(0, A).$$

Proof. Using (2.12) and setting $\theta = 0$ in (3.9), we deduce (3.10) at once.

We refrain from attempting to give any historical account of the classical Euler–Maclaurin formula for $A = I$. Some historical material may be found in a paper of Ostrowski [38]. Other versions of the formula

have been found by Duncan [21] and Mahler [34]. There is also another classical formula which Jordan [27] calls Boole's first summation formula; see Boole's treatise [14] (1st ed., p. 95; 3rd ed., p. 102). In fact, this result appears to be due to Euler; see Burkhardt's article in the Encyklopädie [15], p. 1335. In our work, the formula results from taking $k = 2$ and $A = I_1 = \{(-1)^n\}$. Nörlund [37], p. 34, proves a result from which Boole's result can be obtained by addition. In Nörlund's development, the result is obtained by means of the Euler polynomials in much the same way as the ordinary Euler–Maclaurin formula is approached through the classical Bernoulli polynomials. It is also of interest to note that Euler's transformation of series

$$\sum_{n=0}^{\infty}(-1)^n f(n) = \sum_{m=0}^{\infty}\frac{(-1)^m}{2^{m+1}}\Delta^m f(0),$$

where f satisfies suitable conditions, can be obtained from Boole's formula by replacing the derivatives by their forward difference approximations $\Delta^m f(0)$. For a discussion of this transformation, see Knopp's book [28], p. 244, or Hildebrand's book [25], p. 157.

In the proof below we cite two results proved later, namely, (4.9) and (9.11). It is clear, however, that no circularity is involved. We give the result corresponding to Corollary 3.4 rather than the one associated with other corollaries because of the greater simplicity.

COROLLARY 3.5 (Boole's summation formula). *Let $f \in C^{(2r)}[2M, 2N]$, where $r \geqslant 1$ and M and N are integers. Then*

$$\sum_{n=2M}^{2N}(-1)^n f(n) = \frac{1}{2}\{f(2N)+f(2M)\} +$$
$$+ \sum_{j=1}^{r}(4^j-1)\frac{B_{2j}}{(2j)!}\{f^{(2j-1)}(2N) - f^{(2j-1)}(2M)\} + E_r(0, I_1),$$

where

$$E_r(0, I_1) = -\int_{2M}^{2N}\{4^r P_{2r}(x/2) - P_{2r}(x)\} f^{(2r)}(x)\,dx.$$

Proof. It is clear that $P_0(x, I_1) \equiv 0 = B_0(I_1)$ and that $B_1(I_1) = -1/2$. For $n \geqslant 2$, (4.9) and (9.11) yield

$$P_n(x, I_1) = 2^{n-1}\left\{P_n\left(\frac{x}{2}\right) - P_n\left(\frac{x+1}{2}\right)\right\} = 2^n P_n\left(\frac{x}{2}\right) - P_n(x).$$

Consequently, (2.12) yields for $n \geqslant 2$

$$B_n(I_1) = (-1)^n n! P_n(0, I_1) = (-1)^n n!\{2^n P_n(0) - P_n(0)\} = (2^n - 1)B_n,$$

where B_n denotes the nth ordinary Bernoulli number. Since $B_n = 0$ for odd $n \geq 2$, we obtain the above result after replacing k by 2, a_n by $(-1)^n$, r by $2r$, and j by $2j$ in (3.10), and then adding $f(2N)$ to both sides.

It is fairly clear that Corollary 3.5 is obtainable from the classical Euler–Maclaurin formula. Indeed, we need only observe that

$$(3.11)\qquad \sum_{n=2M}^{2N}(-1)^n f(n) = \sum_{m=M}^{N-1} f(2m) - \sum_{m=M}^{N-1} f(2m+1) + f(2N) = 2\sum_{m=M}^{N-1} f(2m) - \sum_{n=2M}^{2N-1} f(n) + f(2N),$$

and then apply Corollary 3.4 with $A = I$ to both sums on the far right side of (3.11). In fact, the whole theory of the periodic Euler–Maclaurin formula can be developed from the classical form of Corollary 3.3 for general $\theta \in [0, 1]$. For, by subdividing the range of summation into residue classes modulo k, we get

$$\sum_{n=Mk}^{Nk-1} a_n f(n+\theta) = \sum_{j=0}^{k-1}\sum_{m=M}^{N-1} a_{mk+j} f(mk+j+\theta) = \sum_{j=0}^{k-1} a_j \sum_{m=M}^{N-1} g(m+\theta_j),$$

where $\theta_j = (j+\theta)/k \in [0, 1]$ and $g(x) = f(kx)$. This approach is adopted in the work of Rosser and Schoenfeld [40].

4. The periodic Poisson summation formula. Define the sequence $B = \{b_n\}$, $-\infty < n < \infty$, by

$$(4.1)\qquad b_n = \frac{1}{k}\sum_{j=0}^{k-1} a_j e^{-2\pi ijn/k}.$$

These are the finite Fourier series coefficients of $\{a_n\}$. Observe that $b_0 = B_0(A)$. Clearly, B also has period k. Note that (4.1) holds if and only if

$$(4.2)\qquad a_n = \sum_{j=0}^{k-1} b_j e^{2\pi ijn/k}, \qquad -\infty < n < \infty.$$

Also, if we replace b_n by c_n and a_j by b_j in (4.1), then we find that $c_n = a_{-n}/k$.

THEOREM 4.1. *For $r \geq 1$ and x real,*

$$(4.3)\qquad \tfrac{1}{2}\{P_r(x+, A) + P_r(x-, A)\} = -\sum_{n=1}^{\infty} (k/2\pi in)^r \{b_n e^{2\pi inx/k} + (-1)^r b_{-n} e^{-2\pi inx/k}\}.$$

Unless $r = 1$ and x is an integer such that $a_x \neq 0$, the left side of (4.3) may be replaced by $P_r(x, A)$.

Proof. Since $P_r(x, A)$ is of bounded variation on every finite interval, $P_r(x, A)$ may be expanded in a Fourier series:

$$(4.4)\qquad \tfrac{1}{2}\{P_r(x+, A) + P_r(x-, A)\} = \lim_{N\to\infty}\sum_{n=-N}^{N} c_{n,r} e^{2\pi inx/k},$$

where

$$c_{n,r} = \frac{1}{k}\int_0^k P_r(x, A) e^{-2\pi inx/k}\, dx.$$

From Proposition 2.1 we see that for $r \geq 1$,

$$(4.5)\qquad c_{0,r} = 0.$$

Using Definition 1 and then inverting the order of integration, we have for $r \geq 2$ and $n \neq 0$,

$$c_{n,r} = \frac{1}{k}\int_0^k \left\{\int_0^x P_{r-1}(u, A)\, du + \frac{(-1)^r}{r!} B_r(A)\right\} e^{-2\pi inx/k}\, dx = \frac{1}{k}\int_0^k P_{r-1}(u, A)\, du \int_u^k e^{-2\pi inx/k}\, dx = -\frac{1}{2\pi in}\int_0^k P_{r-1}(u, A)(1 - e^{-2\pi inu/k})\, du = \frac{k}{2\pi in} c_{n,r-1},$$

where we have again employed Proposition 2.1. Hence, for $n \neq 0$ and $r \geq 1$,

$$(4.6)\qquad c_{n,r} = (k/2\pi in)^{r-1} c_{n,1}.$$

There remains the calculation of $c_{n,1}$ for $n \neq 0$. From Definition 1 and an integration by parts, we get

$$c_{n,1} = \frac{1}{k}\int_0^k \{B_0(A)x - B_1(A) - \mathscr{A}(x)\} e^{-2\pi inx/k}\, dx = -\frac{B_0(A)k}{2\pi in} - \frac{1}{k}\sum_{j=0}^{k} a_j \int_j^k e^{-2\pi inx/k}\, dx = -\frac{1}{2\pi in}\sum_{j=0}^{k-1} a_j e^{-2\pi inj/k} = -\frac{k}{2\pi in} b_n,$$

where we have used the definition of $B_0(A)$ and (4.1). Hence, for $n \neq 0$ and $r \geq 1$, (4.6) and the above yield

$$(4.7)\qquad c_{n,r} = -(k/2\pi in)^r b_n.$$

Substituting (4.5) and (4.7) into (4.4) and then replacing n by $-n$ for $n<0$, we arrive at (4.3), and the proof is complete, on using (2.11) and the fact that $P_r(x, A)$ is continuous for $r\geqslant 2$.

If $A=I$ and $k=1$, we note that (4.3) yields for $r\geqslant 1$

$$(4.8)\quad \tfrac{1}{2}\{P_r(x+, I)+P_r(x-, I)\} = -\sum_{n=1}^{\infty}(2\pi in)^{-r}\{e^{2\pi inx}+(-1)^r e^{-2\pi inx}\} = \tfrac{1}{2}\{P_r(x+)+P_r(x-)\},$$

which is a familiar result; for example, see Knopp's book [28], p. 522. Thus, (4.8) gives the simple proof mentioned earlier that $P_r(x, I)=P_r(x)$ and $B_r(I)=B_r$ for $r\geqslant 2$.

COROLLARY 4.2. *For* $r\geqslant 1$,

$$(4.9)\quad P_r(x, A) = k^{r-1}\sum_{j=0}^{k-1} a_j P_r\left(\frac{x-j}{k}\right) = k^{r-1}\sum_{m=0}^{k-1} a_{-m}P_r\left(\frac{x+m}{k}\right).$$

Proof. For $r\geqslant 1$ and non-integral x, $P_r(x, A)=\lim_{N\to\infty} S_N(x, A)$, where, by (4.3) and (4.1),

$$(4.10)\quad S_N(x, A) \equiv -\sum_{n=1}^{N}(k/2\pi in)^r\{b_n e^{2\pi inx/k}+(-1)^r b_{-n}e^{-2\pi inx/k}\}$$
$$= -\frac{1}{k}\sum_{j=0}^{k-1} a_j\sum_{n=1}^{N}(k/2\pi in)^r\{e^{2\pi in(x-j)/k}+(-1)^r e^{-2\pi in(x-j)/k}\}$$
$$= k^{r-1}\sum_{j=0}^{k-1} a_j S_N\left(\frac{x-j}{k}, I\right).$$

On letting $N\to\infty$, we obtain the first form of (4.9) if x is not an integer. As the first two expressions in (4.9) are right-continuous at integers, it follows that the first form of (4.9) is valid for integral x as well. The last expression in (4.9) is easily obtained by setting $m=k-j$ and using periodicity.

It is known that the partial sums $S_N(x, I)$ are bounded for all N and real x and that they converge uniformly on bounded intervals except, possibly, in the neighborhoods of integers. For $r\geqslant 2$ this is trivial, and for $r=1$ see, for example, Titchmarsh's text [41], pp. 42–43. By (4.10), the same is therefore true of $S_N(x, A)$. Thus, as a result of a well-known theorem [41], p. 41, we can multiply the series on the right side of (4.3) by a function integrable on a bounded interval $[u, v]$ and then integrate term by term over $[u, v]$.

We now define

$$(4.11)\quad \beta = \max_{0\leqslant j\leqslant k-1}|b_j|$$

and

$$(4.12)\quad \varrho_1 = \tfrac{1}{2}\sum_{j=0}^{k-1}|a_j|,\qquad \varrho_r = 2\beta(k/2\pi)^r\zeta(r)\quad \text{if}\quad r\geqslant 2.$$

COROLLARY 4.3. *For* $r\geqslant 1$ *and all real* x *we have* $|P_r(x, A)|\leqslant\varrho_r$. *Furthermore, for the integral of* (3.5),

$$|F_r(d;\theta, A)|\leqslant\varrho_r\int_d^{\infty}|f^{(r)}(x)|\,dx,$$

provided that the integral on the right converges. Also, for the error term $E_r(\theta, A)$ *of Theorem* 3.1, *we have*

$$|E_r(\theta, A)|\leqslant\varrho_r\int_c^{d}|f^{(r)}(x)|\,dx;$$

this result also holds for $E_r(\theta, A)$ *in Corollaries* 3.3 *and* 3.4 *if we set* $c=Mk$ *and* $d=Nk$.

Proof. As $P_1(x)=x-[x]-\frac{1}{2}$, it follows that $|P_1(x)|\leqslant\frac{1}{2}$, so that (4.9) yields $|P_1(x, A)|\leqslant\varrho_1$. For $r\geqslant 2$, we use (4.3) to get $|P_r(x, A)|\leqslant\varrho_r$. The estimates for $F_r(d;\theta, A)$ and $E_r(\theta, A)$ now follow from the definitions of these quantities.

The above estimates for the error terms E_r and F_r are rather crude. For the classical case $A=I$, better estimates are available. In particular, see Ostrowski's paper [38] where a detailed study is made of $E_r(0, I)$ when c and d are integers.

We now derive our periodic version of the Poisson summation formula.

THEOREM 4.4. *If* f *is of bounded variation on* $[c, d]$, *then*

$$(4.13)\quad \tfrac{1}{2}\sum_{c\leqslant n\leqslant d}{}' a_n\{f(n+)+f(n-)\} = b_0\int_c^d f(x)\,dx+\sum_{n=1}^{\infty}\int_c^d(b_n e^{2\pi inx/k}+b_{-n}e^{-2\pi inx/k})f(x)\,dx,$$

where here, and in the sequel, the dash ′ *on the summation sign indicates that if* c *is an integer then the first term of the sum on the left is* $a_c f(c+)$, *and if* d *is an integer then the last term of that sum is* $a_d f(d-)$.

Proof. The proof uses an extension of an idea of L. J. Mordell [35]. We use the ordinary Poisson summation formula in the form given by Landau ([29], Vol. 2, p. 274): if $F(x)$ is of bounded variation on $[c, d]$, then

$$\frac{1}{2}\sum_{c\le n\le d}{}' \{F(n+)+F(n-)\} = \lim_{N\to\infty}\sum_{\nu=-N}^{N}\int_c^d F(x)e^{2\pi i\nu x}\,dx. \tag{4.14}$$

Put $F(x) = G(x, B)f(x)$, where $f(x)$ is of bounded variation on $[c, d]$ and

$$G(x, B) = \sum_{j=0}^{k-1} b_j e^{2\pi i jx/k}.$$

Since $G(n, B) = a_n$ by (4.2), we find that (4.14) gives

$$\frac{1}{2}\sum_{c\le n\le d}{}' a_n\{f(n+)+f(n-)\} = \lim_{N\to\infty}\sum_{\nu=-N}^{N}\int_c^d G(x, B)f(x)e^{2\pi i\nu x}\,dx$$
$$= \lim_{N\to\infty}\sum_{\nu=-N}^{N}\sum_{j=0}^{k-1} b_j\int_c^d e^{2\pi ix(\nu+j/k)}f(x)\,dx.$$

If we now put $m = j+\nu k$, we find that

$$\frac{1}{2}\sum_{c\le n\le d}{}' a_n\{f(n+)+f(n-)\} = \lim_{N\to\infty}\sum_{m=-Nk}^{Nk+k-1} b_m\int_c^d e^{2\pi imx/k}f(x)\,dx \tag{4.15}$$
$$= \lim_{M\to\infty}\sum_{m=-M}^{M} b_m\int_c^d e^{2\pi imx/k}f(x)\,dx,$$

since by the Riemann–Lebesgue lemma,

$$\lim_{|Q|\to\infty}\sum_{m=Q+\alpha_Q}^{Q+\beta_Q} b_m\int_c^d e^{2\pi imx/k}f(x)\,dx = 0$$

for bounded integers α_Q and β_Q. From (4.15), it is now easy to deduce (4.13).

DEFINITION 2. A is *even* if $a_{-n} = a_n$, $-\infty<n<\infty$. A is *odd* if $a_{-n} = -a_n$, $-\infty<n<\infty$. If either of these holds, we shall write $a_{-n} = \gamma a_n$, so that $\gamma = +1$ if A is even, and $\gamma = -1$ if A is odd.

Observe that A is even if and only if B is even. For if A is even,

$$b_n = \frac{1}{k}\sum_{j=0}^{k-1} a_j e^{-2\pi ijn/k} = \frac{1}{k}\sum_{j=1}^{k} a_{k-j}e^{-2\pi in(k-j)/k} = \frac{1}{k}\sum_{j=1}^{k} a_j e^{2\pi ijn/k} = b_{-n}.$$

The converse is proved similarly. Likewise, A is odd if and only if B is odd. Note that if A is odd, $a_0 = b_0 = 0$.

The following corollary now follows easily from Theorem 4.1.

COROLLARY 4.5. *Let $r\ge 1$, x be real, and A be even or odd. If $(-1)^r\gamma = 1$, then*

$$\tfrac{1}{2}\{P_r(x+, A)+P_r(x-, A)\} = -2\sum_{n=1}^{\infty}(k/2\pi in)^r b_n\cos(2\pi nx/k);$$

and if $(-1)^r\gamma = -1$, then

$$\tfrac{1}{2}\{P_r(x+, A)+P_r(x-, A)\} = -2i\sum_{n=1}^{\infty}(k/2\pi in)^r b_n\sin(2\pi nx/k).$$

COROLLARY 4.6. *Let $r\ge 0$, A be even or odd, and $(-1)^r\gamma = -1$. Then $B_r(A) = 0$, unless $r = 1$ and A is even in which case $B_1(A) = -\frac{1}{2}a_0$.*

Proof. If $r = 0$, then A is odd, so that, as already observed, $B_0(A) = b_0 = 0$. For $r = 1$, the result is clear if we put $x = 0$ in Corollary 4.5 and then use (2.11) and (2.3). For $r\ge 2$, the result follows from (2.12) and Corollary 4.5.

The above corollary generalizes the familiar fact that $B_3 = B_5 = B_7 = \ldots = 0$.

The following corollary is immediate from Theorem 4.4.

COROLLARY 4.7. *Let f be of bounded variation on $[c, d]$. If A is even,*

$$\frac{1}{2}\sum_{c\le n\le d}{}' a_n\{f(n+)+f(n-)\} = b_0\int_c^d f(x)\,dx+2\sum_{n=1}^{\infty} b_n\int_c^d f(x)\cos(2\pi nx/k)\,dx;$$

if A is odd,

$$\frac{1}{2}\sum_{c\le n\le d}{}' a_n\{f(n+)+f(n-)\} = 2i\sum_{n=1}^{\infty} b_n\int_c^d f(x)\sin(2\pi nx/k)\,dx.$$

We now indicate another proof of Theorem 3.1, the periodic Euler–Maclaurin formula in the case $\theta = 0$. Let $f\in C^{(r)}[c, d]$ where $r\ge 1$. From Theorem 4.4, an integration by parts, and Theorem 4.1, we obtain

$$\sum_{c\le n\le d}{}' a_n f(n) = b_0\int_c^d f(x)\,dx+\sum_{n=1}^{\infty}\int_c^d (b_n e^{2\pi inx/k}+b_{-n}e^{-2\pi inx/k})f(x)\,dx \tag{4.16}$$
$$= b_0\int_c^d f(x)\,dx-\tfrac{1}{2}\{P_1(d+, A)+P_1(d-, A)\}f(d)+$$
$$+\tfrac{1}{2}\{P_1(c+, A)+P_1(c-, A)\}f(c)-$$
$$-\sum_{n=1}^{\infty}(k/2\pi in)\int_c^d (b_n e^{2\pi inx/k}-b_{-n}e^{-2\pi inx/k})f'(x)\,dx.$$

By our remarks following Corollary 4.2, we may invert the order of summation and integration on the far right side of (4.16). Using Theorem 4.1 and (2.11), we may then write (4.16) as

$$\sum_{c<n\le d} a_n f(n) = b_0\int_c^d f(x)\,dx-P_1(d, A)f(d)+$$
$$+P_1(c, A)f(c)+\int_c^d P_1(x, A)f'(x)\,dx.$$

Since $b_0 = B_0(A)$, the above is simply (3.2) with $\theta = 0$, and the remainder of the proof is precisely the same as before.

5. A reciprocity theorem for some exponential sums. The following reciprocity theorem contains the reciprocity theorem for generalized Gaussian sums as a special case.

THEOREM 5.1. *Let a, b and c be integers such that $ac \neq 0$ and $ack+b$ is even. Then,*

$$(5.1)\qquad \sum_{n=0}^{|c|k-1} a_n e^{\pi i a n^2/ck+\pi i b n/ck} = (|c|k/|a|)^{1/2} e^{-\pi i b^2/4ack+\frac{1}{4}\pi i \operatorname{sgn}(ac)} \sum_{n=0}^{|a|k-1} b_n e^{-\pi i c n^2/ak-\pi i b n/ak}.$$

A proof of Theorem 5.1 (with $c > 0$) by contour integration can be found in [10], Theorem 1, by Berndt. The proof we give here uses our periodic Poisson summation formula. The proof is in the same spirit as that of Dirichlet [20] where he used the Poisson summation formula to evaluate ordinary Gaussian sums. (Dirichlet's proof can also be found in Davenport's book [18], pp. 14–17.)

Proof of Theorem 5.1. Let $f(x) = \exp(\pi i a x^2/ck+\pi i b x/ck)$. Since $ack+b$ is even, it is easily verified that $f(n+|c|k) = f(n)$, if n is an integer. Hence, Theorem 4.4 yields

$$S \equiv \sum_{n=0}^{|c|k-1} a_n e^{\pi i a n^2/ck+\pi i b n/ck} = \lim_{N\to\infty} \sum_{n=-N}^{N} b_n \int_0^{|c|k} e^{2\pi i n x/k+\pi i a x^2/ck+\pi i b x/ck}\,dx$$
$$= \sum_{n=-\infty}^{\infty}{}^{*} b_n e^{-\pi i a(b/2a+nc/a)^2/ck} \int_0^{|c|k} e^{\pi i a(x+b/2a+nc/a)^2/ck}\,dx,$$

where for the remainder of the proof we write $\sum_{n=-\infty}^{\infty}{}^{*}$ for $\lim_{N\to\infty}\sum_{n=-N}^{N}$. Now replace $x+b/2a+nc/a$ by cku to obtain

$$S = ck \sum_{n=-\infty}^{\infty}{}^{*} b_n e^{-\pi i(b/2+nc)^2/ack} \int_{b/2ack+n/ak}^{\operatorname{sgn} c+b/2ack+n/ak} e^{\pi i acku^2}\,du.$$

Put $n = akm+r$, $-\infty < m < \infty$, $0 \leqslant r \leqslant |a|k-1$, and obtain after some manipulation

$$S = cke^{-\pi i b^2/4ack} \sum_{r=0}^{|a|k-1} b_r e^{-\pi i c r^2/ak-\pi i b r/ak} \times$$
$$\times \sum_{m=-\infty}^{\infty}{}^{*} e^{-\pi i ackm^2-\pi i bm} \int_{m+b/2ack+r/ak}^{m+\operatorname{sgn} c+b/2ack+r/ak} e^{\pi i acku^2}\,du.$$

Observe that $\exp(-\pi i ackm^2-\pi i bm) = 1$. If m is even, this is obvious; if m is odd, this follows from the fact that $ack+b$ is even. Hence, the above equation can be simplified to

$$(5.2)\qquad S = cke^{-\pi i b^2/4ack} \sum_{r=0}^{|a|k-1} b_r e^{-\pi i c r^2/ak-\pi i b r/ak} \operatorname{sgn} c \int_{-\infty}^{\infty} e^{\pi i acku^2}\,du$$
$$= (2|c|k/|a|)^{1/2} e^{-\pi i b^2/4ack} \sum_{r=0}^{|a|k-1} b_r e^{-\pi i c r^2/ak-\pi i b r/ak} \int_{-\infty}^{\infty} e^{2\pi i y^2 \operatorname{sgn}(ac)}\,dy,$$

where we put $y = (|ac|k/2)^{1/2}u$. The integral on the right side of (5.2) may be evaluated by letting $k = a_0 = c = 1$, $b = 0$, and $a = 2\varepsilon$ where $\varepsilon = \pm 1$. Then (5.2) reduces to

$$1 = \sum_{r=0}^{1} e^{-\pi i \varepsilon r^2/2} \int_{-\infty}^{\infty} e^{2\pi i y^2 \varepsilon}\,dy$$

or

$$(5.3)\qquad \int_{-\infty}^{\infty} e^{2\pi i y^2\varepsilon}\,dy = 2^{-1/2} e^{\pi i \varepsilon/4}.$$

If we substitute (5.3) into (5.2) with $\varepsilon = \operatorname{sgn}(ac)$, we arrive at (5.1) at once, and the proof is complete.

COROLLARY 5.2 (Reciprocity theorem for generalized Gaussian sums). *Let a, b and c be integers such that $ac \neq 0$ and $ac+b$ is even. Then*

$$\sum_{n=0}^{|c|-1} e^{\pi i a n^2/c+\pi i b n/c} = |c/a|^{1/2} e^{-\pi i b^2/4ac+\frac{1}{4}\pi i \operatorname{sgn}(ac)} \sum_{n=0}^{|a|-1} e^{-\pi i c n^2/a-\pi i b n/a}.$$

Proof. Let $k = a_0 = 1$ in Theorem 5.1; then Corollary 5.2 follows immediately.

For other implications of Theorem 5.1, see Berndt's paper [10].

6. Periodic Lerch and zeta-functions

DEFINITION 3. Let $\sigma = \operatorname{Re}(s) > 1$ and let x and a be real. Then the *periodic Lerch function* $\varphi(x, a, s; A)$ is defined by

$$\varphi(x, a, s; A) = \sum_{n=0}^{\infty}{}' a_n e^{2\pi i n x/k}(n+a)^{-s},$$

where the prime $'$ indicates that if a is a non-positive integer, then the term corresponding to $n = -a$ is omitted from the sum. The periodic Hurwitz zeta-function $\zeta(s, a; A)$ and the periodic zeta-function $\zeta(s; A)$ are defined by

$$\zeta(s, a; A) = \varphi(0, a, s; A) = \sum_{n=0}^{\infty}{}' a_n(n+a)^{-s}$$

and

$$\zeta(s;A) = \zeta(s,0;A) = \sum_{n=0}^{\infty}{}' a_n n^{-s} = \sum_{n=1}^{\infty} a_n n^{-s}.$$

If $A = I$ and $0 < a \leqslant 1$, then $\varphi(x, a, s; A)$ and $\zeta(s, a; A)$ reduce to the classical Lerch function $\varphi(x, a, s)$ and classical Hurwitz zeta-function $\zeta(s, a)$, respectively. Also, if $A = I$, $\zeta(s; A)$ reduces to the Riemann zeta-function $\zeta(s)$. Furthermore, if $A = \chi = \{\chi(n)\}$, where $\chi(n)$ is a character of modulus k, then $\zeta(s; A) = L(s, \chi)$, the classical Dirichlet L-function.

Note that on setting $n = mk+r$, $0 \leqslant m < \infty$, $0 \leqslant r \leqslant k-1$, we get

$$(6.1)\qquad \varphi(x, a, s; A) = k^{-s} \sum_{r=0}^{k-1} a_r e^{2\pi i r x/k} \sum_{m=0}^{\infty}{}' e^{2\pi i m x}\big(m+(r+a)/k\big)^{-s} = k^{-s} \sum_{r=0}^{k-1} a_r e^{2\pi i r x/k} \varphi\big(x, (r+a)/k, s; I\big).$$

Before stating and proving the functional equation for $\varphi(x, a, s; A)$, we need to define some sequences associated with A. Let $A_1 = \{a_n'\}$, where $a_n' = a_{n+1}$; $A^* = \{a_n^*\}$, where $a_n^* = a_{-n}$; and $A_1^* = \{a_n''\}$, where $a_n'' = a_{-n-1}$. Thus, A_1, A^*, and A_1^* all have period k when A does.

THEOREM 6.1. *If x and a are real, then the function $\varphi(x, a, s; A)$ has an analytic continuation into the entire s-plane where $\varphi(x, a, s; A)$ is holomorphic except, possibly, for a simple pole at $s = 1$. The pole exists only if x is an integer and $b_{-x} \neq 0$, and in this case the residue of $\varphi(x, a, s; A)$ is b_{-x}. Furthermore, if $0 \leqslant a \leqslant 1$ and $x \leqslant 1$, then for all s*

$$\varphi(x, a, 1-s; A) = (k/2\pi)^s \Gamma(s)\{e^{\pi i s/2 - 2\pi i a x/k}\varphi(-a, x, s; B) + e^{-\pi i s/2 + 2\pi i a(1-x)/k}\varphi(a, 1-x, s; B_1^*)\}.$$

Proof. In Theorem 3.1 let $f(u) = e^{2\pi i u x/k}u^{-s}$, $\theta = a$, $c = 1$, and $r = 1$. Then,

$$(6.2)\qquad \sum_{1<n+a\leqslant d} a_n f(n+a) = B_0(A)\int_1^d f(u)\,du - P_1(d-a, A)f(d) + P_1(1-a, A)f(1) + \int_1^d P_1(u-a, A)f'(u)\,du.$$

Now,

$$(6.3)\qquad f'(u) = (2\pi i x/k)e^{2\pi i u x/k}u^{-s} - s e^{2\pi i u x/k}u^{-s-1} = (2\pi i x/k)g(u; x, s) - s g(u; x, s+1),$$

where we now write $g(u; x, s) = e^{2\pi i u x/k}u^{-s} = f(u)$ in order to indicate the dependence on the parameters x and s. Define also an entire function

$E(s)$ by

$$E(s) = \begin{cases} e^{2\pi i a x/k} \sum\limits_{0\leqslant n\leqslant 1-a}{}' a_n e^{2\pi i n x/k}(n+a)^{-s}, & \text{if } a \leqslant 1,\\ -e^{2\pi i a x/k} \sum\limits_{1-a<n<0} a_n e^{2\pi i n x/k}(n+a)^{-s}, & \text{if } a > 1,\end{cases}$$

and define for $\sigma > 1$

$$G(x, s) = \int_1^{\infty} g(u; x, s)\,du$$

and

$$H(x, s) = \int_1^{\infty} P_1(u-a, A)g(u; x, s)\,du.$$

Recalling that $B_0(A) = b_0$, letting $d \to \infty$ in (6.2), and employing the above notation, we find that for $\sigma > 1$

$$(6.4)\qquad e^{2\pi i a x/k}\varphi(x, a, s; A) = E(s) + b_0 G(x, s) + P_1(1-a, A)f(1) + (2\pi i x/k)H(x, s) - sH(x, s+1).$$

If $x = 0$ and $\sigma > 1$, then

$$(6.5)\qquad G(0, s) = \int_1^{\infty} u^{-s}\,du = 1/(s-1);$$

this provides the analytic continuation of $G(0, s)$ into the entire complex s-plane. If $x \neq 0$ and $Z \geqslant Y$, then an integration by parts yields for large positive Y and $\sigma > 0$

$$\int_Y^Z g(u; x, s)\,du = (k/2\pi i x)u^{-s}e^{2\pi i u x/k}\Big|_{u=Y}^{Z} + s(k/2\pi i x)\int_Y^Z u^{-s-1}e^{2\pi i u x/k}\,du = O\left(\frac{k|s|}{x\sigma}Y^{-\sigma}\right).$$

Hence, $G(x, s)$ is a holomorphic function of s for $\sigma > 0$ provided that $x \neq 0$. Furthermore, if $x \neq 0$ and $0 < \sigma < 1$,

$$\int_0^{\infty} g(u; x, s)\,du = \int_0^{\infty} e^{2\pi i u x/k}u^{-s}\,du = \Gamma(1-s)(2\pi|x|/k)^{s-1}e^{\frac{1}{2}\pi i(1-s)\operatorname{sgn} x} = \Gamma(1-s)(2\pi x/k)^{s-1}e^{\frac{1}{2}\pi i(1-s)},$$

where the principal value of $(2\pi x/k)^{s-1}$ is used. (See, for example, Titchmarsh's book [41], pp. 107–108.) Hence, if $x \neq 0$ and $0 < \sigma < 1$,

$$(6.6)\qquad G(x, s) = -\int_0^1 g(u; x, s)\,du + \Gamma(1-s)(2\pi x/k)^{s-1}e^{\frac{1}{2}\pi i(1-s)}.$$

As the right side of (6.6) is holomorphic for $\sigma < 1$, we see that (6.6) provides an analytic continuation for $G(x, s)$ into the half-plane $\sigma < 1$, and since we have previously observed that $G(x, s)$ is analytic for $\sigma > 0$, we conclude that for $x \neq 0$, $G(x, s)$ is an entire function.

The analytic continuations of $H(x, s)$ and $H(x, s+1)$ are a bit more difficult to establish. As before, we use $\sum_{n=-\infty}^{\infty}{}^{*}$ as an abbreviation for $\lim_{N\to\infty} \sum_{n=-N}^{N}$. From (4.3), we have for $\sigma > 1$

$$(6.7)\quad H(x, s) = -\int_1^\infty g(u; x, s) \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}{}^{*} (k/2\pi in) b_n e^{2\pi in(u-a)/k}\, du$$

$$= H_1(x, s) + H_2(x, s),$$

where

$$(6.8)\quad H_1(x, s) = -\int_1^\infty g(u; x, s) \sum_{0<|n|\leqslant |x|} (k/2\pi in) b_n e^{2\pi in(u-a)/k}\, du$$

$$= -\sum_{0<|n|\leqslant |x|} (k/2\pi in) b_n e^{-2\pi ina/k} G(n+x, s).$$

Since $G(n+x, s)$, $0 < |n| \leqslant |x|$, has an analytic continuation to the entire s-plane, it follows that $H_1(x, s)$ can be analytically continued to the entire s-plane. Moreover, from (6.5) and (6.6), $H_1(x, s)$ is holomorphic everywhere with the possible exception of a simple pole at $s = 1$. There is a simple pole only if x is a non-zero integer and $b_{-x} \neq 0$ in which case the residue of $H_1(x, s)$ is $(k/2\pi ix) b_{-x} e^{2\pi iax/k}$.

For $\sigma > 1$, we have

$$H_2(x, s) = -\int_1^\infty \sum_{|n|>|x|}{}^{*} (k/2\pi in) b_n e^{-2\pi ian/k} g(u; n+x, s)\, du,$$

and we wish to show that we can change the order of summation and integration. By the remarks following Corollary 4.2, the inversion is valid over a finite interval. Hence, for $Y > 1$,

$$(6.9)\quad \int_1^Y \sum_{|n|>|x|}{}^{*} (k/2\pi in) b_n e^{-2\pi ian/k} g(u; n+x, s)\, du$$

$$= \sum_{|n|>|x|}{}^{*} (k/2\pi in) b_n e^{-2\pi ian/k} G(n+x, s) -$$

$$- \sum_{|n|>|x|}{}^{*} (k/2\pi in) b_n e^{-2\pi ian/k} \int_Y^\infty g(u; n+x, s)\, du.$$

We wish to show that the second sum on the right side of (6.9) tends to 0 as Y tends to ∞. Since $|n| > |x|$, this is easily done upon an integration by parts. Thus, letting $Y \to \infty$ in (6.9), we conclude that for $\sigma > 1$

$$(6.10)\quad H_2(x, s) = -\sum_{|n|>|x|}{}^{*} (k/2\pi in) b_n e^{-2\pi ian/k} G(n+x, s).$$

(For a more complete discussion of the type of argument considered above, see [8], p. 406, by Berndt or Titchmarsh's treatise [42], p. 15.) Moreover, by an integration by parts, we easily deduce that for $|n| > |x|$ and $\sigma > 0$,

$$G(n+x, s) = O\left(\frac{|s|}{\sigma(|n|-|x|)}\right).$$

Hence, for each $\varepsilon > 0$, the series on the right side of (6.10) converges uniformly for $\sigma \geqslant \varepsilon > 0$. Thus, $H_2(x, s)$ can be analytically continued to the half-plane $\sigma > 0$ and is analytic there.

Thus, putting (6.8) and (6.10) into (6.7), we find that for $\sigma > 0$

$$(6.11)\quad H(x, s) = -\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}{}^{*} (k/2\pi in) b_n e^{-2\pi ian/k} G(n+x, s).$$

Moreover, $H(x, s)$ is holomorphic for $\sigma > 0$ with the possible exception of a simple pole at $s = 1$; the pole exists only if x is a non-zero integer and $b_{-x} \neq 0$, and in this case $H(x, s)$ has the residue $(k/2\pi ix) b_{-x} e^{2\pi iax/k}$. Referring to (6.4), we see that $\varphi(x, a, s; A)$ is holomorphic for $\sigma > 0$ with the possible exception of a simple pole at $s = 1$. The pole exists if x is an integer and $b_{-x} \neq 0$, and the residue is b_{-x}.

Let $\delta_x = 1/x$ if x is a non-zero integer and let $\delta_x = 0$ otherwise. Then using (6.5) and (6.6), we find that (6.11) becomes for $0 < \sigma < 1$

$$(6.12)\quad H(x, s) = \delta_x (k/2\pi i) b_{-x} e^{2\pi iax/k}/(s-1) -$$

$$- \sum_{\substack{n=-\infty\\ n\neq 0,-x}}^{\infty}{}^{*} (k/2\pi in) b_n e^{-2\pi ian/k} \Big\{ -\int_0^1 g(u; n+x, s)\, du +$$

$$+ \Gamma(1-s)\{2\pi(n+x)/k\}^{s-1} e^{\frac{1}{2}\pi i(1-s)} \Big\}$$

$$= \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}{}^{*} (k/2\pi in) b_n e^{-2\pi ian/k} \int_0^1 g(u; n+x, s)\, du -$$

$$- \Gamma(1-s)(2\pi/k)^{s-2} e^{-\pi is/2} \sum_{\substack{n=-\infty\\ n\neq 0,-x}}^{\infty}{}^{*} b_n e^{-2\pi ian/k} (n+x)^{s-1}/n$$

$$= -\int_0^1 P_1(u-a, A) g(u; x, s)\, du -$$

$$- \Gamma(1-s)(2\pi/k)^{s-2} e^{-\pi is/2} \sum_{\substack{n=-\infty\\ n\neq 0,-x}}^{\infty}{}^{*} b_n e^{-2\pi ian/k} (n+x)^{s-1}/n,$$

by (4.3), where the inversion in order of summation and integration is justified by the comments after Corollary 4.2. The far right side of (6.12) is clearly holomorphic for all $\sigma < 1$. Hence, $H(x, s)$ has an analytic continuation into the full s-plane where it is holomorphic with the possible exception of a simple pole at $s = 1$. By (6.4), the same is therefore true of $\varphi(x, a, s; A)$. (Observe that in (6.4) a possible pole of $H(x, s+1)$ at $s = 0$ is canceled by a factor of s.) It remains to prove the functional equation of $\varphi(x, a, s; A)$.

Now let $\sigma < 0$. Then (6.12) shows that

$$(2\pi ix/k)H(x, s) - sH(x, s+1)$$
$$= -(2\pi/k)^{s-1}\Gamma(1-s)e^{-\pi is/2}\sideset{}{^*}\sum_{\substack{n=-\infty\\ n\neq 0,-x}}^{\infty} b_n e^{-2\pi ian/k}(n+x)^{s-1}\{ix - i(n+x)\}/n -$$
$$-\int_0^1 P_1(u-a, A)\{(2\pi ix/k)g(u; x, s) - sg(u; x, s+1)\}\,du$$
$$= i(2\pi/k)^{s-1}\Gamma(1-s)e^{-\pi is/2}\sideset{}{^*}\sum_{\substack{n=-\infty\\ n\neq 0,-x}}^{\infty} b_n e^{-2\pi ian/k}(n+x)^{s-1} - \int_0^1 P_1(u-a, A)f'(u)\,du$$

by (6.3). On splitting the sum above and using (6.4), we get for $\sigma < 0$

$$(6.13)\quad e^{2\pi iax/k}\varphi(x, a, s; A)$$
$$= E(s) + P_1(1-a, A)f(1) - \int_0^1 P_1(u-a, A)f'(u)\,du + b_0 G(x, s) +$$
$$+ i(2\pi/k)^{s-1}\Gamma(1-s)e^{-\pi is/2}\times$$
$$\times\Big\{\sum_{\substack{n=1\\ n\neq -x}}^{\infty} b_n e^{-2\pi ian/k}(n+x)^{s-1} + \sum_{\substack{m=1\\ m\neq x}}^{\infty} b_{-m}e^{2\pi iam/k}(-m+x)^{s-1}\Big\}.$$

If $x \leqslant 1$, then for $m \neq x$ and $m \geqslant 1$ we have $x - m < 0$. Thus,

$$(x-m)^{s-1} = |x-m|^{s-1}e^{\pi i(s-1)} = -e^{\pi is}(m-x)^{s-1}.$$

Hence, for $x \leqslant 1$ and $\sigma < 0$, the last sum on the right side of (6.13) is

$$-e^{\pi is}\sideset{}{'}\sum_{m=1}^{\infty} b_{-m}e^{2\pi iam/k}(m-x)^{s-1} = -e^{\pi is + 2\pi ia/k}\sideset{}{'}\sum_{n=0}^{\infty} b_{-n-1}e^{2\pi ian/k}(n+1-x)^{s-1}$$
$$= -e^{\pi is + 2\pi ia/k}\varphi(a, 1-x, 1-s; B_1^*).$$

So for $x \leqslant 1$ and $\sigma < 0$,

$$(6.14)\quad e^{2\pi iax/k}\varphi(x, a, s; A)$$
$$= Q + i(2\pi/k)^{s-1}\Gamma(1-s)\Big\{e^{-\pi is/2}\sideset{}{'}\sum_{n=1}^{\infty} b_n e^{-2\pi ian/k}(n+x)^{s-1} -$$
$$- e^{\pi is/2 + 2\pi ia/k}\varphi(a, 1-x, 1-s; B_1^*)\Big\},$$

where, for $0 \leqslant a \leqslant 1$ and $\sigma < 0$, (2.7) gives

$$Q = E(s) + P_1(1-a, A)f(1) - \int_0^1 \{b_0(u-a) - B_1(A)\}f'(u)\,du +$$
$$+ a_0\int_a^1 f'(u)\,du + b_0 G(x, s)$$
$$= E(s) + P_1(1-a, A)f(1) - \{b_0(1-a) - B_1(A)\}f(1) +$$
$$+ b_0\int_0^1 f(u)\,du + a_0\{f(1) - f(a)\} + b_0 G(x, s)$$
$$= E(s) - a_0 f(a) + \{P_1(1-a, A) - b_0(1-a) + B_1(A) + a_0\}f(1) +$$
$$+ b_0\Big\{G(x, s) + \int_0^1 g(u; x, s)\,du\Big\} = b_0\Big\{G(x, s) + \int_0^1 g(u; x, s)\,du\Big\},$$

regardless of whether $a = 0$ or $0 < a \leqslant 1$. Furthermore, from (6.5) and (6.6), we get

$$Q = \begin{cases} b_0 i(2\pi x/k)^{s-1}\Gamma(1-s)e^{-\pi is/2}, & \text{if } x \neq 0,\\ 0, & \text{if } x = 0.\end{cases}$$

Thus, regardless of whether $x = 0$ or $x \neq 0$, we conclude from (6.14) that for $0 \leqslant a \leqslant 1$, $x \leqslant 1$, and $\sigma < 0$

$$e^{2\pi iax/k}\varphi(x, a, s; A) = i(2\pi/k)^{s-1}\Gamma(1-s)\{e^{-\pi is/2}\varphi(-a, x, 1-s; B) -$$
$$- e^{\pi is/2 + 2\pi ia/k}\varphi(a, 1-x, 1-s; B_1^*)\}.$$

By analytic continuation, the above equation holds for all s if $0 \leqslant a \leqslant 1$ and $x \leqslant 1$. On replacing s by $1-s$ we get the stated result.

COROLLARY 6.2 (Lerch's functional equation). *Let $0 < a \leqslant 1$ and $0 < x < 1$. Then $\varphi(x, a, s)$ has an analytic continuation into the full complex s-plane and is an entire function of s. Furthermore, for all s,*

$$\varphi(x, a, 1-s) = (2\pi)^{-s}\Gamma(s)\{e^{\pi is/2 - 2\pi iax}\varphi(-a, x, s) +$$
$$+ e^{-\pi is/2 + 2\pi ia(1-x)}\varphi(a, 1-x, s)\}.$$

Proof. Recall that if $0 < a \leqslant 1$ and $A = I$, then $\varphi(x, a, s; A) = \varphi(x, a, s)$, Lerch's function. The result now is an immediate consequence of Theorem 6.1.

COROLLARY 6.3 (Hurwitz's formula). *If $0 < a \leqslant 1$ and $\sigma > 1$, then*

$$\zeta(1-s, a) = 2(2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}\cos(\pi s/2 - 2\pi na)\cdot n^{-s}.$$

Proof. In Theorem 6.1 set $x=0$. Then for $0\leqslant a\leqslant 1$ and $\sigma>1$, it follows easily that

$$(6.15)\qquad \zeta(1-s, a; A) = \varphi(0, a, 1-s; A)$$
$$= (k/2\pi)^s\Gamma(s)\Big\{e^{\pi is/2}\sum_{n=1}^{\infty} b_n e^{-2\pi ian/k}n^{-s} + e^{-\pi is/2}\sum_{n=1}^{\infty} b_{-n}e^{2\pi ian/k}n^{-s}\Big\}.$$

If we now let $A=I$ and $k=1$, the result immediately follows.

Note that if we set $a=1$ in Corollary 6.3, we obtain the functional equation of $\zeta(s)$.

Corollary 6.4. *Let $r\geqslant 2$ be an integer and $0\leqslant a\leqslant 1$. Then*

$$(6.16)\qquad \zeta(1-r, a; A) = (-1)^{r-1}(r-1)!P_r(-a, A)$$

and

$$(6.17)\qquad \zeta(1-r; A) = -\frac{1}{r}B_r(A).$$

Proof. In (6.15) set $s=r$ and obtain

$$\zeta(1-r, a; A) = i^r\Gamma(r)\sum_{n=1}^{\infty}(k/2\pi n)^r\{b_n e^{-2\pi ian/k}+(-1)^r b_{-n}e^{2\pi ian/k}\}$$
$$= (-1)^{r-1}\Gamma(r)P_r(-a, A),$$

by (4.3). This yields (6.16). Setting $a=0$ in (6.16) and applying (2.12), we get (6.17).

Note that if we let $A=I$, (6.16) and (6.17) reduce to the familiar results $\zeta(1-r, a) = -B_r(a)/r$ and $\zeta(1-r) = -B_r/r$.

Corollary 6.5. *The periodic zeta-function $\zeta(s; A)$ has an analytic continuation into the entire s-plane where it is holomorphic with the possible exception of a simple pole at $s=1$ where the residue is $b_0 = B_0(A)$. Furthermore,*

$$\zeta(1-s; A) = (k/2\pi)^s\Gamma(s)\{e^{\pi is/2}\zeta(s; B)+e^{-\pi is/2}\zeta(s; B^*)\}.$$

Proof. The result follows from (6.15) if we put $a=0$.

Corollary 6.6. *If χ is a primitive character modulo $k>1$ and*

$$G(\chi) = \sum_{j=1}^{k-1}\chi(j)e^{2\pi ij/k},$$

then $L(s,\chi)$ is an entire function, and for all s

$$L(1-s, \chi) = (k/2\pi)^s\Gamma(s)k^{-1}G(\chi)\{\chi(-1)e^{\pi is/2}+e^{-\pi is/2}\}L(s, \bar\chi).$$

Proof. Let $a_n = \chi(n)$ and note that

$$b_n = \frac{1}{k}\sum_{j=1}^{k-1}\chi(j)e^{-2\pi inj/k} = \frac{1}{k}\bar\chi(-n)G(\chi),$$

where we have employed the factorization theorem for Gaussian sums associated with a primitive character; see Davenport's book [18], p. 67. It follows that $b_n^* = b_{-n} = \bar\chi(n)G(\chi)/k$. The result now follows from Corollary 6.5.

We remark that the above proof of the functional equation for $L(s,\chi)$ is entirely new. In [9] Berndt essentially derived Theorem 6.1 when $a_n = \chi(n)$ and χ is primitive. However, in that paper he used the functional equation of $L(s,\chi)$ to derive Theorem 3.1 for $a_n=\chi(n)$ and $\theta=0$.

The method used above for proving Theorem 6.1 is an extension of a well-known method for proving the functional equation of $\zeta(s)$. (See, for example, Titchmarsh's book [42], pp. 13–15.) The same method has been previously used by Berndt to obtain Corollary 6.2 [8] and Corollary 6.3 [7]. We remark that if $0\leqslant x\leqslant 1$ and $0<a\leqslant 1$, then by assuming Corollary 6.2 and Corollary 6.3, we may prove a slightly weaker version of Theorem 6.1 with the aid of (6.1) and (4.2).

In the next section we shall employ periodic theta functions to derive the functional equation of another class of Dirichlet series.

We next show how the periodic Euler–Maclaurin formula can be used to evaluate many periodic zeta-functions at certain positive integral arguments. Let

$$(6.18)\qquad M_r(A) = \sum_{j=0}^{k-1} a_j j^r$$

be the power sum associated with A. (If $r=0$, we shall understand that $M_0(A) = kB_0(A)$.) In Theorem 3.1, let $c=0$, $d=k$, $\theta=0$, and $f(x) = x^r$. We then obtain for $r\geqslant 1$

$$(6.19)\qquad M_r(A)+a_k k^r = B_0(A)k^{r+1}/(r+1)+$$
$$+\sum_{j=1}^{r}(-1)^j P_j(0, A)r!k^{r-j+1}/(r-j+1)!+(-1)^{r+1}r!\int_0^k P_r(x, A)\,dx.$$

This last integral is zero by Proposition 2.1.

From Theorem 4.1, we have for $j\geqslant 2$

$$(6.20)\qquad P_j(0, A) = -\sum_{n=1}^{\infty}(k/2\pi in)^j\{b_n+(-1)^j b_{-n}\}$$
$$= -(k/2\pi i)^j\{\zeta(j; B)+(-1)^j\zeta(j; B^*)\}.$$

Using (2.8) and (6.20) in (6.19), we arrive at

$$(6.21)\qquad M_r(A) = B_0(A)k^{r+1}/(r+1)+B_1(A)k^r-$$
$$-r!k^{r+1}\sum_{j=2}^{r}\frac{(-2\pi i)^{-j}}{(r-j+1)!}\{\zeta(j; B)+(-1)^j\zeta(j; B^*)\}$$

if $r \geqslant 1$. If $r = 1$, the sum is understood to be 0, and in this case (6.21) is just (2.2). From (6.21), we see that we can evaluate $\zeta(j; B) + (-1)^j \zeta(j; B^*)$ recursively.

We may also note that (2.8), (2.11), and Theorem 4.1 give

$$(6.22)\quad -B_1(A) - \tfrac{1}{2}a_0 = P_1(0, A) + \tfrac{1}{2}a_0 = \tfrac{1}{2}\{P_1(0-, A) + P_1(0+, A)\}$$
$$= -\sum_{n=1}^{\infty} \frac{k}{2\pi i n}(b_n - b_{-n}) = -\frac{k}{2\pi i}\{\zeta(1; B) - \zeta(1; B^*)\},$$

with the last equation valid provided that $a_0 = 0$.

Suppose now that B is even or odd. If B is odd, then necessarily A is odd; so $a_0 = 0$, $b_0 = 0$, and (6.22) gives

$$(6.23)\qquad \zeta(1; B) = \frac{\pi i}{k} B_1(A).$$

For B even or odd, (6.21) may be written as

$$(6.24)\quad M_r(A) = B_0(A)k^{r+1}/(r+1) + B_1(A)k^r -$$
$$- r!\,k^{r+1} \sum_{j=2}^{r} \frac{(-2\pi i)^{-j}}{(r-j+1)!}\{1 + (-1)^j \gamma\}\zeta(j; B).$$

From (6.24), we see that if $(-1)^m \gamma = 1$, we can evaluate $\zeta(m; B)$ recursively by taking $r = m, m-2, m-4, \ldots$

We give a few examples. Let $A = I$ and $r = 2$; since $B_1 = -1/2$, (6.24) reduces to $0 = 1/3 - 1/2 + \pi^{-2}\zeta(2)$, or $\zeta(2) = \pi^2/6$. Let $a_n = \chi(n)$, where χ is even and non-principal, and $r = 2$. Then $B_0(\chi) = 0$, and from (2.2),

$$B_1(\chi) = \frac{1}{k}\sum_{n=1}^{k-1} n\chi(n) = 0,$$

which is easily seen by replacing n by $k-n$. Since $b_n = \overline{\chi}(-n)G(\chi)/k$, we find that (6.24) yields for even, primitive χ modulo $k > 1$

$$M_2(\chi) = \frac{k^2}{\pi^2} G(\chi) L(2, \overline{\chi}).$$

Using (6.24), we can evaluate recursively $L(2n, \chi)$, $n \geqslant 1$, when χ is even and primitive. If χ is odd and primitive, then by (6.23) and (2.2)

$$L(1, \overline{\chi}) = -\pi i B_1(A)/G(\chi) = -\pi i M_1(\chi)/kG(\chi).$$

Using (6.24), we may then evaluate $L(2n+1, \chi)$, $n \geqslant 0$, recursively when χ is odd and primitive. For a slightly fuller discussion of the evaluation of L-functions by this method, consult the works of Ayoub [3], Berndt [6], [9], or Rosser and Schoenfeld [40].

In Corollary 6.4 we gave one connection between periodic Bernoulli numbers and the values of periodic zeta-functions. Another can be given as follows. If $(-1)^r\gamma = 1$ and $r \geqslant 2$, we find that if we combine (2.12) and (6.20),

$$(6.25)\qquad \zeta(r; B) = \tfrac{1}{2}(-1)^{r+1}(2\pi i/k)^r B_r(A)/r!.$$

Lastly, it is well known that the ordinary Euler–Maclaurin formula can be used to evaluate certain other infinite series. In the same way, Theorem 3.1 can be used to evaluate analogous infinite series with periodic coefficients. For another method of summing infinite series with periodic coefficients, see Berndt's paper [11].

7. Periodic theta-functions

DEFINITION 4. Let g and h be real and $\sigma > 0$. Then the *periodic theta-function* $\theta(s, g, h; A)$ is defined by

$$\theta(s, g, h; A) = \sum_{n=-\infty}^{\infty} a_n e^{-\pi s(n+g)^2/k + 2\pi i n h/k}.$$

We now derive a transformation formula for $\theta(s, g, h; A)$. In the case $A = I$, the theorem reduces to a famous result of P. Epstein [22].

THEOREM 7.1. *We have*

$$(7.1)\qquad \theta(s, g, h; A) = (k/s)^{1/2} e^{-2\pi i g h/k}\theta(1/s, h, -g; B),$$

where the principal branch of the square root is chosen.

Proof. In the periodic Poisson summation formula of Theorem 4.4 put $f(x) = \exp\{-\pi(x+g)^2/ks + 2\pi i x h/k\}$, where $s > 0$ is real. As before, we shall write $\sum\limits_{n=-\infty}^{\infty}{}^{*}$ for $\lim\limits_{N\to\infty}\sum\limits_{n=-N}^{N}$. We find that

$$(7.2)\qquad \sum_{n=-M}^{M}{}' a_n e^{-\pi(n+g)^2/ks + 2\pi i n h/k}$$
$$= \sum_{n=-\infty}^{\infty}{}^{*} b_n \Big\{\int_{-\infty}^{\infty} - \int_{|x|\geqslant M}\Big\} e^{-\pi(x+g)^2/ks + 2\pi i x(n+h)/k}\,dx.$$

We next wish to show that

$$\lim_{M\to\infty}\sum_{n=-\infty}^{\infty}{}^{*} b_n \int_{|x|\geqslant M} e^{-\pi(x+g)^2/ks} e^{2\pi i x(n+h)/k}\,dx = 0.$$

This is accomplished in a familiar fashion by two integrations by parts. (See, e.g., Davenport's book [18], p. 65.) Hence, letting M tend to ∞

in (7.2), we find that

$$(7.3)\qquad \theta(1/s,g,h;A) = \sum_{n=-\infty}^{\infty}{}^{*} b_n \int_{-\infty}^{\infty} e^{-\pi(x+g)^2/ks+2\pi ix(n+h)/k}\,dx$$

$$= se^{-2\pi igh/k}\sum_{n=-\infty}^{\infty}{}^{*} b_n e^{-2\pi ing/k}\int_{-\infty}^{\infty} e^{-\pi y^2s/k+2\pi iys(n+h)/k}\,dy,$$

where we have put $x+g = ys$. Now,

$$\int_{-\infty}^{\infty} e^{-\pi y^2 s/k+2\pi iys(n+h)/k}\,dy = e^{-\pi s(n+h)^2/k}\int_{-\infty}^{\infty} e^{-\pi s(y-i(n+h))^2/k}\,dy$$

$$= e^{-\pi s(n+h)^2/k}\int_{-\infty}^{\infty} e^{-\pi sy^2/k}\,dy,$$

by an application of Cauchy's theorem to $\exp(-\pi sy^2/k)$ integrated over the rectangle with vertices $y = \pm R$ and $y = \pm R - i(n+h)$ with R then tending to ∞. Thus, (7.3) becomes

$$(7.4)\qquad \theta(1/s,g,h;A) = se^{-2\pi igh/k}\sum_{n=-\infty}^{\infty}{}^{*} b_n e^{-\pi s(n+h)^2/k-2\pi ing/k}\int_{-\infty}^{\infty} e^{-\pi sy^2/k}\,dy$$

$$= J(ks)^{1/2}e^{-2\pi igh/k}\sum_{n=-\infty}^{\infty} b_n e^{-\pi s(n+h)^2/k-2\pi ing/k},$$

where

$$J = \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx = \pi^{-1/2}\Gamma(1/2).$$

By letting $k = s = 1$ and $g = h = 0$ in (7.4), we find that $J = 1$ (and hence $\Gamma(1/2) = \pi^{1/2}$). Theorem 7.1 for $s > 0$ now follows from (7.4) upon replacing s by $1/s$. By analytic continuation, (7.1) is valid for all s with $\sigma > 0$.

For $\sigma > 1/2$, define

$$\psi(s,g,h;A) = \sum_{n=-\infty}^{\infty}{}' a_n e^{2\pi inh/k}|n+g|^{-2s},$$

where the prime on the summation sign indicates that if g is an integer, then the term corresponding to $n = -g$ is to be omitted. By appealing to a general theorem of S. Bochner [12], p. 338, we may with the help of (7.1) deduce the functional equation of $\psi(s,g,h;A)$ given below at once.

The proof is facilitated by defining

$$\theta_1(s,g,h;A) = \sum_{n=-\infty}^{\infty}{}' a_n e^{-\pi s(n+g)^2/k+2\pi inh/k} = \theta(s,g,h;A) - \delta_g a_{-g} e^{-2\pi igh/k},$$

where $\delta_g = 1$ if g is an integer and $\delta_g = 0$ otherwise. Then (7.1) can be expressed as the "modular" relation

$$\theta_1(x,g,h;A) = (k/x)^{1/2}e^{-2\pi igh/k}\theta_1(1/x,h,-g;B) + \delta_h b_{-h}(k/x)^{1/2} - \delta_g a_{-g}e^{-2\pi igh/k}.$$

Finally, the Mellin transform for $\sigma > 1/2$

$$\int_0^{\infty} x^{s-1}\theta_1(x,g,h;A)\,dx = (k/\pi)^s\Gamma(s)\psi(s,g,h;A)$$

expresses the relationship between θ_1 and ψ and enables Bochner's Theorem 3 to be used. Of course, the functional equation can also be proved directly in a simple way by using the methods of Section 6.

COROLLARY 7.2. *The Dirichlet series $\psi(s,g,h;A)$ possesses an analytic continuation into the entire s-plane where it is holomorphic with the possible exception of a simple pole at $s = 1/2$; the pole exists only if h is an integer and $b_{-h} \neq 0$ in which case the residue is b_{-h}. Furthermore, $\psi(s,g,h;A)$ satisfies the functional equation*

$$(\pi/k)^{-s}\Gamma(s)\psi(s,g,h;A) = k^{1/2}e^{-2\pi igh/k}(\pi/k)^{s-1/2}\Gamma(\tfrac{1}{2}-s)\psi(\tfrac{1}{2}-s,h,-g;B).$$

In the case $A = I$, Corollary 7.2 reduces to another result of Epstein [23].

We next give another proof, based upon Theorem 7.1, of the reciprocity law, Theorem 5.1.

THEOREM 7.3. *Let a and c be integers such that $ac \neq 0$, and let x and y be real numbers such that $ack+2(x+y)$ is an even integer. Then,*

$$(7.5)\qquad \sum_{n=0}^{|c|k-1} a_n e^{\pi ia(n+x/a)^2/ck+2\pi iny/ck}$$

$$= |ck/a|^{1/2}e^{-2\pi ixy/ack+\frac{1}{4}\pi i\,\mathrm{sgn}(ac)}\sum_{n=0}^{|a|k-1} b_n e^{-\pi ic(n+y/c)^2/ak-2\pi inx/ak}.$$

Although this result appears to be more general than Theorem 5.1, it is not. For if we put $2(x+y) = b$, (7.5) is transformed into (5.1).

G. Landsberg [30] was evidently the first person to apply the theory of theta-functions to Gaussian sums when he derived the reciprocity law for ordinary Gaussian sums. His proof is reproduced in R. Bellman's book [4]. M. Lerch [32] generalized Landsberg's method to obtain the reciprocity theorem for a slightly more general class of Gaussian sums than the ordinary Gaussian sums. However, in fact, Landsberg [30], equation (14), p. 242, obtains a version of Theorem 7.3 less explicit than we do; also Lerch's result is covered by Landsberg's as well. A type of reciprocity law for some expressions which generalize Gaussian sums has been proved by Bochner [13] using theta-relations.

The proof given below does not differ in spirit from those mentioned above, and so some details will be omitted.

Proof of Theorem 7.3. If $s = k\varepsilon - ia/c$, $\varepsilon > 0$, we have

$$(7.6)\quad \theta(s, g, h; A) = \sum_{n=-\infty}^{\infty} a_n e^{-\pi\varepsilon(n+g)^2 + \pi ia(n+g)^2/ck + 2\pi inh/k}$$
$$= \sum_{r=0}^{|c|k-1} a_r e^{2\pi irh/k} \sum_{m=-\infty}^{\infty} e^{-\pi\varepsilon(mck+r+g)^2 + \pi ia(mck+r+g)^2/ck + 2\pi imch},$$

where we have put $n = ckm + r$, $-\infty < m < \infty$, $0 \leqslant r \leqslant |c|k-1$. Let $g = x/a$ and $h = y/c$. Using the fact that $ack + 2(x+y)$ is even, we find that (7.6) becomes

$$(7.7)\quad \theta(s, g, h; A) = \sum_{r=0}^{|c|k-1} a_r e^{\pi ia(r+x/a)^2/ck + 2\pi iry/ck} \sum_{m=-\infty}^{\infty} e^{-\pi\varepsilon(mck+r+g)^2}$$
$$\sim (1/|c|k\varepsilon^{1/2}) \sum_{r=0}^{|c|k-1} a_r e^{\pi ia(r+x/a)^2/ck + 2\pi iry/ck}$$

as $\varepsilon \to 0+$, since the inside sum is just

$$\theta\big(\varepsilon c^2k^3, (r+g)/ck, 0; I\big) \sim (|c|k\varepsilon^{1/2})^{-1}$$

by (7.1).

On the other hand, putting $\eta = \varepsilon/(1 + i\varepsilon ck/a)$, we obtain

$$\theta(1/s, h, -g; B)$$
$$= \sum_{n=-\infty}^{\infty} b_n e^{-\pi(n+h)^2/k(k\varepsilon - ia/c) - 2\pi ing/k} = \sum_{n=-\infty}^{\infty} b_n e^{-\pi i(n+h)^2 c/ak - \pi(n+h)^2c^2\eta/a^2 - 2\pi ing/k}$$
$$= \sum_{r=0}^{|a|k-1} b_r e^{-2\pi irg/k} \sum_{m=-\infty}^{\infty} e^{-\pi i(-mak+r+h)^2c/ak - \pi(-mak+r+h)^2c^2\eta/a^2 + 2\pi imag},$$

where we put $n = -mak + r$, $-\infty < m < \infty$, $0 \leqslant r \leqslant |a|k-1$. Again, using the fact that $ack + 2(x+y)$ is even, we find that the above simplifies to

$$(7.8)\quad \theta(1/s, h, -g; B)$$
$$= \sum_{r=0}^{|a|k-1} b_r e^{-\pi ic(r+y/c)^2/ak - 2\pi irx/ak} \sum_{m=-\infty}^{\infty} e^{-\pi(mak-r-y/c)^2c^2\eta/a^2}$$
$$\sim (1/|c|k\varepsilon^{1/2}) \sum_{r=0}^{|a|k-1} b_r e^{-\pi ic(r+y/c)^2/ak - 2\pi irx/ak}$$

as $\varepsilon \to 0+$.

We also have as $\varepsilon \to 0+$,

$$(7.9)\quad (k/s)^{1/2} = (\varepsilon - ia/ck)^{-1/2} \sim |a/ck|^{-1/2} e^{\frac{1}{4}\pi i\,\mathrm{sgn}(ac)}.$$

If we now substitute (7.7)–(7.9) into (7.1), we deduce (7.5) forthwith.

8. The periodic Lipschitz summation formula. We next derive a periodic analogue of a classical summation formula due to Lipschitz [33]. For $a_n = \chi(n)$, where $\chi(n)$ is primitive, a proof of the next theorem has been given in [9] by Berndt.

THEOREM 8.1. *Let* $\mathrm{Re}(z) > 0$ *and* a *be real. If* a *is not an integer or if* a *is an integer and* $b_{-a} = 0$, *assume that* $\sigma > 0$; *otherwise assume that* $\sigma > 1$. *Then*

$$(8.1)\quad \Gamma(s) \sum_{n=-\infty}^{\infty} a_n (z+ni)^{-s} e^{2\pi ina/k} = k(2\pi/k)^s \sum_{n+a>0} b_n (n+a)^{s-1} e^{-2\pi z(n+a)/k},$$

where $(z+ni)^{-s}$ *has its principal value.*

Proof. Apply Theorem 4.4 with $f(u) = (z+ui)^{-s}\exp(2\pi iua/k)$ where $\sigma > 1$. Accordingly, we obtain

$$(8.2)\quad \sum_{c \leqslant n \leqslant d}{}' a_n (z+ni)^{-s} e^{2\pi ina/k} = \sum_{n=-\infty}^{\infty}{}^{*} b_n \int_c^d (z+ui)^{-s} e^{2\pi i(n+a)u/k}\,du,$$

where $\sum_{n=-\infty}^{\infty}{}^{*}$ is defined as in the proof of Theorem 7.1. We proceed, as in that proof, and do two integrations by parts to justify letting $c \to -\infty$ and $d \to \infty$. Thus, letting $c \to -\infty$ and $d \to \infty$ in (8.2), we get for $\sigma > 1$

$$(8.3)\quad \sum_{n=-\infty}^{\infty} a_n (z+ni)^{-s} e^{2\pi ina/k} = k^{1-s} \sum_{n=-\infty}^{\infty}{}^{*} b_n J(s, n+a),$$

where

$$(8.4)\quad J(s, a) = \int_{-\infty}^{\infty} (z/k + vi)^{-s} e^{2\pi iav}\,dv = \begin{cases} 0, & \text{if } a \leqslant 0, \\ \dfrac{(2\pi)^s}{\Gamma(s)} a^{s-1} e^{-2\pi za/k}, & \text{if } a > 0. \end{cases}$$

(The computation of $J(s, a)$ is performed in very clear, ample detail in H. Rademacher's book [39], pp. 78–79.) If we use (8.4) in (8.3), we arrive at (8.1) for $\sigma > 1$ since the * may be dropped from the summation sign.

For fixed z with $\mathrm{Re}(z) > 0$, the right side of (8.1) can clearly be analytically continued to an entire function of s. To prove our assertions on the validity of (8.1), we let M, N be positive integers and set $n = mk + r$, $-M \leqslant m \leqslant N-1$, $1 \leqslant r \leqslant k$, to obtain

$$S_{M,N} = \sum_{n=-Mk+1}^{Nk} a_n e^{2\pi ina/k} = \sum_{r=1}^{k} a_r e^{2\pi ira/k} \sum_{m=-M}^{N-1} e^{2\pi ima}.$$

Now if a is not an integer, we easily deduce that

$$|S_{M,N}| \leqslant k \max_{1\leqslant r\leqslant k} |a_r|\,|\csc(\pi a)| = O(1)$$

as $N\to\infty$. If a is an integer, then by (4.1) we have

$$S_{M,N} = kb_{-a}(M+N). \tag{8.5}$$

So, if a is not an integer, or if a is an integer and $b_{-a} = 0$, then $S_{M,N} = O(1)$ as $M, N\to\infty$. From the general theory of Dirichlet series the left side of (8.1) then converges for $\sigma > 0$ and is analytic for $\sigma > 0$. Thus, by analytic continuation, (8.1) is valid for $\sigma > 0$. (On the other hand, if a is an integer and $b_{-a} \neq 0$, then (8.5) does not permit us to deduce that the left side of (8.1) can be continued to the left of $\sigma = 1$.) This completes the proof.

A glance at (8.1) shows that there is an obvious connection with the periodic Lerch function. Observe that in Definition 3, we could take a to be complex and x complex with $\operatorname{Im}(x) \geqslant 0$. If $\operatorname{Im}(x) > 0$, then s can be arbitrary. In the discussion below, we shall assume these modifications in the definition of $\varphi(x, a, s; A)$.

If we replace a by x and z by ia with $\operatorname{Im}(a) < 0$, then (8.1) becomes

$$e^{-\pi is/2}\Gamma(s)\sum_{n=-\infty}^{\infty} a_n(n+a)^{-s}e^{2\pi inx/k} = k(2\pi/k)^s \sum_{n+x>0} b_n(n+x)^{s-1}e^{-2\pi ia(n+x)/k}. \tag{8.6}$$

The sum on the left side of (8.6) may be written as

$$\sum_{m=0}^{\infty} a_{-m-1}(-m-1+a)^{-s}e^{-2\pi i(m+1)x/k} + \varphi(x, a, s; A) = e^{\pi is-2\pi ix/k}\varphi(-x, 1-a, s; A_1^*) + \varphi(x, a, s; A).$$

Hence, (8.6) becomes

$$\Gamma(s)\{e^{\pi is/2-2\pi ix/k}\varphi(-x, 1-a, s; A_1^*) + e^{-\pi is/2}\varphi(x, a, s; A)\} = k(2\pi/k)^s \sum_{n+x>0} b_n(n+x)^{s-1}e^{-2\pi ia(n+x)/k}. \tag{8.7}$$

If we let $m = n+[x]$ on the right side of (8.7) and define a sequence B'_x by $B'_x = \{b_{n+1-[x]}\}$, $-\infty < n < \infty$, then the right side of (8.7) may be expressed in terms of a periodic Lerch function associated with B'_x or B'_{x+1}, depending upon whether x is integral or non-integral.

9. Further properties of periodic Bernoulli numbers, functions, and polynomials. In this section we derive a partial theory of periodic Bernoulli numbers, periodic Bernoulli functions, and periodic Bernoulli polynomials, which will be defined below. We shall emphasize the analogies between our results and the corresponding properties of the ordinary Bernoulli numbers, functions, and polynomials.

In the case in which $a_n = \chi(n)$, a primitive character modulo k, the associated periodic Bernoulli numbers were originally defined by Ankeny, Artin and Chowla [2] and Leopoldt [31]. These numbers, which are called generalized Bernoulli numbers, were, in fact, defined by the aforementioned authors by equation (9.2) below for $a_n = \chi(n)$. Many of the arithmetical properties proved below have been proved by Carlitz [16], Leopoldt [31] and others in the case $a_n = \chi(n)$. Leopoldt [31] has also proven special cases of (6.17) and (6.25).

With β defined by (4.11) and ϱ_r by (4.12), we recall that Corollary 4.3 gave for all real x and $r\geqslant 1$

$$|P_r(x, A)| \leqslant \varrho_r. \tag{9.1}$$

PROPOSITION 9.1. *For* $|y| < 2\pi/k$,

$$\frac{y\sum_{n=0}^{k-1} a_n e^{ny}}{e^{ky}-1} = \sum_{j=0}^{\infty}\frac{B_j(A)}{j!}y^j = e^{B(A)y}, \tag{9.2}$$

where the last expression uses the umbral convention according to which after the formal expansion into power series, the expression $\{B(A)\}^j$ *is to be replaced by* $B_j(A)$.

Proof. In Corollary 3.4 put $M = 0$, $N = 1$, and $f(x) = \exp(xy)$, where $0 < |y| < 2\pi/k$. Accordingly, we obtain

$$\sum_{n=0}^{k-1} a_n e^{ny} = B_0(A)(e^{ky}-1)/y + \sum_{j=1}^{r}\frac{B_j(A)}{j!}y^{j-1}(e^{ky}-1) + E_r(0, A), \tag{9.3}$$

where, by (9.1),

$$|E_r(0, A)| \leqslant \varrho_r\int_0^k |y^r e^{xy}|\,dx \leqslant \varrho_r|y|^{r-1}(e^{k|y|}-1).$$

For $0 < |y| < 2\pi/k$, the definition (4.12) shows that $E_r(0, A)$ tends to 0 as r tends to ∞. Letting r tend to ∞ in (9.3), we arrive at (9.2) after a little rearrangement. Using the definition of $B_0(A)$, we lastly note that (9.2) is valid for $y = 0$ as well.

When $A = I$, (9.2) gives the well-known generating function for the ordinary Bernoulli numbers as given in Fort's book [24], p. 57,

$$\frac{y}{e^y-1} = \sum_{j=0}^{\infty}\frac{B_j}{j!}y^j = e^{By}.$$

For $A = I$, the next result reduces to the familiar result ([24], p. 31, or [28], p. 183)

$$(B+1)^n - B^n = 0, \quad n\geqslant 2,$$

where we again use the umbral convention.

PROPOSITION 9.2. *For $n \geqslant 1$,*

$$(B(A)+k)^n - B(A)^n = nM_{n-1}(A), \tag{9.4}$$

where $M_n(A)$ is defined by (6.18).

Proof. Rewrite (9.2) as

$$y\sum_{j=0}^{k-1} a_j e^{jy} = \sum_{j=0}^{\infty} \frac{B_j(A)}{j!} y^j \sum_{m=1}^{\infty} \frac{(ky)^m}{m!} = \sum_{n=1}^{\infty} \frac{y^n}{n!} \sum_{j=0}^{n-1} \binom{n}{j} B_j(A) k^{n-j}.$$

If we expand e^{jy} in a power series and then equate coefficients of like powers of y on each side, we obtain for $n \geqslant 1$

$$\frac{1}{(n-1)!}\sum_{j=0}^{k-1} a_j j^{n-1} = \frac{1}{n!}\sum_{j=0}^{n-1}\binom{n}{j} B_j(A) k^{n-j},$$

which is plainly equivalent to (9.4).

DEFINITION 5. The *n-th periodic Bernoulli polynomial* $B_n(x, A)$ is defined for $0 \leqslant x < 1$ by $B_n(x, A) = n!P_n(x, A)$. By the remarks after (2.6), or immediately below, $B_n(x, A)$ is, indeed, a polynomial, and the degree is at most n. For values of x outside of $[0, 1)$, $B_n(x, A)$ is defined by analytic continuation, i.e., by the polynomial itself.

Since for $0 \leqslant x < 1$, $n!P_n(x) = B_n(x)$, the nth ordinary Bernoulli polynomial, by Corollary 4.2 for $n \geqslant 1$

$$B_n(x, A) = k^{n-1}\sum_{m=0}^{k-1} a_{-m} B_n\left(\frac{x+m}{k}\right). \tag{9.5}$$

By analytic continuation, (9.5) is valid for all x; (9.5) also holds for $n = 0$. Since $B_n(x)$ has degree n, $B_n(x, A)$ is a polynomial with degree at most n.

By (2.12) we have $B_j(0, A) = (-1)^j B_j(A)$ if $j \neq 1$; and from (2.8), $B_1(0, A) = -B_1(A) - a_0$. Furthermore, by (2.6) we have for $0 < x < 1$ and $0 \leqslant j \leqslant n$

$$\frac{1}{n!} B_n^{(j)}(x, A) = P_n^{(j)}(x, A) = P_{n-j}(x, A) = \frac{1}{(n-j)!} B_{n-j}(x, A).$$

Consequently, the above holds for all x, so that for all x and $0 \leqslant j \leqslant n$

$$\frac{1}{n!} B_n^{(j)}(x, A) = \frac{1}{(n-j)!} B_{n-j}(x, A). \tag{9.6}$$

The next proposition generalizes a well-known formula ([24], p. 28, or [28], p. 525) to which it reduces when $A = I$. (In the formula in Fort's text [24], read $B_{n-\nu}(x)$ for $B_{n-\nu}$.)

PROPOSITION 9.3. *For $n \geqslant 0$ and all x and y,*

$$B_n(x+y, A) = \sum_{j=0}^{n}\binom{n}{j} B_j(y, A) x^{n-j} = \{x + B(y, A)\}^n. \tag{9.7}$$

Proof. By Taylor's theorem and (9.6),

$$B_n(x+y, A) = \sum_{j=0}^{n} \frac{B_n^{(j)}(y, A)}{j!} x^j = \sum_{j=0}^{n} \frac{n!B_{n-j}(y, A)}{j!(n-j)!} x^j,$$

which is just (9.7).

On taking $y = 0$ in (9.7) and using the values of $B_j(0, A)$ obtained below Definition 5, we see that if $n \geqslant 1$

$$\begin{aligned} B_n(x, A) &= \sum_{j=0}^{n}\binom{n}{j}(-1)^j B_j(A) x^{n-j} - na_0 x^{n-1} \\ &= \{x - B(A)\}^n - na_0 x^{n-1}. \end{aligned} \tag{9.8}$$

The next proposition reduces to the well-known generating function for the Bernoulli polynomials $B_n(x)$ when $A = I$ ([24], p. 57).

PROPOSITION 9.4. *For $|y| < 2\pi/k$,*

$$\frac{y\sum_{n=0}^{k-1} a_{-n} e^{(n+x)y}}{e^{ky}-1} = \sum_{j=0}^{\infty} \frac{B_j(x, A)}{j!} y^j = e^{B(x,A)y}.$$

Proof. Multiplying both sides of (9.2) by $\exp(-xy)$ and subsequently using (9.8), we get for $|y| < 2\pi/k$,

$$\begin{aligned} \frac{y\sum_{n=0}^{k-1} a_n e^{(n-x)y}}{e^{ky}-1} &= \sum_{n=0}^{\infty} y^n \sum_{j+m=n} \frac{B_j(A)(-x)^m}{j!m!} \\ &= \sum_{n=0}^{\infty} \frac{(-y)^n}{n!} \sum_{j=0}^{n}\binom{n}{j}(-1)^j B_j(A) x^{n-j} \\ &= \sum_{n=0}^{\infty} \frac{(-y)^n}{n!} B_n(x, A) - a_0 y e^{-xy}, \end{aligned}$$

or

$$\frac{y\sum_{n=1}^{k} a_n e^{(n-x)y}}{e^{ky}-1} = \sum_{n=0}^{\infty} \frac{B_n(x, A)}{n!}(-y)^n.$$

Replacing $-y$ by y and then n by $k-n$ in the above, we find that

$$\sum_{j=0}^{\infty} \frac{B_j(x, A)}{j!} y^j = \frac{-y \sum_{n=1}^{k} a_n e^{-(n-x)y}}{e^{-ky}-1} = \frac{y \sum_{m=0}^{k-1} a_{-m} e^{(m+x)y}}{e^{ky}-1},$$

and we are done.

The next result shows that a periodic Bernoulli number can be written as a linear combination of ordinary Bernoulli numbers.

PROPOSITION 9.5. *For $n = 0$ and $n \geq 2$,*

$$B_n(A) = (-1)^n \sum_{j=0}^{n} \binom{n}{j} B_j k^{j-1} M_{n-j}(A^*) = \frac{1}{k}(-1)^n \{kB + M(A^*)\}^n.$$

Proof. The result is trivial for $n = 0$. For $n \geq 2$, we find that from (9.5) and (9.7)

$$B_n(0, A) = k^{n-1} \sum_{j=0}^{k-1} a_{-j} B_n(j/k)$$
$$= k^{n-1} \sum_{j=0}^{k-1} a_{-j} \sum_{r=0}^{n} \binom{n}{r} B_r (j/k)^{n-r} = \sum_{r=0}^{n} \binom{n}{r} B_r k^{r-1} M_{n-r}(A^*),$$

and the result follows from (2.12) and Definition 5.

The next proposition is analogous to the well-known fact [24], p. 26, that for $n \geq 0$

$$B_n(x+1) - B_n(x) = nx^{n-1}. \tag{9.9}$$

PROPOSITION 9.6. *For $n \geq 0$,*

$$B_n(x+k, A) - B_n(x, A) = n \sum_{j=0}^{k-1} a_{-j}(x+j)^{n-1}.$$

Proof. For $n = 0$, the result is trivial. For $n \geq 1$, we have from (9.5) and (9.9),

$$B_n(x+k, A) - B_n(x, A) = k^{n-1} \sum_{j=0}^{k-1} a_{-j} \left\{ B_n\left(\frac{x+j}{k}+1\right) - B_n\left(\frac{x+j}{k}\right) \right\}$$
$$= k^{n-1} \sum_{j=0}^{k-1} a_{-j} n \left(\frac{x+j}{k}\right)^{n-1},$$

and the proposition follows.

Recall the multiplication theorem for the ordinary Bernoulli polynomials [24], p. 35,

$$\sum_{j=0}^{m-1} B_n(x+j/m) = m^{1-n} B_n(mx), \tag{9.10}$$

where $m \geq 1$ and $n \geq 0$. We now show that we can replace $B_n(z)$ by $P_n(z)$ in (9.10). Let

$$g(x) = \sum_{j=0}^{m-1} P_n(x+j/m) \quad \text{and} \quad h(x) = m^{1-n} P_n(mx).$$

It is easily seen that $g(x)$ and $h(x)$ both have period $1/m$. If $0 \leq x < 1/m$, we have from (9.10),

$$g(x) = \sum_{j=0}^{m-1} B_n(x+j/m) = m^{1-n} B_n(mx) = m^{1-n} P_n(mx).$$

By periodicity, we then have for all x,

$$\sum_{j=0}^{m-1} P_n(x+j/m) = m^{1-n} P_n(mx). \tag{9.11}$$

We now prove a multiplication theorem for periodic Bernoulli functions.

PROPOSITION 9.7. *For $m, n \geq 1$,*

$$\sum_{j=0}^{mk-1} P_n(x+j/m, A) = kB_0(A) m^{1-n} P_n(mx).$$

Proof. From Corollary 4.2 and (9.11),

$$\sum_{j=0}^{mk-1} P_n(x+j/m, A) = k^{n-1} \sum_{r=0}^{k-1} a_r \sum_{j=0}^{mk-1} P_n(x/k + j/mk - r/k)$$
$$= m^{1-n} \sum_{r=0}^{k-1} a_r P_n(mx - mr),$$

and the proposition follows immediately.

The remaining results pertain to the special cases when A is even or odd. Recall that $\gamma = 1$ if A is even, and that $\gamma = -1$ if A is odd. The next two propositions generalize the facts, obvious from (4.3), that

$$P_n(1-x) = (-1)^n P_n(x) \quad \text{and} \quad B_n(1-x) = (-1)^n B_n(x), \tag{9.12}$$

where $n = 0$ or $n \geq 2$.

PROPOSITION 9.8. *If $a_j = \gamma a_{-j}$, then for $n \geq 2$*

$$P_n(k-x, A) = (-1)^n \gamma P_n(x, A).$$

If, in addition, $0 \leqslant a < 1$, *then*

$$\zeta(1-n, a; A) = -\frac{1}{n}\gamma B_n(a, A).$$

Proof. From Corollary 4.2 and (9.12), for $n \geqslant 2$,

$$P_n(-x, A) = k^{n-1}\sum_{m=0}^{k-1} a_{-m}P_n\left(\frac{-x+m}{k}\right) = k^{n-1}\sum_{j=1}^{k} a_{j-k}P_n\left(1-\frac{x+j}{k}\right)$$

$$= (-1)^n k^{n-1}\sum_{j=0}^{k-1} a_j P_n\left(\frac{x+j}{k}\right) = (-1)^n\gamma P_n(x, A).$$

The second conclusion follows from Corollary 6.4 and Definition 5.

In a similar fashion, one can use (9.5), (9.12) and (9.9) to prove

PROPOSITION 9.9. *If* $a_j = \gamma a_{-j}$, *then for* $n \geqslant 2$,

$$B_n(-x, A) = (-1)^n\gamma B_n(x, A) + (-1)^n a_0 n x^{n-1}$$

and

$$B_n(k-x, A) = (-1)^n\gamma\left\{B_n(x, A) - n\sum_{j=1}^{k-1} a_j(x-j)^{n-1}\right\}.$$

The second conclusion follows from the first on using Proposition 9.6 and the fact that $(\gamma-1)a_0 = 0$ in both cases.

Hence, if $a_0 = 0$ and $(-1)^n\gamma = 1$, then $B_n(x, A)$ is even; if $a_0 = 0$ and $(-1)^n\gamma = -1$, then $B_n(x, A)$ is odd.

The following result is a generalization of a formula known in the case $A = I$ in which case $\zeta(s; A) = \zeta(s)$. (See the handbook of Abramowitz and Stegun [1], p. 807.) The result also complements (6.25).

PROPOSITION 9.10. *If* $n \geqslant 2$ *and* $(-1)^n\gamma = -1$, *then*

$$\zeta(n; B) = \frac{i}{2k}(2\pi i/k)^n \int_0^k P_n(x, A)\cot(\pi x/k)\,dx. \tag{9.13}$$

Proof. On integrating the second formula of Corollary 4.5 termwise, we find that

$$\int_0^k P_n(x, A)\cot(\pi x/k)\,dx \tag{9.14}$$

$$= -2i\sum_{j=1}^{\infty}(k/2\pi ij)^n b_j \int_0^k \sin(2\pi jx/k)\cot(\pi x/k)\,dx.$$

Now for $j \geqslant 1$,

$$\sin(2\pi jx/k)\cot(\pi x/k) = \sum_{r=-j+1}^{j}\cos(2\pi rx/k). \tag{9.15}$$

If we integrate both sides of (9.15) over $[0, k]$, we obtain the value k. Substituting this value in (9.14), we arrive at (9.13) forthwith.

Our last result gives an integral representation of the periodic Bernoulli numbers when A is even or odd.

PROPOSITION 9.11. *For* $m, n \geqslant 1$,

$$\int_0^1 P_m(kx, A)P_n(x)\,dx = \frac{(-1)^{n+1}\gamma}{k^n(m+n)!}B_{m+n}(A). \tag{9.16}$$

Proof. We shall give the proof when $(-1)^m\gamma = -1$ and n is odd. The proofs in the three remaining cases are analogous except that when $(-1)^{m+n}\gamma = -1$ we use Corollary 4.6 as well.

From Corollary 4.5 and an inversion in order of summation and integration,

$$\int_0^1 P_m(kx, A)P_n(x)\,dx$$

$$= -\frac{4k^m}{(2\pi i)^{m+n}}\sum_{\mu=1}^{\infty}\sum_{\nu=1}^{\infty} b_\mu \mu^{-m}\nu^{-n}\int_0^1 \sin(2\pi\mu x)\sin(2\pi\nu x)\,dx$$

$$= -\frac{4k^m}{(2\pi i)^{m+n}}\sum_{\mu=1}^{\infty} b_\mu \mu^{-m-n}\int_0^1 \sin^2(2\pi\mu x)\,dx$$

$$= -\frac{2k^m}{(2\pi i)^{m+n}}\zeta(m+n; B) = \frac{(-1)^{m+n}}{k^n(m+n)!}B_{m+n}(A),$$

by (6.25). This is plainly equivalent to (9.16) since $(-1)^m = -\gamma$.

10. Applications to trigonometric functions. Some of the results obtained here bear a strong resemblance to Proposition 9.1 and could, indeed, be obtained from it by suitable specialization and manipulation. Nevertheless, we proceed directly.

PROPOSITION 10.1. *Let* y *and* z *be complex numbers with* $|y| < 2\pi/k$. *Then*

$$y\sum_{n=0}^{k-1} a_n\cos(ny+z) \tag{10.1}$$

$$= \{\sin(ky+z) - \sin z\}\sum_{j=0}^{\infty}\frac{(-1)^j B_{2j}(A)}{(2j)!}y^{2j} +$$

$$+ \{\cos(ky+z) - \cos z\}\sum_{j=0}^{\infty}\frac{(-1)^j B_{2j+1}(A)}{(2j+1)!}y^{2j+1}$$

$$= \{\sin(ky+z) - \sin z\}\cos\{B(A)y\} + \{\cos(ky+z) - \cos z\}\sin\{B(A)y\}.$$

In particular, if A is even,

$$\cos\{B(A)y\} = \frac{y\sum_{n=0}^{k}{}' a_n\cos(ny+z)}{\sin(ky+z)-\sin z} \tag{10.2}$$

$$= \frac{y\sum_{n=0}^{k} a_n\cos(ny+z)}{\sin(ky+z)-\sin z} - a_0\frac{y}{2}\cot\frac{ky}{2}$$

$$= \frac{y\sum_{n=1}^{k-1} a_n\cos(ny+z)}{\sin(ky+z)-\sin z} + a_0\frac{y}{2}\cot\frac{ky}{2},$$

where the prime $'$ on the summation sign indicates that the first and last terms of the sum are to be halved. And, if A is odd,

$$\sin\{B(A)y\} = \frac{y\sum_{n=0}^{k-1} a_n\cos(ny+z)}{\cos(ky+z)-\cos z}. \tag{10.3}$$

Proof. In Corollary 3.4, put $M=0$, $N=1$, and $f(x)=\cos(xy+z)$, where y and z are real. As in the proof of Proposition 9.1, $E_r(0, A)$ tends to 0 as r tends to ∞, provided that $|y|<2\pi/k$. Since

$$f^{(2j)}(x) = (-1)^j y^{2j}\cos(xy+z) \quad \text{and} \quad f^{(2j-1)}(x) = (-1)^j y^{2j-1}\sin(xy+z),$$

(10.1) follows very easily from (3.10) for y and z real with $|y|<2\pi/k$. By analytic continuation, (10.1) holds for complex y and z with $|y|<2\pi/k$. The special case (10.3) follows from (10.1) and Corollary 4.6. For even A, the first equation in (10.2) is also obtained by using Corollary 4.6. The remaining equations are obtained after observing that the terms corresponding to $n=0$ and to $n=k$ in the second expression of (10.2) contribute to this a total of

$$a_0\frac{y}{2}\,\frac{\cos(ky+z)+\cos z}{\sin(ky+z)-\sin z} = a_0\frac{y}{2}\,\frac{2\cos(\frac12 ky+z)\cos\frac12 ky}{2\cos(\frac12 ky+z)\sin\frac12 ky} = a_0\frac{y}{2}\cot\tfrac12 ky.$$

It is obvious that (10.1), (10.2) and (10.3) can be written in a variety of forms by changing z. For example, if z is replaced by $z-\pi/2$, then every term of the kind $\cos(ny+z)$ is replaced by $\sin(ny+z)$, and every term of the kind $\sin(ky+z)$ is replaced by $-\cos(ky+z)$. All of the equations assume a particularly simple form when $z=0$.

It is remarkable that all of the expressions in (10.2) and (10.3) are independent of z. As a consequence of this, and the remarks in the preceding

paragraph, we obtain for even A

$$\frac{\sum_{n=0}^{k}{}'' a_n\cos(ny+z_1)}{\sin(ky+z_1)-\sin z_1} = \frac{\sum_{n=0}^{k}{}'' a_n\cos(ny+z_2)}{\sin(ky+z_2)-\sin z_2} \tag{10.4}$$

$$= \frac{\sum_{n=0}^{k}{}'' a_n\sin(ny+z_3)}{\cos z_3-\cos(ky+z_3)} = \frac{\sum_{n=0}^{k}{}'' a_n\sin(ny+z_4)}{\cos z_4-\cos(ky+z_4)},$$

where the $''$ on the summation signs means that the range of summation is either $[0, k]$ with the first and last terms halved, or is $[0, k]$ or $[1, k-1]$ without this restriction; however, the identical interpretation is to be used in all four sums. The result (10.4) is valid for arbitrary complex z_1, z_2, z_3 and z_4, as well as for arbitrary y, since the range of y can be extended by analytic continuation. Similarly, for all complex z_1, z_2, z_3, z_4 and y, we get for odd A

$$\frac{\sum_{n=0}^{k-1} a_n\cos(ny+z_1)}{\cos(ky+z_1)-\cos z_1} = \frac{\sum_{n=0}^{k-1} a_n\cos(ny+z_2)}{\cos(ky+z_2)-\cos z_2} \tag{10.5}$$

$$= \frac{\sum_{n=0}^{k-1} a_n\sin(ny+z_3)}{\sin(ky+z_3)-\sin z_3} = \frac{\sum_{n=0}^{k-1} a_n\sin(ny+z_4)}{\sin(ky+z_4)-\sin z_4}.$$

Furthermore, we may replace y by iy and z by iz in (10.1) to obtain the following result for hyperbolic functions:

$$y\sum_{n=0}^{k-1} a_n\cosh(ny+z) \tag{10.6}$$

$$= \{\sinh(ky+z)-\sinh z\}\sum_{j=0}^{\infty}\frac{B_{2j}(A)}{(2j)!}y^{2j} +$$

$$+\{\cosh(ky+z)-\cosh z\}\sum_{j=0}^{\infty}\frac{B_{2j+1}(A)}{(2j+1)!}y^{2j+1}$$

$$= \{\sinh(ky+z)-\sinh z\}\cosh\{B(A)y\} + \{\cosh(ky+z)-\cosh z\}\sinh\{B(A)y\},$$

provided that $|y|<2\pi/k$. Of course, (10.6) can also be proved directly from Corollary 3.4. The same substitutions convert (10.2) into the following result for even A:

$$\text{(10.7)}\quad \cosh\{B(A)y\} = \frac{y\sum_{n=0}^{k}{}' a_n\cosh(ny+z)}{\sinh(ky+z)-\sinh z} = \frac{y\sum_{n=0}^{k} a_n\cosh(ny+z)}{\sinh(ky+z)-\sinh z} - a_0\frac{y}{2}\coth\frac{ky}{2}$$
$$= \frac{y\sum_{n=1}^{k-1} a_n\cosh(ny+z)}{\sinh(ky+z)-\sinh z} + a_0\frac{y}{2}\coth\frac{ky}{2},$$

which is valid if $|y| < 2\pi/k$. For the same y, (10.3) yields for odd A

$$\text{(10.8)}\qquad \sinh\{B(A)y\} = \frac{y\sum_{n=0}^{k-1} a_n\cosh(ny+z)}{\cosh(ky+z)-\cosh z}.$$

We can specialize (10.2) by setting $A = I$ and thereby obtain the well-known expansion ([28], p. 204)

$$\frac{y}{2}\cot\frac{y}{2} = \sum_{j=0}^{\infty}\frac{(-1)^j B_{2j}}{(2j)!}y^{2j} = \cos(By),$$

provided that $|y| < 2\pi$. Similarly, (10.7) yields for $|y| < 2\pi$

$$\frac{y}{2}\coth\frac{y}{2} = \sum_{j=0}^{\infty}\frac{B_{2j}}{(2j)!}y^{2j} = \cosh(By).$$

The formulas (10.4), (10.5) and their analogues for the hyperbolic functions can all be proved by elementary methods. We illustrate this by proving the following consequence of (10.4) for even A,

$$\text{(10.9)}\qquad \frac{\sum_{n=0}^{k} a_n\cos(ny)}{\sin(ky)} = \frac{\sum_{n=0}^{k} a_n\sin(ny)}{1-\cos(ky)}.$$

To prove this, we note that since A is even

$$\sum_{n=0}^{k} a_n\cos(ny) = \sum_{m=0}^{k} a_m\cos(k-m)y$$
$$= \sum_{m=0}^{k} a_m\{\cos(ky)\cos(my)+\sin(ky)\sin(my)\}.$$

A slight rearrangement of the above yields (10.9).

11. Formulas for numerical integration. It is well known that the trapezoidal rule can be derived from the ordinary Euler–Maclaurin formula. See, for example, Munro's paper [36]. In this section, we indicate how the Newton–Cotes quadrature formulas and various other quadrature formulas can be derived from special cases of our periodic Euler–Maclaurin formula, though we shall work out the details in only a few cases.

THEOREM 11.1 (Simpson's parabolic rule). *Let $f \in C^{(r)}[0, 2N]$, where $r \geqslant 4$ and N is a positive integer. Then,*

$$\text{(11.1)}\quad \int_0^{2N} f(x)\,dx = \frac{1}{3}\{f(0)+4f(1)+2f(2)+4f(3)+\ldots+2f(2N-2)+4f(2N-1)+f(2N)\}+$$
$$+\frac{1}{3}\sum_{j=4}^{r}(2^j-4)\frac{B_j}{j!}\{f^{(j-1)}(2N)-f^{(j-1)}(0)\}+$$
$$+\frac{(-1)^r}{3}\int_0^{2N}\{4P_r(x)-2^rP_r(x/2)\}f^{(r)}(x)\,dx.$$

The result also holds for $r = 1, 2, 3$ provided that the sum $\sum_{j=4}^{r}$ is interpreted to be 0.

Proof. Let $a_n = 2$ if n is even and let $a_n = 4$ if n is odd. Thus, $k = 2$, $\gamma = 1$, $B_0(A) = 3$, $B_1(A) = -1$. Applying Corollary 3.4, we find that

$$\text{(11.2)}\quad 3\int_0^{2N} f(x)\,dx$$
$$= 2f(0)+4f(1)+2f(2)+4f(3)+\ldots+2f(2N-2)+4f(2N-1)+$$
$$+\{f(2N)-f(0)\}-\sum_{j=2}^{r}\frac{B_j(A)}{j!}\{f^{(j-1)}(2N)-f^{(j-1)}(0)\}-E_r(0, A).$$

Now from Corollary 4.2 and (9.11), for $n \geqslant 1$,

$$\text{(11.3)}\quad P_n(x, A) = 2^{n-1}\{2P_n(x/2)+4P_n(\{x+1\}/2)\} = 4P_n(x)-2^nP_n(x/2).$$

It follows from (11.3) and (2.12) that for $n \geqslant 2$,

$$\text{(11.4)}\qquad \frac{B_n(A)}{n!} = (-1)^n(4-2^n)P_n(0) = (-1)^n(4-2^n)\frac{B_n}{n!}.$$

Hence, if $n = 2$ or if $n \geqslant 3$ is odd, then $B_n(A) = 0$. (The latter fact also follows from Corollary 4.6.) If we use (11.3) and (11.4) in (11.2), we arrive at (11.1).

THEOREM 11.2 (Simpson's or Newton's composite three-eighths rule). *Let $f\in C^{(r)}[0,3N]$, where $r\geqslant 4$ and N is a positive integer. Then,*

$$\int_0^{3N} f(x)\,dx = \tfrac{3}{8}\{f(0)+3f(1)+3f(2)+2f(3)+\ldots+2f(3N-3)+3f(3N-2)+3f(3N-1)+f(3N)\}+$$

$$+\frac{3}{8}\sum_{j=4}^{r}(3^{j-1}-3)\frac{B_j}{j!}\{f^{(j-1)}(3N)-f^{(j-1)}(0)\}+\frac{3(-1)^r}{8}\int_0^{3N}\{3P_r(x)-3^{r-1}P_r(x/3)\}f^{(r)}(x)\,dx.$$

The result also holds for $r=1,2,3$ with the understanding that the sum $\sum_{j=4}^{r}$ is zero.

Proof. Let $a_n=3$ if $n\equiv 1,2 \pmod 3$ and $a_n=2$ if $n\equiv 0 \pmod 3$. Thus, $k=3$, $\gamma=1$, $B_0(A)=8/3$, and $B_1(A)=-1$. From Corollary 4.2 and (9.11), for $n\geqslant 1$,

$$P_n(x,A)=3^{n-1}\{2P_n(x/3)+3P_n(\{x+1\}/3)+3P_n(\{x+2\}/3)\}=3P_n(x)-3^{n-1}P_n(x/3).$$

Hence, using (2.12), we have for $n\geqslant 2$,

$$\frac{B_n(A)}{n!}=(-1)^n(3-3^{n-1})P_n(0)=(-1)^n(3-3^{n-1})\frac{B_n}{n!}.$$

Thus, if $n=2$ or if $n\geqslant 3$ is odd, $B_n(A)=0$. The result now follows as before by using Corollary 3.4.

THEOREM 11.3 (Weddle's composite rule). *Let $f\in C^{(r)}[0,6N]$, where $r\geqslant 6$ and N is a positive integer. Then,*

$$\int_0^{6N} f(x)\,dx=\frac{3}{10}\{f(0)+5f(1)+f(2)+6f(3)+f(4)+5f(5)+2f(6)+\ldots+f(6N-2)+5f(6N-1)+f(6N)\}-$$

$$-\frac{3}{10}\sum_{j=6}^{r}(3^{j-1}-2^{j+1}+5)\frac{B_j}{j!}\{f^{(j-1)}(6N)-f^{(j-1)}(0)\}+\frac{3(-1)^r}{10}\int_0^{6N}\{5P_r(x)-2^{r+1}P_r(x/2)+3^{r-1}P_r(x/3)\}f^{(r)}(x)\,dx.$$

The result also holds for $1\leqslant r\leqslant 5$ on setting the sum $\sum_{j=6}^{r}$ equal to 0.

Proof. Let $a_n=2$ if $n\equiv 0 \pmod 6$, $a_n=5$ if $n\equiv \pm 1 \pmod 6$, $a_n=1$ if $n\equiv \pm 2 \pmod 6$, and $a_n=6$ if $n\equiv 3 \pmod 6$. Now $k=6$, $\gamma=1$, $B_0(A)=10/3$, and $B_1(A)=-1$. On using Corollary 4.2 and applying (9.11) several times, we get for $n\geqslant 1$

$$P_n(x,A)=5P_n(x)-2^{n+1}P_n(x/2)+3^{n-1}P_n(x/3)$$

and

$$\frac{B_n(A)}{n!}=\{5-2^{n+1}+3^{n-1}\}\frac{B_n}{n!}.$$

We note that $B_2(A)=B_4(A)=0$ and $B_n(A)=0$ if n is odd. Proceeding as before, we reach the desired result.

Of course, many other formulas for numerical integration, besides the Newton–Cotes formulas, can be developed from the periodic Euler–Maclaurin formula. For example, if we know that the main contribution to an integral arises from a certain subinterval of integration, then we can choose the sequence A to reflect this.

Added in proof. For applications to quadratic residues and Bessel functions, see Bruce C. Berndt, *Periodic Bernoulli numbers, summation formulas and applications*, Advanced Seminar on Special Functions (to appear).

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An asymptotic inequality concerning primes in contours for the case of quadratic number fields

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Introduction. Our main result is an extension to quadratic number fields of the result that, assuming Schinzel's Hypothesis H, $\pi(x+y)$ sometimes exceeds $\pi(x)+\pi(y)$ (cf. [6], [15]).

To be concrete and definite, we will at first concentrate on the Gaussian integers. Later we show what modifications are needed to carry over to other quadratic number fields.

For any "reasonable" bounded region S in the complex plane we ask whether, as S expands homothetically, there must appear Gaussian integers y for which the translate by y of our region contains more prime Gaussian integers than the region itself.

Let us temporarily assume Hypothesis H. We may then state our principle result in the following form:

THEOREM 1. *If S is not "logarithmically centered on zero"* (definition to follow), *then*

(1) *For all sufficiently large x there exist arbitrarily large Gaussian integers y for which the translate $xS+y$ contains more prime Gaussian integers than xS.*

Remark. Since on the average there will be fewer primes in $xS+y$ as $|y|$ increases, (1) states that there are exceptions to this average behavior, and that thick clusters of primes will occur arbitrarily far from the origin. Of course, these clusters may be few and far between.

DEFINITION. A *logarithmic center* of a region S is a complex number α which minimizes $f(\alpha) = \int_S \log|z-\alpha|\,d\text{area}$; S is *logarithmically centered* if zero is a logarithmic center of S.

Remark. It is easy to show that minimizing $f(\alpha)$ maximizes (as $x\to\infty$),

$$\mathrm{Li}[x(S-\alpha)] = \frac{2}{\pi}\int_{x(S-\alpha)\cap\{|z|>2\}} \frac{d\text{area}}{\log|z|}. \tag{2}$$